Differential Geometry of Singular Spaces and Reduction of Symmetries
Lecture 2

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- The family $C^\infty(S)$ of functions on S is called a differential structure on S.

- A map $\varphi : S \longrightarrow T$ between differential spaces S and T is smooth if its pull-back φ^* maps $C^\infty(T)$ to $C^\infty(S)$. Moreover, $\varphi : S \longrightarrow T$ diffeomorphism if it is invertible and $\varphi^{-1} : T \longrightarrow S$ is smooth.

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- In the following, we consider only subcartesian differential spaces.

Local charts and derivations

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- All local properties of S can be studied in terms of inclusion maps $\varphi(U) \hookrightarrow \mathbb{R}^n$.
- However, we have to remember that $\varphi(U)$ need not be open.
- A derivation on S is a derivation X of the differential structure $C^\infty(S)$ of S.
- In other words, $X : C^\infty(S) \to C^\infty(S)$ is a linear map satisfying Leibnitz's rule

$$X(f_1 f_2) = X(f_1)f_2 + f_1 X(f_2).$$

Derivations on subcartesian spaces

- If $\varphi : S \to T$ is a diffeomorphism, then a derivation X of $C^\infty(S)$ pushes-forward to a derivation $\varphi_* X : C^\infty(T) \to C^\infty(T)$ such that

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- For each open set U in S, the derivation X of $C^\infty(S)$ defines a derivation $X_{|U}$ of $C^\infty(U)$ such that for every $f \in C^\infty(S)$,

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where ${}_{|U}$ denotes the restriction to U.

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Theorem

If S is a subset of $\mathbb{R}^n$, and X is a derivation of $C^\infty(S)$, then for each $x \in S$, there exists a neighbourhood U of x in $\mathbb{R}^n$ and a derivation Y of $C^\infty(\mathbb{R}^n)$ such that

$$X(F_{|S})_{|S\cap U} = (Y(F))_{|S\cap U}$$

Proof

Let u be a derivation of $\mathcal{C}^\infty(S)$ at $x \in S \subseteq \mathbb{R}^n$. For each $F \in \mathcal{C}^\infty(\mathbb{R}^n)$ the restriction $F_{|S}$ of F to S is in $\mathcal{C}^\infty(S)$. It is easy to see that the map $\mathcal{C}^\infty(\mathbb{R}^n) \to \mathbb{R} : F \mapsto u(F_{|S})$ is a derivation at x of $\mathcal{C}^\infty(\mathbb{R}^n)$.

We denote the natural coordinate functions on $\mathbb{R}^n$ by $x^1, ..., x^n : \mathbb{R}^n \to \mathbb{R}$. Every derivation Y of $\mathcal{C}^\infty(\mathbb{R}^n)$ is of the form $\sum_{i=1}^n F^i \frac{\partial}{\partial x^i}$, where $F^i = Y(x^i)$ for $i = 1, ...n$. Let X be a derivation of $\mathcal{C}^\infty(S)$ and $F \in \mathcal{C}^\infty(\mathbb{R}^n)$. For each $x \in S$, the derivation $X(x)$ of $\mathcal{C}^\infty(S)$ at x gives a derivation of $\mathcal{C}^\infty(\mathbb{R}^n)$ at x. Hence,

$$X(F_{|S})(x) = X(x)(F_{|S}) = \sum_{i=1}^n \frac{\partial F}{\partial x^i}(x)(X(x)(x^i_{|S})) = \sum_{i=1}^n \frac{\partial F}{\partial x^i}(x)(X(x^i_{|S}))(x).$$

This is valid for every $x \in S$.

Proof cd.

Hence,

$$X(F_{|S}) = \sum_{i=1}^{n} \frac{\partial F}{\partial x^i}(X(x^i_{|S})).$$

For $i = 1, ..., n$, the coefficients $X(x^i_{|S})$ are in $\mathcal{C}^\infty(S)$. Since S is a differential subspace of $\mathbb{R}^n$, for each $x \in S$ there exists a neighbourhood U of x in $\mathbb{R}^n$ and functions $F^1, ..., F^n \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that $X(x^i_{|S})_{|U\cap S} = F^i_{|U\cap S}$ for each $i = 1, ..., n$. Hence,

$$X(F_{|S})_{|U\cap S} = \left(\sum_{i=1}^{n} F^i \frac{\partial F}{\partial x^i}\right)_{|U\cap S}.$$

Since $F^1, ..., F^n$ are smooth functions on $\mathbb{R}^n$, it follows that $Y = \sum_{i=1}^{n} F^i \frac{\partial}{\partial x^i}$ is a vector field on $\mathbb{R}^n$. □

Integration of derivations

- Let X be a derivation of $C^\infty(S)$, and let I be an interval in $\mathbb{R}$. A smooth map $c : I \longrightarrow S$ is an integral curve of X if

$$\frac{d}{dt} f(c(t)) = (X(f))(c(t))$$

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Theorem

Let S be a subcartesian space, and let X be a derivation of $C^{\infty}(S)$. For every $x \in S$, there exists a unique maximal integral curve $c : I \longrightarrow S$ of X such that $c(0) = x$.

Proof

(i) Local existence. For the sake of simplicity assumme that S is a subset of $\mathbb{R}^n$. By the preceeding Theorem, there exists a neighbourhood U of $x \in \mathbb{R}^n$ and a vector field Y on $\mathbb{R}^n$ such that

$$X(F_{|S})_{|S\cap U} = (Y(F))_{|S\cap U}$$

for every $F \in C^\infty(\mathbb{R}^n)$.
Let c_0 be an integral curve in $\mathbb{R}^n$ of the vector field Y such that $c_0(0) = x$. Let I_x be the connected component of $c_0^{-1}(\mathbb{R})$ containing 0, and $c : I_x \to R$ the curve in S obtained by the restriction of c_0 to I_x. Clearly, $c(0) = x$. For each $t_0 \in I_x$ and each $f \in \mathcal{C}^\infty(S)$ there exists a neighbourhood U of $c(t_0)$ in S and a function $F \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that $f_{|U} = F_{|U}$. Therefore,

$$\frac{d}{dt} f(c(t))_{|t=t_0} = \frac{d}{dt} F(c(t))_{t=t_0} = (Y(F))(c(t_0)) = (X(f))(c(t_0)),$$

which implies that $c : I_x \to S$ is an integral curve of X through x.

Proof

(ii) Smoothness. It follows from the theory of differential equations that the integral curve c_0 in $\mathbb{R}^n$ of a smooth vector field Y is smooth. Hence, $c = c_{0|I_x}$ is smooth. Since local a local chart $\varphi : U \to \varphi(U) \subseteq \mathbb{R}^n$ chart gives a diffeomorphism of an open neighbourhood U of $x \in S$ on its image in $\mathbb{R}^n$, the inverse φ^{-1} is smooth and the composition $c = \varphi^{-1} \circ c_x$ is smooth.
(iii) Local uniqueness. This follows from the local uniqueness of solutions of first order differential equations in $\mathbb{R}^n$.
(iv) Maximality. Follows the same arguments as for manifolds.
(v) Global uniqueness. Follows the same arguments as for manifolds. □

Vector fields

Let X be a derivation of $\mathcal{C}^\infty(S)$. We denote by $(\exp tX)(x)$ the point on the maximal integral curve of X through x corresponding to the value t of the parameter. Given $x \in S$, $(\exp tX)(x)$ is defined for t in an interval I_x containing zero, and $(\exp 0X)(x) = x$. If t, s, and $t+s$ are in I_x, $s \in I_{(\exp tX)(x)}$, and $t \in I_{(\exp sX)(x)}$, then

$$(\exp(t+s)X)(x) = (\exp sX)((\exp tX)(x)) = (\exp tX)((\exp sX)(x)).$$

In the case when S is a manifold, the map $\exp tX$ is a local one-parameter group of local diffeomorphisms of S. For a subcartesian space S, the $\exp tX$ might fail to be a local diffeomorphism.

Definition

A derivation X is a vector field on S if $\exp tX$ is a local 1-parameter group of local diffeomorphisms of S.

Let $\mathfrak{F}$ be a family of vector fields on a subcartesian space S and x_0 a point in S. Let $X_1, ..., X_n$ be vector fields in $\mathfrak{F}$. Consider a piecewise smooth curve given by first following the integral curve of X_1 through x_0 for time τ_1, next following the integral curve of X_2 through $x_1 = (\exp \tau_1 X_1)(x_0)$ for time τ_2, after that following the integral curve of X_3 through $x_2 = (\exp \tau_2 X_2)(x_1)$ for some τ_3, and so on. For each $i = 1, ..., n$, let J_i be the closed interval in $\mathbb{R}$ with endpoints 0 and τ_i. In other words, $J_i = [0, \tau_i]$ if $\tau_i > 0$ and $J_i = [\tau_i, 0]$ if $\tau_i < 0$. Note that $\tau_i < 0$ means that the integral curve of X_i is followed in the negative time direction. Clearly, for every i, J_i is contained in the domain $I_{x_{i-1}}$ of the maximal integral curve of X_i originating at x_{i-1}.. The range of the curve is

$$\bigcup_{i=1}^{n} \{(\exp t_i X_i)(x_{i-1}) \in S \mid t \in J_i\}.$$

Definition

The orbit O_{x_0} of the family $\mathfrak{F}$ of vector fields is the union of all the ranges described above that contains x_0.

Theorem

Let O_{x_0} be the orbit through x_0 of a family $\mathfrak{F}$ of vector fields on a subcartesian space S. For each $X \in \mathfrak{F}$ and $f \in \mathcal{C}^\infty(S)$, the integral curve of fX through x_0 is contained in O_{x_0}. Similarly, if X, Y are in $\mathfrak{F}$, then the integral curve of $(\exp X)_* Y$ through x_0 is contained in O_{x_0}.

Proof. For $f \in \mathcal{C}^\infty(S)$ and $X \in \mathfrak{F}$, integral curves of X and fX differ by parametrization, provided $f \neq 0$. Integral curves of fX originating at the points for which $f = 0$ are constant. Hence, integral curves of fX originating at x_0 are contained in the orbit O_{x_0} of $\mathfrak{F}$.
We have the equality

$$\exp(s(\exp X)_* Y)(x_0) = \exp(X) \circ \exp(sY) \circ \exp(-X)(x_0)$$

whenever both sides are defined. Hence, the following integral curves $t \mapsto \exp(-tX)(x_0)$, $t \mapsto \exp(tY)[\exp(X)(x_0)]$ and $t \mapsto \exp(tX)[\exp(sY)[\exp(-X)(x_0)]]$ are well defined and contained in O_{x_0}. Moreover, the point $\exp(s(\exp X)_* Y))(x_0)$ can be obtained from x_0 by following the curves c_1, c_2 and c_3. Therefore, $\exp(s(\exp X)_* Y))(x_0)$ is contained in O_{x_0}.

Definition

A family $\mathfrak{F}$ of vector fields on S is locally complete if, for every $X, Y \in \mathfrak{F}$ and $x \in S$, there exists an open neighbourhood U of x and $Z \in \mathfrak{F}$ such that $((\exp X)_* Y)_{|U} = Z_{|U}$.

Theorem

Every family $\mathfrak{F}$ of vector fields on a subcartesian space S can be extended to a locally complete family $\widetilde{\mathfrak{F}}$ with the same orbits.

Proof. If $\mathfrak{F}$ is not locally complete, we can find vector fields X and Y in $\mathfrak{F}$ and $x_0 \in S$, such that there is no neighbourhood U of x_0 and $Z \in \mathfrak{F}$ satisfying $((\exp X)_* Y)_{|U} = Z_{|U}$. Since X is a vector field on S, there is a neighbourhood V of x_0 in S such that $\exp X$ restricts to a diffeomorphism of V onto its image. Hence, $(\exp X)_* Y$ is well defined on V. There exists $f \in \mathcal{C}^\infty(S)$ and open neighbourhoods U_1 and U_2 of x_0 in S such that $\overline{U}_1 \subset U_2 \subset V$, $f_{|U_1} = 1$ and $f_{|S \setminus U_2} = 0$. Define a vector field Z by $Z_{|V} = f(\exp X)_* Y$ and $Z_{|S\setminus U_2} = 0$. By Theorem above, orbits of the family $\mathfrak{F}_1 = \mathfrak{F} \cup \{Z\}$ are the same as orbits of $\mathfrak{F}$. □

Sussmann's Theorem

Theorem

Each orbit O of a family $\mathfrak{F}$ of vector fields on a subcartesian space S is a manifold. Moreover, in the manifold topology of O, the differential structure on O induced by its inclusion in S coincides with its manifold differential structure.

Proof. By Theorem above, there exists a locally complete family of vector fields on S with the same set of orbits as $\mathfrak{F}$. Hence, without loss of generality we may assume that the family $\mathfrak{F}$ is locally complete.
(i) **Notation.** In order to simplify the presentation we introduce the following notation. For $k > 0$, let $\mathbf{X} = (X^1, ..., X^k) \in \mathfrak{F}^k$, $\mathbf{t} = (t_1, ..., t_k)$ and

$$\exp(\mathbf{tX})(x) = \left(\exp(t_k X^k) \circ ... \circ \exp(t_1 X^1)\right)(x).$$

Given $\mathbf{X}$, the expression for $\exp(\mathbf{tX})(x)$ is defined for all $(\mathbf{t}, x)$ in an open subset $\Omega(\mathbf{X})$ of $\mathbb{R}^k \times S$. Let $\Omega_{\mathbf{t}}(\mathbf{X})$ denote the set of all $x \in S$ such that $(\mathbf{t}, x) \in \Omega(\mathbf{X})$.

In other words, $\Omega_{\mathbf{t}}(\mathbf{X})$ is the set of all x for which $\exp(\mathbf{tX})(x)$ is defined. Moreover, we denote by $\Omega_{\mathbf{X}}^{x} \subseteq \mathbb{R}^k$ the set of $\mathbf{t} \in \mathbb{R}^k$ such that $\exp(\mathbf{tX})(x)$ is defined and set

$$\exp_x \mathbf{X} : \Omega_{\mathbf{X}}^{x} \to S : \mathbf{t} \mapsto \exp(\mathbf{tX})(x).$$

By construction, if $x \in O$, then $\exp_x \mathbf{X}$ is smooth and its range is contained in O.

(ii**) Rank of a locally complete family of vector fields.** For each $x \in O$, the rank of $\mathfrak{F}$ at x, denoted $\mathrm{rank}\, \mathfrak{F}_x$, is the number of vector fields $X^1, ..., X^m$ in $\mathfrak{F}$ such that $X^1(x), ..., X^m(x)$ form a basis of the subspace of $T_x S$ spanned by values at x of vector fields in $\mathfrak{F}$. Since linear independence is an open property, it follows that if $X^1, ..., X^m$ are linearly independent at x, then they are linearly independent in a neighbourhood of x. The assumption that the family $\mathfrak{F}$ is locally complete ensures that the rank of $\mathfrak{F}$ is constant on O.

(iii) Covering of the orbit by manifolds. Given $x \in O$, there exists $\mathbf{X} = (X^1, ..., X^m) \in \mathfrak{F}^m$ and a neighbourhood V of x in S such that $\{X^1, ..., X^m\}$ is a maximal linearly independent subset of $\mathfrak{F}_{|V}$. For each $i = 1, ..., m$, and $u \in \mathbb{R}$,

$$u\frac{d}{dt}\exp(tX^i)(x) = uX^i(\exp(tX^i)(x)).$$

Hence, for each $\mathbf{u} = (u_1, ..., u_m) \in \mathbb{R}^m$,

$$T\exp_x \mathbf{X}(\mathbf{u}) = u_1 X^1(x) + ... + u_m X^m(x).$$

The vectors $X_1(x), ..., X_m(x)$ are linearly independent, which implies that the derived map $T\exp_x \mathbf{X} : \mathbb{R}^m \to T_x S$ is one-to-one. Since S is subcartesian, without loss of generality we may assume that there exists a smooth map $\varphi : V \to \mathbb{R}^n$ that induces a diffeomorphism of V onto its image $\varphi(V) \subseteq \mathbb{R}^n$.

By the first Theorem stated here, for every $i = 1, \ldots, m$, the vector field $\varphi_* X^i$ on $\varphi(V)$ locally extends to a vector field Y^i on $\mathbb{R}^n$. Shrinking V, if necessary, we may assume that all vector fields $\varphi_* X^i$ are restrictions to $\varphi(V)$ of vector fields Y^i on $\mathbb{R}^n$. Let $y = \varphi(x)$ and $\mathbf{Y} = (Y_1, \ldots, Y_m)$. As before, we denote by $\exp_y \mathbf{Y}$ the map from the neighbourhood of $\mathbf{0} \in \mathbb{R}^m$ to $\mathbb{R}^n$ given by

$$\exp_y(\mathbf{Y})(\mathbf{t}) = \left(\exp(t_m Y^m)\circ \ldots \circ \exp(t_1 Y^1)\right)(y).$$

Linear independence at x of the vector fields $X^1, \ldots, X^m$ implies that the vector fields $Y_1, \ldots, Y_n$ are linearly independent at y. Hence, there exists a neighbourhood W of $\mathbf{0}$ in $\mathbb{R}^m$ such that $\exp_y \mathbf{Y}(W)$ is a submanifold of $\mathbb{R}^n$ and $\exp_y \mathbf{Y}$, restricted to W, gives a diffeomorphism $\exp_y \mathbf{Y}_{|W} : W \to \exp_y \mathbf{Y}(W)$. Since $y = \varphi(x) \in \varphi(V)$ and for $i = 1, \ldots, m$, the restriction to $\varphi(V)$ of Y^i gives the vector field $\varphi_* X^i$ on $\varphi(V)$, the set $\exp_y Y(W)$ is contained in $\varphi(V)$, and it is the image of $W \subseteq \mathbb{R}^m$ under the map

$$\exp_{\varphi_{(x)}}(\varphi_* \mathbf{X}) : W \to \varphi(V) : \mathbf{t} \to \left(\exp(t_m \varphi_*^m X^1)\circ \ldots \circ \exp(t_1 \varphi_* X^1)\right)(\varphi(x)).$$

In other words,

$$\exp_y \mathbf{Y}(W) = \exp_{\varphi_{(x)}}(\varphi_* \mathbf{X})(W) \subseteq \varphi(V).$$

The differential structure of $\exp_y Y(W)$ is generated by restrictions to $\exp_y Y(W)$ of smooth functions in $\mathcal{C}^\infty(\mathbb{R}^m)$. The differential structure of $\varphi(V)$ is also generated by restrictions to $\varphi(V)$ of smooth functions in $\mathcal{C}^\infty(\mathbb{R}^m)$. Hence, equation above implies that $\exp_{\varphi_{(x)}}(\varphi_* \mathbf{X})(W)$ is a manifold in the differential structure generated by restrictions of smooth functions on $\varphi(V)$. We say that $\exp_{\varphi_{(x)}}(\varphi_* \mathbf{X})(W)$ is a submanifold of $\varphi(V)$. Moreover, $\exp_{\varphi_{(x)}}(\varphi_* \mathbf{X})_{|W} : W \to \exp_{\varphi_{(x)}}(\varphi_* \mathbf{X})(W)$ is a diffeomorphism. Since φ is a diffeomorphism of V on its image $\varphi(V)$ and V is open in S, it follows that $\exp_x(\mathbf{X})(W)$ is a submanifold of S and $\exp_x(\mathbf{X})_{|W}$ is a diffeomorphism of W onto U.
The construction above can be repeated for each point x in the orbit O, a finite collection $\mathbf{X}$ of vector fields in $\mathfrak{F}$, and a neighbourhood W of $\mathbf{0} \in \mathbb{R}^m$ where m is the number of vector fields in $\mathbf{X}$ that satisfy the assumptions made above. In this way we get a family of sets $\exp_x \mathbf{X}(W)$ in O covering O.

In other words,

$$O = \bigcup_{x \in O} \bigcup_{\mathbf{X}} \bigcup_{W} \exp_x \mathbf{X}(W).$$

Each $\exp_x \mathbf{X}(W)$ is a submanifold of S diffeomorphic to W.

(iv) Topology of the orbit. We have shown that the orbit O is covered by a family $\{\exp_x \mathbf{X}(W)\}$ of subsets of O, where $x \in O$, $\mathbf{X} = (X_1, ..., X_m) \in \mathfrak{F}^m$ is a frame field for $\mathfrak{F}$ in a neighbourhood of x and W is a neighbourhood of $\mathbf{0} \in \mathbb{R}^m$ such that $\exp_x X$ is a diffeomorphism of W on its image. We want to take this family of subsets of O to be a basis for the topology of O. For this definition to make sense, we must verify that if $x_0 \in \exp_{x_1} \mathbf{X}_1(W_1) \cap \exp_{x_2} \mathbf{X}_2(W_2)$, then there exists a frame field $\mathbf{X}_0$ for $\mathfrak{F}$ in a neighbourhood of x_0 and an open neighbourhood W_0 of $\mathbf{0}$ in $\mathbb{R}^m$ such that

$$\exp_{x_0} \mathbf{X}_0(W_0) \subseteq \exp_{x_1} \mathbf{X}_1(W_1) \cap \exp_{x_2} \mathbf{X}_2(W_2).$$

This can be verified by direct computation.

Therefore, we can take the family $\{\exp_x \mathbf{X}(W)\}$ of subsets of the orbit O, where $x \in O$, $\mathbf{X} = (X_1, ..., X_m) \in \mathfrak{F}^m$ is a frame field for $\mathfrak{F}$ in a neighbourhood of x and W is a neighbourhood of $\mathbf{0} \in \mathbb{R}^m$ such that $\exp_x \mathbf{X}$ is a diffeomorphism of W on its image, as a basis of a topology $\mathcal{T}$ on O. In this topology, O is a connected topological space locally homeomorphic to $\mathbb{R}^m$. Note that the topology $\mathcal{T}$ of O may be finer than its subspace topology.

(v) Differential structure of the orbit. The orbit O is covered by open sets $\{\exp_x \mathbf{X}(W)\}$, each of which is diffeomorphic to an open neighbourhood of $\mathbf{0} \in \mathbb{R}^m$. Moreover, if the intersection $U_{12} = \exp_{x_1} \mathbf{X}_1(W_1) \cap \exp_{x_2} \mathbf{X}_2(W_2)$ is not empty, then it is an open subset of O and $\exp_{x_1} \mathbf{X}_1 \circ (\exp_{x_2} \mathbf{X}_2)^{-1}$ is a diffeomorphism of $\exp_{x_2} \mathbf{X}_2(U_{12})$ onto $\exp_{x_1} \mathbf{X}_1(U_{12})$. Hence, O is a smooth manifold diffeomorphic to $\mathbb{R}^m$.

For each function $f \in \mathcal{C}^\infty(O)$ and each $x \in O$, the restriction of f to $\exp_x \mathbf{X}(W)$ is smooth. But $\exp_x \mathbf{X}(W)$ is a submanifold of S. It means that if $f \in \mathcal{C}^\infty(O)$, then for each $x \in O$ there exists a neighbourhood $U = \exp_x \mathbf{X}(W)$ of x in O and a function $h \in \mathcal{C}^\infty(S)$ such that $f_{|U} = h_{|U}$.
Conversely, let $f : O \to \mathbb{R}$ be such that for each $x \in O$, there exists an open neighbourhood U of x in O and $h \in \mathcal{C}^\infty(S)$ such that $f_{|U} = h_{|U}$.
Consider $\exp_{x_0} \mathbf{X}_0(W_0) \subseteq O$. By hypothesis, for each $x \in \exp_{x_0} \mathbf{X}_0(W_0)$, there exists an open neighbourhood U of x in O and $h \in \mathcal{C}^\infty(S)$ such that $f_{|U} = h_{|U}$. In particular, the restrictions of f and h to the open neighbourhood $U \cap \exp_{x_0} \mathbf{X}_0(W_0)$ of x in $\exp_{x_0} \mathbf{X}_0(W_0)$ coincide. Hence, the restriction of f to $\exp_{x_0} \mathbf{X}_0(W_0)$ is smooth. This holds for every open set $\exp_{x_0} \mathbf{X}_0(W_0)$ of our covering of O by manifolds. Hence, f is smooth in the manifold differential structure $\mathcal{C}^\infty(O)$ of the orbit.
We have shown that the manifold differential structure of the orbit O coincides with the differential structure of O induced by its inclusion into S. □

Stratified subcartesian spaces

A stratification of a subcartesian space S is a partition of S by a locally finite family $\mathfrak{M}$ of locally closed connected submanifolds M, called strata of $\mathfrak{M}$, which satisfy the following

Frontier Condition. For $M, M' \in \mathfrak{M}$, if $M' \cap \overline{M} \neq \emptyset$, then either $M' = M$ or $M' \subset \overline{M} \backslash M$.

We showed that every subcartesian space S admits a partition $\mathfrak{O}$ by orbits of the family $\mathfrak{X}(S)$ of all vector fields on S, which we denote by $\mathfrak{O}$. It is of interest to see under what conditions this partition of S is a stratification.

The partition $\mathfrak{O}$ of a subcartesian space S by orbits of the family $\mathfrak{X}(S)$ of all vector fields on S satisfies Frontier Condition.

Proof. Let O and O' be orbits of $\mathfrak{X}(S)$. Suppose $x \in O' \cap \overline{O}$ with $O' \neq O$. We first show that $O' \subset \overline{O}$. Note that the orbit O is invariant under the family of one-parameter local groups of local diffeomorphisms of S generated by vector fields. Since, $x \in \overline{O}$, it follows that, for every vector field X on S, $\exp(tX)(x)$ is in $\overline{O}$ if it is defined. But, O' is the orbit of $\mathfrak{X}(S)$ through x. Hence, $O' \subset \overline{O}$.

Stratifications of S can be partially ordered by inclusion. If $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are two stratifications of S, we say that $\mathfrak{M}_1$ is a refinement of $\mathfrak{M}_2$ and write $\mathfrak{M}_1 \geq \mathfrak{M}_2$, if, for every $M_1 \in \mathfrak{M}_1$, there exists $M_2 \in \mathfrak{M}_2$ such that $M_1 \subseteq M_2$. We say that $\mathfrak{M}$ is a minimal (coarsest) stratification of S if it is not a refinement of a different stratification of S. If S is a manifold, then the minimal stratification of S consists of a single manifold $M = S$.

If $(S, \mathfrak{M})$ is a stratified subcartesian space and N is a manifold, the product $S \times N$ is stratified by the family $\mathfrak{M}_{S\times N} = \{M \times N \mid M \in \mathfrak{M}\}$. If U is an open subset of a stratified space $(S, \mathfrak{M})$, we can consider a family $\mathfrak{M}_U = \{M \cap U \mid U \in \mathfrak{M}\}$. In general, $\mathfrak{M}_U$ need not be a stratification of U.

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A stratification $\mathfrak{M}$ of a subcartesian space S is locally trivial if, for every $M \in \mathfrak{M}$ and each $x \in M$,
(i) there exists an open neighbourhood U of x in S such that $\mathfrak{M}_U$ is a stratification of U,
(ii) there exists a subcartesian stratified space $(S', \mathfrak{M}')$ with a distinguished point $y \in S'$ such that the singleton $\{y\} \in \mathfrak{M}'$, and
(iii) there is an isomorphism $\varphi : (U, \mathfrak{M}_U) \to ((M \cap U) \times S', \mathfrak{M}'_{(M \cap U) \times S'})$ such that $\varphi(x) = (x, y)$.
Let $\mathfrak{M}$ be a stratification of a subcartesian space S.

Definition

We say that $\mathfrak{M}$ admits local extension of vector fields if, for each $M \in \mathfrak{M}$, for each vector field X_M on M and for each point $x \in M$, there exists a neighbourhood V of x in M, and a vector field X on S such that $X_{|V} = X_{M|V}$. In other words, the vector field X is an extension to S of the restriction of X_M to V.

Theorem

Every locally trivial stratification of a subcartesian space S admits local extensions of vector fields.

Proof. Let X_M be a vector field on $M \in \mathfrak{M}$. Since M is locally trivial, given $x_0 \in M$, there exists a neighbourhood U of x_0 in M, a stratified differential space $(S', \mathfrak{M}')$ with a distinguished point $y \in S'$ such that the singleton $\{y_0\} \in \mathfrak{M}'$, and an isomorphism $\varphi : U \longrightarrow (M \cap U) \times S'$of stratified subcartesian spaces such that $\varphi(x_0) = (x_0, y_0)$.
Let $\exp(tX_M)$ be the local one-parameter group of local diffeomorphisms of M generated by X_M, and let $X_{(M\cap U)\times S'}$ be a derivation of $\mathcal{C}^\infty((M \cap U) \times S')$ defined by

$$(X_{(M\cap U)\times S'} h)(x, y) = \frac{d}{dt} h(\exp(tX_M)(x), y)_{|t=0},$$

for every $h \in \mathcal{C}^\infty((M \cap U) \times S')$ and each $(x, y) \in (M \cap U) \times S'$. Since $X_{(M\cap U)\times S'}$ is defined in terms of a local one-parameter group $(x, y) \mapsto (\exp(tX_M)(x), y)$ of diffeomorphisms, it is a vector field on $(M \cap U) \times S'$.

We can use the inverse of the diffeomorphism $\varphi : U \to (M \cap U) \times S'$ to push-forward $X_{(M\cap U)\times S'}$ to a vector field $X_U = (\varphi^{-1})_* X_{(M\cap U)\times S'}$ on U. Choose a function $f_0 \in \mathcal{C}^\infty(S)$ with support in U and such that $f(x) = 1$ for x in some neighbourhood U_0 of x_0 contained in U. Let X be a derivation of $\mathcal{C}^\infty(S)$ extending $f_0 X_U$ by zero outside U. In other words, for every $f \in \mathcal{C}^\infty(S)$, if $x \in U$, then $(Xf)(x) = f_0(x)(X_U f)(x)$, and if $x \notin U_0$, then $(Xf)(x) = 0$. Clearly, X is a vector field on S extending the restriction of X_M to $M \cap U_0$. □

Theorem

Let $\mathfrak{M}$ be a stratification of a subcartesian space S admitting local extensions of vector fields. The partition $\mathfrak{O}$ of S by orbits of the family $\mathfrak{X}(S)$ of all vector fields on S is a stratification of S, and $\mathfrak{M}$ is a refinement of $\mathfrak{O}$. Moreover, if $\mathfrak{M}$ is minimal, then $\mathfrak{M} = \mathfrak{O}$.

Proof. Let $\mathfrak{M}$ be a stratification of S admitting local extensions of vector fields. Since every vector field X_M on a manifold $M \in \mathfrak{M}$ extends locally to a vector field on S and M is connected, it follows that M is contained in an orbit $O \in \mathfrak{O}$.

Every orbit $O \in \mathfrak{O}$ is a union of strata of $\mathfrak{M}$. Since $\mathfrak{M}$ is locally finite, for each $x \in O$, there exists a neigbourhood V of x in S which intersects only a finite number of strata $M_1, ..., M_k$ of $\mathfrak{M}$. Hence, V intersects only a finite number of orbits in $\mathfrak{O}$. Moreover, since strata of $\mathfrak{M}$ form a partition of S, it follows that $V = \bigcup_{i=1}^{k} M_i \cap V$.

Consider $x \in M_1$. Since M_1 is locally closed there exists a neighbourhood U of x contained in V, and such that $M_1 \cap U$ is closed in U. We can relabel the manifolds $M_1, ..., M_k$ so that $O \cap U = \bigcup_{i=1}^{l} M_i \cap U$ for some $l \leq k$. Without loss of generality we may assume that $x \in \overline{M}_i$ for each $i = 2, ..., l$. We want to see if $O \cap U$ is closed in U.

Suppose we have a sequence (y_k) in $O \cap U$ convergent to $y \in U$. Since $O \cap U$ is a finite union of disjoined manifolds, there must be a subsequence of (y_k) contained in one of them. Without loss of generality we may assume that each $y_k \in M_i$ for some $i = 1, ..., l$. We want to show that the limit $y = \lim_{k \to \infty} y_k \in O \cap U$. If $y \in M_i$, then $y \in M_i \cap U \subseteq O \cap U$. If $y \in \overline{M}_i \backslash M_i$, then $y \in M_j$ for some $j = 1, ..., k$. By assumption, $y \in U$ and U intersects only the strata that have x in their closure. If $M_j \subseteq O$ then $y \in O \cap U$. Therefore, $y \notin O \cap U$ implies that M_j is not contained in O. By a construction in the proof of Sussmann's Theorem, $\exp_x \mathbf{X}(W)$ is an m dimensional locally closed submanifold of S. Let U_0 be an open neighbourhood of x in U such that $U_0 \cap \exp_x \mathbf{X}(W)$ is closed in U_0.

As before, we consider a sequence (y_k) in $M_i \cap U_0 \cap \exp_x \mathbf{X}(W) \subseteq O \cap U_0$, which converges to $y \in M_j \cap U_0$. Since $M_j \nsubseteq O$, it follows that $y \notin U_0 \cap \exp_x \mathbf{X}(W) \subseteq U_0 \cap O$. This contradicts the fact that $U_0 \cap \exp_x \mathbf{X}(W)$ is closed in U_0 Therefore, $O \cap U$ is closed in U. Since x is an arbitrary point of the orbit O, it follows that O is locally closed.

We have shown that the partition $\mathfrak{O}$ of S by orbits of the family $X(S)$ of all vector fields on S is locally finite and that each orbit in $\mathfrak{O}$ is locally closed. Also, we showed earlier that that $\mathfrak{O}$ is a stratification of S. By construction, every stratum of the original stratification $\mathfrak{M}$ is contained in a stratum of $\mathfrak{O}$. This implies that $\mathfrak{M} \geq \mathfrak{O}$. If $\mathfrak{M}$ is minimal, then $\mathfrak{M} = \mathfrak{O}$. $\square$

Theorem

The space P/G of orbits of a proper action of a Lie group G on a manifold P is a minimally stratified space that admits local extensions of vector fields.

Proof. Minimal stratification (Bierstone). Local extension of vector fields (Lusala - Śniatycki).