Differential Geometry of Singular Spaces and Reduction of Symmetries

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- In 1998, Koon and Marsden proposed Poisson reduction of non-holonomic systems.
- In a 2007 paper, Yoshimura and Marsden investigated reduction of symmetries of Dirac structures.

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- The following photograph of soap bubbles illustrates the structure of a stratified space.

Singular reduction

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- Finally, I realized that Cushman was using the language of differential geometry in the sense of Sikorski.
- In his 1972 book, Sikorski introduced the notion of a differential structure on a topological space that is given by a class of continuous functions which are deemed to be smooth.

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  2. For every $n \in \mathbb{N}$, $F \in C^\infty(\mathbb{R}^n)$ and $f_1, ..., f_n \in C^\infty(S)$,
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  3. If $h : S \longrightarrow \mathbb{R}$ has the property that for every point $x \in S$, there exists an open neighbourhood U of x in S and a function $f \in C^\infty(S)$ such that
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- A differential space S is a manifold if every point of S has a neighbourhood diffeomorphic to an open subset of $\mathbb{R}^n$.
- A differential space S is subcartesian if it is Hausdorff and every point of S has a neighbourhood diffeomorphic to a subset of $\mathbb{R}^n$.

Proper action

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- Properness of Φ means that for every convergent sequence (p_n) in P and a sequence (g_n) in G such that the sequence $(g_n p_n)$ is convergent in P, there exists a convergent subsequence (g_{n_k}) of G and
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- Properness of the action implies that all isotropy groups

$$G_p = \{g \in G \mid gp = p\}$$

are compact.

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- Proof of this theorem involves all the steps which entered in Cushman's singular reduction theory.
- Our aim is to decode the structure of P/G from the data encoded in its differential structure $C^{\infty}(P/G)$.

Differential equations on subcartesian spaces

- A derivation of $C^\infty(S)$ is a map $X : C^\infty(S) \to C^\infty(S)$ satisfiying Leibniz's rule

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- Let I be an interval in $\mathbb{R}$. A smooth map $c : I \to S$ is an integral curve of a derivation X if
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Let S be a subcartesian space, and let X be a derivation of $C^\infty(S)$. For every $x \in S$, there exists a unique maximal integral curve $c : I \to S$ of X such that $c(0) = x$.

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- Let $\mathfrak{F}$ be a family of vector fields on a subcartesian space S.
- For $x_0 \in S$, the orbit of $\mathcal{F}$ through x_0 is

$$O_{x_0} = \bigcup_{n=1}^{\infty} \bigcup_{X_1,..,X_n} \bigcup_{J_1,...,J_n} \bigcup_{i=1}^{n} \{(\exp t_i X_i)(x_{i-1}) \in S \mid t_i \in J_i\},$$

where the vector fields $X_1, ..., X_n$ are in $\mathfrak{F}$ and, for each $i = 1, ..., n$, the interval $J_i \subset I_{x_{i-1}}$ is either $[0, \tau_i]$ or $[\tau_i, 0]$ with $x_i = (\exp \tau_i X_i)(x_{i-1})$.

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- According to our theorem, each of these spaces has a partition by immersed manifolds that are orbits of the family of all vector fields.
- Moreover, this partition is minimal in the sense that there is no local one-parameter group of local diffeomorphisms that acts transversally to the manifolds of the partition.

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- The process of determination of the structure of the orbit space P/G induced by an invariant geometric structure on P is called reduction of symmetries.

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- The differential structure $C^{\infty}(P/G)$ inherits an isomorphic algebraic structure.

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- The last step is to determine the geometric structure on P/G and

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- By a Theorem of Palais (1961), the properness of the action of G on P ensures that for every point $p \in P$, there exists a slice S_p through p.

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- By a Theorem of Palais (1961), the properness of the action of G on P ensures that for every point $p \in P$, there exists a slice S_p through p.

Corollary

The open neighboourhood GS_p/G of the orbit Gp in P/G is diffeomorphic to S_p/G_p.

Bochner's Linearization Lemma

- Since p is a fixed point of G_p, the derived action of G restricted to G_p preserves T_pP, and induces a linear action

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- By Bochner's Linearization Lemma (1945), one can choose S_p so that the action of G_p on S_p is equivalent to the the restriction of Ψ_p to an open G_p-invariant neighbourhood U_p of 0 in a subspace E_p of T_pP.

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Corollary

S_p/G_p is diffeomorphic to the orbit space U_p/G_p of the linear action Ψ_p of G_p on $E_p \supseteq U_p$.

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- By Weyl's Nullstellensatz (1946), the ring of algebraic invariants of a linear action of a compact group is finitely generated.

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- Therefore, P/G is subcartesian.

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- The next step is to determine the geometric structure on P/G on the basis of the algebraic structure of $C^\infty(P/G)$.

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- Thus, the orbit space P/G is stratified. Each stratum is a Poisson manifold singularly foliated by symplectic manifolds.

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Theorem

Each stratum of the orbit type stratification of P/G is a Poisson manifold singularly foliated by symplectic manifolds that are orbits of the family of all Poisson vector fields on P/G.