MOMENTUM MAPS & CLASSICAL FIELDS

– Overview –

MARK J. GOTAY

Pacific Institute for the Mathematical Sciences

Joint work over the years 1979–2009 with:

- Jim Isenberg (Eugene)
- Jerry MARSDEN (Pasadena)
- Richard Montgomery (Santa Cruz)
- Jędrzej Śniatycki (Calgary)
- Phil Yasskin (College Station)

Pacific Institute for the Mathematical Sciences

Goals:

- Study the Lagrangian and Hamiltonian structures of classical field theories (CFTs) with constraints

Pacific Institute for the Mathematical Sciences

Goals:

- Study the Lagrangian and Hamiltonian structures of classical field theories (CFTs) with constraints
- Explore connections between initial value constraints & gauge transformations

Goals:

- Study the Lagrangian and Hamiltonian structures of classical field theories (CFTs) with constraints
- Explore connections between initial value constraints & gauge transformations
- Tie together & understand many different and apparently unrelated facets of CFTs

Goals:

- Study the Lagrangian and Hamiltonian structures of classical field theories (CFTs) with constraints
- Explore connections between initial value constraints & gauge transformations
- Tie together & understand many different and apparently unrelated facets of CFTs
- Focus on roles of gauge symmetry and momentum maps

Pacific Institute for the Mathematical Sciences

Methods & Tools:

- calculus of variations

Methods & Tools:

- calculus of variations
- initial value analysis, Dirac constraint theory

Methods & Tools:

- calculus of variations
- initial value analysis, Dirac constraint theory
- multisymplectic geometry, multimomentum maps

Pacific Institute for the Mathematical Sciences

Methods & Tools:

- calculus of variations
- initial value analysis, Dirac constraint theory
- multisymplectic geometry, multimomentum maps
- Noether's theorem

Pacific Institute for the Mathematical Sciences

Methods & Tools:

- calculus of variations
- initial value analysis, Dirac constraint theory
- multisymplectic geometry, multimomentum maps
- Noether's theorem
- energy-momentum maps

Pacific Institute for the Mathematical Sciences

Generic Properties of Classical Field Theories

Gleaned from extensive study of standard examples:

- electromagnetism, Yang–Mills
- gravity
- strings
- relativistic fluids
- topological field theories ...

Pioneers: Choquet-Bruhat, Lichnerowicz, Dirac–Bergmann, Arnowit–Deser–Misner (ADM), Fischer–Marsden...

Generic Properties of Classical Field Theories

1. *The Euler–Lagrange equations are underdetermined: there are not enough evolution equations to propagate all field components.*

Generic Properties of Classical Field Theories

1. *The Euler–Lagrange equations are underdetermined: there are not enough evolution equations to propagate all field components.*

— This is because the theory has gauge freedom.

— The corresponding gauge group is known at the outset.

Pacific Institute for the Mathematical Sciences

Generic Properties of Classical Field Theories

1. *The Euler–Lagrange equations are underdetermined: there are not enough evolution equations to propagate all field components.*

— This is because the theory has gauge freedom.

— The corresponding gauge group is known at the outset.

— Kinematic fields have no significance.

— Dynamic fields ψ, conjugate momenta ρ have physical meaning.

2. *The Euler – Lagrange equations are overdetermined: they include constraints*

$$\Phi^i(\psi, \rho) = 0 \tag{1}$$

on the choice of initial data.

2. *The Euler – Lagrange equations are overdetermined: they include constraints*

$$\Phi^i(\psi, \rho) = 0 \qquad (1)$$

on the choice of initial data.

— So initial data $(\psi(0), \rho(0))$ cannot be freely specified

Pacific Institute for the Mathematical Sciences

2. *The Euler – Lagrange equations are overdetermined: they include constraints*

$$\Phi^i(\psi, \rho) = 0 \tag{1}$$

on the choice of initial data.

— So initial data $(\psi(0), \rho(0))$ cannot be freely specified

— Elliptic system (typically)

— Assume all constraints are first class in the sense of Dirac

Pacific Institute for the Mathematical Sciences

3. *The* Φ^i *generate gauge transformations of* (ψ, ρ) *via the canonical symplectic structure on the space of Cauchy data.*

3. *The Φ^i generate gauge transformations of (ψ, ρ) via the canonical symplectic structure on the space of Cauchy data.*

— So the presence of constraints $\longleftrightarrow$ gauge freedom

4. *The Hamiltonian (with respect to a slicing) has the form*

$$H = \int_\Sigma \sum_i \alpha_i \Phi^i(\psi, \rho)\, d\Sigma$$

depending linearly on the atlas fields α_i

4. *The Hamiltonian (with respect to a slicing) has the form*

$$H = \int_\Sigma \sum_i \alpha_i \Phi^i(\psi, \rho)\, d\Sigma$$

depending linearly on the atlas fields α_i

Atlas fields:

— closely related to kinematic fields

— arbitrarily specifiable

— “drive” the entire gauge ambiguity of the CFT

Pacific Institute for the Mathematical Sciences

5. *The evolution equations for the dynamic fields* (ψ, ρ) *take the adjoint form*

$$\frac{d}{d\lambda}\begin{pmatrix}\psi \\ \rho\end{pmatrix} = \mathbb{J} \cdot \sum_i \left[D\Phi^i(\psi(\lambda), \rho(\lambda)) \right]^* \alpha_i. \tag{2}$$

Pacific Institute for the Mathematical Sciences

5. *The evolution equations for the dynamic fields* (ψ, ρ) *take the adjoint form*

$$\frac{d}{d\lambda}\begin{pmatrix}\psi \\ \rho\end{pmatrix} = \mathbb{J} \cdot \sum_i \Big[D\Phi^i(\psi(\lambda), \rho(\lambda))\Big]^* \alpha_i. \qquad (2)$$

— λ is a slicing parameter ("time")

— $\mathbb{J}$ is a compatible almost complex structure; * an L^2-adjoint

Pacific Institute for the Mathematical Sciences

5. *The evolution equations for the dynamic fields* (ψ, ρ) *take the adjoint form*

$$\frac{d}{d\lambda}\begin{pmatrix}\psi \\ \rho\end{pmatrix} = \mathbb{J} \cdot \sum_i \left[D\Phi^i(\psi(\lambda), \rho(\lambda))\right]^* \alpha_i. \qquad (2)$$

— λ is a slicing parameter ("time")

— $\mathbb{J}$ is a compatible almost complex structure; * an L^2-adjoint

— hyperbolic system (typically)

Pacific Institute for the Mathematical Sciences

Adjoint form displays, in the clearest and most concise way, the interrelations between the

- dynamics
- initial value constraints, and
- gauge ambiguity of a theory

6. *The Euler–Lagrange equations are equivalent, modulo gauge transformations, to the combined evolution equations* (2) *and constraint equations* (1).

6. *The Euler–Lagrange equations are equivalent, modulo gauge transformations, to the combined evolution equations* (2) *and constraint equations* (1)*.*

— The constraints are preserved by the evolution equations

Pacific Institute for the Mathematical Sciences

7. *The space of solutions of the field equations is not necessarily smooth. It may have quadratic singularities occurring at symmetric solutions.*

7. *The space of solutions of the field equations is not necessarily smooth. It may have quadratic singularities occurring at symmetric solutions.*

— symplectic reduction

— linearization stability

— quantization

Philosophy

Noether's theorem and the Dirac analysis of constraints do much to predict and explain features 1–6.

Philosophy

Noether's theorem and the Dirac analysis of constraints do much to predict and explain features 1–6.

— I wish to go further and provide (realistic) sufficient conditions which guarantee that they must occur in a CFT.

— I provide such criteria for 1–6 and lay the groundwork for 7.

A key objective is thus to derive the adjoint formalism for CFTs.

Pacific Institute for the Mathematical Sciences

Example: (Abelian) Chern–Simons Theory

A topological field theory on 3-dimensional "spacetime" $X = \mathbb{R} \times \Sigma$.

Example: (Abelian) Chern–Simons Theory

A topological field theory on 3-dimensional "spacetime" $X = \mathbb{R} \times \Sigma$.

— Fields are 1-forms $A = (A_0, \boldsymbol{A})$ on X.

Example: (Abelian) Chern–Simons Theory

A topological field theory on 3-dimensional "spacetime" $X = \mathbb{R} \times \Sigma$.

— Fields are 1-forms $A = (A_0, \boldsymbol{A})$ on X.

— Lagrangian density is $\mathcal{L} = dA \wedge A$.

Example: (Abelian) Chern–Simons Theory

A topological field theory on 3-dimensional “spacetime” $X = \mathbb{R} \times \Sigma$.

— Fields are 1-forms $A = (A_0, \boldsymbol{A})$ on X.

— Lagrangian density is $\mathcal{L} = dA \wedge A$.

— Gauge group is $\mathrm{Diff}(X)$: $\eta \cdot A = \eta_* A$

Pacific Institute for the Mathematical Sciences

Example: (Abelian) Chern–Simons Theory

A topological field theory on 3-dimensional "spacetime" $X = \mathbb{R} \times \Sigma$.

— Fields are 1-forms $A = (A_0, \boldsymbol{A})$ on X.

— Lagrangian density is $\mathcal{L} = dA \wedge A$.

— Gauge group is Diff(X): $\eta \cdot A = \eta_* A$

— E–L equations: $dA = 0$.

Pacific Institute for the Mathematical Sciences

— Kinematic fields: A_0

- Kinematic fields: A_0
- Dynamic fields: $\boldsymbol{A}$

Pacific Institute for the Mathematical Sciences

- Kinematic fields: A_0
- Dynamic fields: $\boldsymbol{A}$
- IV constraint: $\Phi(\boldsymbol{A}, \rho) = (dA)_{12} = 0$

Pacific Institute for the Mathematical Sciences

- Kinematic fields: A_0
- Dynamic fields: $\boldsymbol{A}$
- IV constraint: $\Phi(\boldsymbol{A}, \boldsymbol{\rho}) = (dA)_{12} = 0$
- Gauge generators: $(D_i a)\dfrac{\delta}{\delta A^i}$

Pacific Institute for the Mathematical Sciences

— Kinematic fields: A_0

— Dynamic fields: $\boldsymbol{A}$

— IV constraint: $\Phi(\boldsymbol{A}, \boldsymbol{\rho}) = (dA)_{12} = 0$

— Gauge generators: $(D_i a)\dfrac{\delta}{\delta A^i}$

— Slicing of $\Lambda^1(X) \to X$ generated by:

$$\frac{d}{d\lambda} = \zeta^\mu \frac{\partial}{\partial x^\mu} - A_\nu \, \zeta^\nu{}_{,\mu} \frac{\partial}{\partial A_\mu}$$

- Kinematic fields: A_0
- Dynamic fields: $\boldsymbol{A}$
- IV constraint: $\Phi(\boldsymbol{A}, \boldsymbol{\rho}) = (dA)_{12} = 0$
- Gauge generators: $(D_i a)\dfrac{\delta}{\delta A^i}$
- Slicing of $\Lambda^1(X) \to X$ generated by:

$$\frac{d}{d\lambda} = \zeta^\mu \frac{\partial}{\partial x^\mu} - A_\nu\, \zeta^\nu{}_{,\mu} \frac{\partial}{\partial A_\mu}$$

- Hamiltonian: $H = -2\displaystyle\int_\Sigma (dA)_{12}(\zeta^\mu A_\mu)\, d\Sigma$

— Atlas field: $\zeta^{\mu} A_{\mu}$

— Atlas field: $\zeta^\mu A_\mu$

— Evolution equations in adjoint form reduce to:

$$\frac{d}{d\lambda}\begin{pmatrix} A_i \\ \rho^i \end{pmatrix} = \begin{pmatrix} D_i(\zeta^\mu A_\mu) \\ \epsilon^{0ij} D_j(\zeta^\mu A_\mu) \end{pmatrix}$$

This is equivalent to $(dA)_{0i} = 0$.

— N.B. We also have primary constraints $\rho^0 = 0$ and $\rho^i = \epsilon^{0ij} A_j$.

Traditional Approaches to CFT

Traditional Approaches to CFT

— Group-theoretical

Traditional Approaches to CFT

— Group-theoretical

- concerned with the gauge covariance of a CFT
- Lagrangian-oriented
- covariant
- based on Noether's theorem

— Canonical

— Canonical

- initial value analysis
- Hamiltonian-oriented
- not covariant
- based on space + time decomposition

Connections:

— These two aspects of a mechanical system are linked by the momentum map.

Connections:

— These two aspects of a mechanical system are linked by the momentum map.

— One would like to have an analogous connection in CFT relating gauge symmetries to initial value constraints.

Caveat!

The standard notion of a momentum map associated to a symplectic group action usually cannot be carried over to spacetime covariant field theory, because:

Caveat!

The standard notion of a momentum map associated to a symplectic group action usually cannot be carried over to spacetime covariant field theory, because:

— spacetime diffeomorphisms move Cauchy surfaces,

Caveat!

The standard notion of a momentum map associated to a symplectic group action usually cannot be carried over to spacetime covariant field theory, because:

— spacetime diffeomorphisms move Cauchy surfaces, and

— the Hamiltonian formalism is only defined relative to a fixed Cauchy surface

Prime Example: Einstein's theory of vacuum gravity

— the gauge group is the spacetime diffeomorphism group

Prime Example: Einstein's theory of vacuum gravity

— the gauge group is the spacetime diffeomorphism group

— the only remnants of this group on the instantaneous (i.e., space + time split) level are the superhamiltonian $\mathfrak{H}$ and supermomenta $\mathfrak{J}$

Pacific Institute for the Mathematical Sciences

Prime Example: Einstein's theory of vacuum gravity

— the gauge group is the spacetime diffeomorphism group

— the only remnants of this group on the instantaneous (i.e., space + time split) level are the superhamiltonian $\mathfrak{H}$ and supermomenta $\mathfrak{J}$

- interpreted as the generators of temporal and spatial deformations of a Cauchy surface
- these deformations do not form a group
- nor are $\mathfrak{H}$ and $\mathfrak{J}$ components of a momentum map.

This circumstance forces us to work on the covariant level.

The Way Out: Multisymplectic Field Theory

We must construct a covariant counterpart to the instantaneous Hamiltonian formalism.

In the spacetime covariant (or multisymplectic) framework we develop here—an extension and refinement of the formalism of Kijowski and Szczyrba— the gauge group does act.

So we can define a covariant (or multi-) momentum map on the corresponding covariant (or multi-) phase space.

Pacific Institute for the Mathematical Sciences

The Energy-Momentum Map

Key fact: The covariant momentum map induces an energy-momentum map Φ on the instantaneous phase space.

The Energy-Momentum Map

Key fact: The covariant momentum map induces an energy-momentum map Φ on the instantaneous phase space.

— Bridges the covariant & instantaneous formalisms

The Energy-Momentum Map

Key fact: The covariant momentum map induces an energy-momentum map Φ on the instantaneous phase space.

— Bridges the covariant & instantaneous formalisms

— Φ is the crucial object reflecting the gauge transformation covariance of a CFT in the instantaneous picture.

The Energy-Momentum Map

Key fact: The covariant momentum map induces an energy-momentum map Φ on the instantaneous phase space.

- Bridges the covariant & instantaneous formalisms
- Φ is the crucial object reflecting the gauge transformation covariance of a CFT in the instantaneous picture.
- In ADM gravity, $\Phi = -(\mathfrak{H}, \mathfrak{J})$, so that the superhamiltonian and supermomenta are the components of the energy-momentum map.

A recurrent theme is that the energy-momentum map encodes essentially all the dynamical information carried by a CFT: its

— Hamiltonian (Item 4)

A recurrent theme is that the energy-momentum map encodes essentially all the dynamical information carried by a CFT: its

- Hamiltonian (Item 4)
- gauge freedom (Item 3)

A recurrent theme is that the energy-momentum map encodes essentially all the dynamical information carried by a CFT: its

— Hamiltonian (Item 4)

— gauge freedom (Item 3)

— initial value constraints (Item 2)

Pacific Institute for the Mathematical Sciences

A recurrent theme is that the energy-momentum map encodes essentially all the dynamical information carried by a CFT: its

- Hamiltonian (Item 4)
- gauge freedom (Item 3)
- initial value constraints (Item 2)
- stress-energy-momentum tensor
- ...

Indeed:

Energy-Momentum Theorem

The constraints (1) are given by the vanishing of the energy-momentum map associated to the gauge group of the theory.

Φ thus synthesizes the group-theoretical and canonical approaches to CFT.

Pacific Institute for the Mathematical Sciences

Other Highlights

- parametrization theory (à la Kuchař)
- covariantization theory (à la Yang–Mills)
- stress-energy-momentum tensors
- ‘removing’ second class constraints (à la Stückelberg)

Pacific Institute for the Mathematical Sciences