Riemannian cubics on Lie groups and orbits

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Plan of the talk (1)

Have a Lie group G (of **transformations**) acting transitively on a manifold Q (the **object manifold**:

$$G \times Q \to Q, \quad (g, q) \mapsto gq.$$

Introduce a **right-invariant metric** γ_G on G and project the metric to Q to obtain the **normal metric** γ_Q.

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$\rightsquigarrow$ Occurs naturally:
Image/shape matching ($G =$ Diff, $Q =$ shapes);
quantum mechanics ($G = SU(n+1)$, $Q = \mathbb{CP}^n$ with Fubini-Study metric);
unit sphere ($G = SO(3)$, $Q = S^2$ with usual metric)

Plan of the talk (2)

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- Consider a **higher-order variational principle** in both spaces. The solution curves are called **Riemannian cubics**. Generalization of cubic polynomials and cubic splines to Riemannian manifolds.

  $\rightsquigarrow$ Interpolation problems where a certain degree of smoothness is required (piecewise geodesic interpolation leads to discontinuous velocities). This may be unnatural (shapes), or inconvenient (quantum control, camera configurations).

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- Study the relationship between optimal curves on G and optimal curves on Q, for the variational principle defined on the respective spaces.

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Two main questions:

- Which cubics on Q can be lifted horizontally to cubics on G?
- Which cubics on G project to cubics on Q? (more difficult)

Group actions and normal metrics

Actions

- **Lie group G, object manifold Q.**
- **Transitive group action** $\boxed{G \times Q \to Q,\ (g,q) \mapsto gq}$.
- **Infinitesimal action** $\boxed{\xi_Q(q) := \partial_{\varepsilon=0} \exp(\varepsilon\xi) q}$, where $\xi \in \mathfrak{g}$

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Metrics

- **Right-invariant** Riemannian metric γ_G on G; $\boxed{\gamma_G(u_g, v_g) = \gamma_e(u_g g^{-1}, v_g g^{-1})}$, where γ_e restriction of γ_G to $T_eG \times T_eG$.
- **Normal metric** γ_Q on Q given by

$$\boxed{\gamma_Q(u_q, u_q) = \inf_{\{\xi \in \mathfrak{g} \,|\, \xi_Q(q) = u_q\}} \gamma_e(\xi, \xi).}$$

- Will use **raising/lowering** bundle maps $\flat$ and $\sharp$ for both metrics. That is, $\flat : TG \to T^*G$ or $\flat : TQ \to T^*Q$; and $\sharp = \flat^{-1}$.

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Examples

- **Lie group**: $Q = G$, left multiplication; γ_e; obtain right-invariant metric.
- **Unit sphere**: $G = SO(3)$; $Q = S^2$; $\gamma_e(\mathbf{\Omega}, \mathbf{\Omega}) = \mathbf{\Omega} \cdot \mathbf{\Omega}$ ($\cong$ 2-level QM system)
- **Landmark matching**: $G = \mathrm{Diff}(\Omega)$; $Q = \mathbb{R}^3$; $\gamma_e = \ldots$
- **Image matching**: $G = \mathrm{Diff}(\Omega)$; $Q = \mathcal{F}(\Omega)$; γ_e inner product on $\mathfrak{X}(\Omega)$.
- **Quantum mechanics**: $G = SU(n+1)$, $Q = \mathbb{CP}^n$; $\gamma_e(A, B) = -2\,\mathrm{tr}(AB)$

Horizontality

Horizontal space

- Lie algebra of the isotropy subgroup of $q \in Q$ denoted by $\boxed{\mathfrak{g}_q}$ (**vertical space**)
- The **horizontal space** at q is $\boxed{\mathfrak{g}_q^{\perp}}$.
- **Horizontal projection** $\boxed{\mathrm{H}_q}$ orthogonal projection onto $\mathfrak{g}_q^{\perp}$.
- Normal metric can be written as $\boxed{\gamma_Q(u_q, u_q) = \gamma_e(\mathrm{H}_q(\xi), \mathrm{H}_q(\xi))}$ for any ξ with $\xi_Q(q) = u_q$.

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Examples

- **Lie group**: $\mathfrak{g}_g^\perp = \mathfrak{g}$
- **Unit sphere**: $\mathfrak{so}(3)_{\mathbf{x}} = \{\mathbf{\Omega} \text{ with } \mathbf{\Omega} \times \mathbf{x} = 0\} \Rightarrow \{\mathbf{\Omega} \text{ with } \mathbf{\Omega} = \lambda \mathbf{x}\}$.
  Therefore, $\boxed{\mathfrak{so}(3)_{\mathbf{x}}^\perp = \mathbf{x}^\perp}$. (2-dimensional)

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Momentum map

- Denote by $J : T^*Q \to \mathfrak{g}^*$ the **cotangent lift momentum map** for the action of G and Q. Then $\boxed{J^\sharp(\alpha_q) \in \mathfrak{g}_q^\perp.}$ Indeed,

$$\gamma_e(J^\sharp(\alpha_q), \xi) = \langle J(\alpha_q), \xi \rangle_{\mathfrak{g}^* \times \mathfrak{g}} = \langle \alpha_q, \xi_Q(q) \rangle_{T^*Q \times TQ} .$$

- For S^2, $J^\sharp(\mathbf{p}, \mathbf{x}) = \mathbf{x} \times \mathbf{p}$. This is in $\mathbf{x}^\perp = \mathfrak{so}(3)_{\mathbf{x}}^\perp$.

Riemannian submersion

- Fix a **reference object** $a \in Q$ and define the **projection mapping**

$$\boxed{\Pi : G \to Q, \quad g \mapsto ga}$$

  This map is a **Riemannian submersion**. This means that vectors tangent to the object manifold Q can be measured by lifting them horizontally to TG and measuring the resulting horizontal vectors using γ_G.
- $V_g G = \ker T_g \Pi$ and $H_g G = V_g^\perp$ give orthogonal decomposition of TG into horizontal and vertical subbundles $\boxed{TG = HG \oplus VG}$.
- If $\Pi(g) = q$, then

$$V_g G = (\mathfrak{g}_q) g, \quad H_g G = (\mathfrak{g}_q^\perp) g.$$

- A curve $g(t) \in G$ is **horizontal** if $\dot{g} \in H_g G$. This is equivalent to $\xi = \dot{g} g^{-1} \in \mathfrak{g}_q^\perp$.

Riemannian cubics

- Denote by $\boxed{D_t}$ the covariant derivative of the Levi–Civita connection. In coordinates $(D_t\dot{q})^k = \ddot{q}^k + \Gamma^k_{ij}\dot{q}^i\dot{q}^j$.
- Consider the second-order variational problem $\delta\mathcal{J} = 0$ for

$$\boxed{\mathcal{J}[q] = \int_0^1 \|D_t\dot{q}\|_Q^2 \mathrm{dt},}$$

  with respect to curves with fixed end-point velocities.
- The Euler–Lagrange equation is [Noakes et al. 1989], [Crouch & Silva Leite 1995]

$$\boxed{D_t^3\dot{q}(t) + R\left(D_t\dot{q}(t), \dot{q}(t)\right)\dot{q}(t) = 0,}$$

  where R is the curvature tensor $R(X,Y)Z := \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Y - \nabla_{[X,Y]}Z$.
- Solutions to this equation are called **Riemannian cubics**.

Cubics for normal metrics

Goal: Derive the equation for cubics in such a way that the horizontal generator of the curve appears. This will be helpful for analysing horizontal lifting properties.

- The **horizontal generator** of a curve $q(t) \in Q$ is the unique curve $\xi(t) \in \mathfrak{g}$ with $\dot{q} = \xi_Q(q)$ ("generator") and $\xi \in \mathfrak{g}_q^\perp$ ("horizontal").
- Define the map $\bar{J} : TQ \to \mathfrak{g}$ by $\bar{J} := \sharp \circ J \circ \flat$, which "marries" G-action and metric. Then the horizontal generator of a curve $q(t) \in Q$ is given by $\boxed{\bar{J}(\dot{q})}$.
- Formula for the covariant derivative following from the horizontal lifting property of geodesics: $\boxed{D_t\dot{q} = (\dot{\bar{J}} + \mathrm{ad}^\dagger_{\bar{J}}\,\bar{J})_Q(q),}$ where $\mathrm{ad}^\dagger_\nu = \sharp \circ \mathrm{ad}^*_\nu \circ \flat$.
- Fact: $\dot{\bar{J}} + \mathrm{ad}^\dagger_{\bar{J}}\,\bar{J}$ is horizontal, that is, in $\mathfrak{g}_q^\perp$.

$\rightsquigarrow$ Rewrite the Lagrangian:

$$\|D_t\dot{q}\|^2_Q = \|(\dot{\bar{J}} + \mathrm{ad}^\dagger_{\bar{J}}\,\bar{J})_Q(q)\|^2_Q \overset{\text{normal metric}}{=} \|\dot{\bar{J}} + \mathrm{ad}^\dagger_{\bar{J}}\|^2_e$$

$\rightsquigarrow$ Euler–Lagrange equation:

$$\boxed{\left[\left(\frac{\delta \bar{J}}{\delta q}\right)^* - \frac{D}{Dt} \circ \left(\frac{\delta \bar{J}}{\delta \dot{q}}\right)^*\right]\left(\dot{\eta}^\flat + (\mathrm{ad}_{\bar{J}}\,\eta)^\flat - \mathrm{ad}^*_\eta\,\bar{J}^\flat\right) = 0,}$$

where $\eta := \dot{\bar{J}} + \mathrm{ad}^\dagger_{\bar{J}}\,\bar{J}$ and $\frac{\delta \bar{J}}{\delta q}, \frac{\delta \bar{J}}{\delta \dot{q}} : TQ \to \mathfrak{g}$ $\rightsquigarrow$ horizontal generator has appeared $\rightsquigarrow$ Examples.

Cubics on Lie groups

$$\left[\left(\frac{\delta \bar{J}}{\delta q}\right)^* - \frac{D}{Dt}\circ\left(\frac{\delta \bar{J}}{\delta \dot{q}}\right)^*\right]\left(\dot{\eta}^\flat + (\mathrm{ad}_{\bar{J}}\,\eta)^\flat - \mathrm{ad}^*_\eta\,\bar{J}^\flat\right) = 0,$$

- A computation yields

$$\left[\left(\frac{\delta \bar{J}}{\delta g}\right)^* - \frac{D}{Dt}\circ\left(\frac{\delta \bar{J}}{\delta \dot{g}}\right)^*\right]\mu = (TR_{g^{-1}})^*\big(-\partial_t - \mathrm{ad}^*_{\bar{J}}\big)\mu$$

  for any curves $g(t) \in G$ and $\mu(t) \in \mathfrak{g}^*$.
- The Euler–Lagrange equation is therefore

$$(\partial_t + \mathrm{ad}^*_{\bar{J}})\left[\dot{\eta}^\flat - (\mathrm{ad}_{\bar{J}}\,\eta)^\flat + \mathrm{ad}^*_\eta\,\bar{J}^\flat\right] = 0,$$

  where $\eta := \dot{\bar{J}} + \mathrm{ad}^\dagger_{\bar{J}}\,\bar{J}$ and $\bar{J} := \bar{J}(g,\dot{g}) = \dot{g}g^{-1}$.
- If γ_G is **bi-invariant** this recovers the **NHP equation** [Noakes et al. 1989]

$$\dddot{\bar{J}} + [\ddot{\bar{J}}, \bar{J}] = 0.$$

- Alternative derivation on Lie groups proceeds via **second-order Euler–Poincaré reduction**: Lagrangian $L(g,\dot{g}) = \frac{1}{2}\|D_t\dot{g}\|^2 = \frac{1}{2}\|(\dot{\bar{J}} + \mathrm{ad}^\dagger_{\bar{J}}\,\bar{J})g\|^2_g$ is right-invariant with reduced Lagrangian $\ell(\bar{J},\dot{\bar{J}}) = \frac{1}{2}\|\dot{\bar{J}} + \mathrm{ad}^\dagger_{\bar{J}}\,\bar{J}\|^2_e$. Second-order Euler–Poincaré equation is

$$(\partial_t + \mathrm{ad}^*_{\bar{J}})\left(\frac{\delta\ell}{\delta\bar{J}} - \partial_t\frac{\delta\ell}{\delta\dot{\bar{J}}}\right) = 0.$$

  For bi-invariance, $\ell(\bar{J},\dot{\bar{J}}) = \frac{1}{2}\|\dot{\bar{J}}\|^2_e \rightsquigarrow$ NHP equation.

Cubics on symmetric spaces

$$\left[\left(\frac{\delta \bar J}{\delta q}\right)^* - \frac{D}{Dt}\circ\left(\frac{\delta \bar J}{\delta \dot q}\right)^*\right]\left(\dot\eta^\flat + (\mathrm{ad}_{\bar J}\,\eta)^\flat - \mathrm{ad}^*_\eta\,\bar J^\flat\right) = 0,$$

Assume that γ_G is bi-invariant, and let Q be a **symmetric space** for G. This means that $[\mathfrak{g}_q^\perp, \mathfrak{g}_q^\perp] \subset \mathfrak{g}_q$ for all $q \in Q$.

- The Euler–Lagrange equation is equivalent to $\mathrm{H}_q(\dddot{\bar J} + 2[\ddot{\bar J}, \bar J]) = 0$.
- In addition it is true for any curve $q(t) \in Q$ that $\mathrm{V}_q(\dddot{\bar J} + 2[\ddot{\bar J}, \bar J]) = 0$.
- **Therefore:** A curve $q(t) \in Q$ is a Riemannian cubic $\iff$ $\dddot{\bar J} + 2[\ddot{\bar J}, \bar J] = 0$. Derived in a different way in [Crouch & Silva Leite 1995].
- Recall that cubics on the group G satisfy the NHP equation $\dddot{\bar J} + [\ddot{\bar J}, \bar J] = 0$.

Cubics on symmetric spaces

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- Recall that cubics on the group G satisfy the NHP equation $\dddot{\bar{J}} + [\ddot{\bar{J}}, \bar{J}] = 0$.

$\rightsquigarrow$ make use of similarity to analyse horizontal lifting properties.

Horizontal lifts of cubics on symmetric spaces

Goal: Find the cubics on Q that can be lifted horizontally to cubics on G.

- Recall the Riemannian submersion $\Pi : G \to Q$, $g \mapsto ga$ with reference object a.
- Let $q(t)$ be a curve in Q with $q(0) = a$. The curve defined by $g(0) = e$ and $\dot{g} = \bar{J}(\dot{q})g$ is horizontal above $q(t)$.
- Therefore, we are looking for the curves $q(t)$, which satisfy $\dddot{\bar{J}} + 2[\ddot{\bar{J}}, \bar{J}] = 0$ (cubic on Q), and **at the same time** $\dddot{\bar{J}} + [\ddot{\bar{J}}, \bar{J}] = 0$ (cubic on G).

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Theorem: A curve $q(t) \in Q$ is a Riemannian cubic and can be lifted horizontally to a Riemannian cubic $g(t) \in G$ **if and only if** it satisfies $\dot{q}(t) = (\xi(t))_Q(q(t))$ for a curve $\xi(t) \in \mathfrak{g}$ of the form

$$\boxed{\xi(t) = \frac{ut^2}{2} + vt + w,}$$

where u, v, w span an Abelian subalgebra that lies in $\mathfrak{g}_{q(0)}^{\perp}$.

Proof: $\boxed{\Rightarrow}$ $\bar{J}$ solves ▭ $\rightsquigarrow [\ddot{\bar{J}}, \bar{J}] = 0 \rightsquigarrow$ it follows (SSP) that $[\bar{J}, \dot{\bar{J}}] = 0 \rightsquigarrow$ from NHP equation: $\ddot{\bar{J}} =$ constant $\rightsquigarrow \bar{J}$ is 2nd order polynomial in t. Coefficients mutually commuting since $[\bar{J}, \dot{\bar{J}}] = [\bar{J}, \ddot{\bar{J}}] = 0$, and horizontal since $\bar{J}$ and $\dot{\bar{J}}$ as well as $\ddot{\bar{J}}$ are horizontal (SSP).

$\boxed{\Leftarrow}$ Assume $q(0) = a$. Start by showing that $\xi(t)$ is horizontal at all times. This makes use of Exp(span(u, v, w)) being an Abelian subgroup, and bi-invariance of γ_G. $\rightsquigarrow$ Curve $g(0) = e$ and $\dot{g} = \xi g$ horizontal lift of $q(t)$. Both $g(t)$ and $q(t)$ are cubics. ■

Horizontal lifts of cubics on symmetric spaces (cont'd)

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- The **rank** of a symmetric space is the dimension of the maximal Abelian Lie subalgebra of $\mathfrak{g}^{\perp}_q$. $\rightsquigarrow$ The bigger the rank, the more vectors are compatible with the Theorem.
- In rank-one symmetric spaces u, v, w are all collinear.

Corollary: In rank-one symmetric spaces (S^2, for example) the only cubics that can be lifted horizontally to cubics are geodesics composed with a cubic polynomial in time.

Proof: Integrate $\dot{q} = \left(\frac{at^2}{2} + bt + c\right) d_Q(q)$. Find $q(t) = e^{\left(\frac{at^3}{6} + \frac{bt^2}{2} + ct\right) d} q(0)$. ■

Horizontal lifts of cubics on symmetric spaces (cont'd)

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$\rightsquigarrow$ include non-horizontal curves on G.

Projections of non-horizontal geodesics

Stay in the symmetric space context for now. That is, γ_G bi-invariant and $[\mathfrak{g}_q^\perp, \mathfrak{g}_q^\perp] \subset \mathfrak{g}_q$.

Question: Which non-horizontal **geodesics** on G project to cubics on Q?

- First, describe geodesics on G. Euler–Poincaré equation is $\dot{\xi} = 0$, with reconstruction relation $\dot{g} = \xi g$.
- Let $q(t)$ be the projected curve $q(t) = \Pi(g(t)) = g(t)a$. Decompose ξ into horizontal and vertical parts

$$\xi = \mathrm{H}_q(\xi) + \mathrm{V}_q(\xi) = \bar{J}(\dot{q}) + \bar{\sigma}.$$

  Here we defined $\bar{\sigma} := \mathrm{V}_q(\xi)$.
- Evolution equations are

$$\boxed{\dot{\bar{J}} = [\bar{\sigma}, \bar{J}], \quad \dot{\bar{\sigma}} = [\bar{J}, \bar{\sigma}].}$$

  $\rightsquigarrow$ have rewritten the geodesic equation on G.
- Recall that in order for $q(t)$ to be a cubic $\boxed{\dddot{\bar{J}} + 2[\ddot{\bar{J}}, \bar{J}] = 0}$ must hold.

Theorem: The projection $q(t)$ of a geodesic $g(t)$ is a Riemannian cubic **if and only if** at time $t = 0$

$$\boxed{[\bar{\sigma}, [\bar{\sigma}, [\bar{\sigma}, \bar{J}]]] + [\bar{J}, [\bar{J}, [\bar{J}, \bar{\sigma}]]] = 0.}$$

Projections of non-horizontal geodesics (cont'd)

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$$[\bar{\sigma}, [\bar{\sigma}, [\bar{\sigma}, \bar{J}]]] + [\bar{J}, [\bar{J}, [\bar{J}, \bar{\sigma}]]] = 0.$$

Proof: Assume $q(0) = a$. We have $\bar{J}(t) = \mathrm{Ad}_{g(t)} \bar{J}(0)$ and $\bar{\sigma}(t) = \mathrm{Ad}_{g(t)} \bar{\sigma}(0) \rightsquigarrow$ if true at $t = 0$, then true at all times. Plugging in the geodesic equation into the equation for cubics one finds

$$\dddot{\bar{J}} + 2[\ddot{\bar{J}}, \bar{J}] = [\bar{\sigma}, [\bar{\sigma}, [\bar{\sigma}, \bar{J}]]] + [\bar{J}, [\bar{J}, [\bar{J}, \bar{\sigma}]]].$$

■

Special cases:

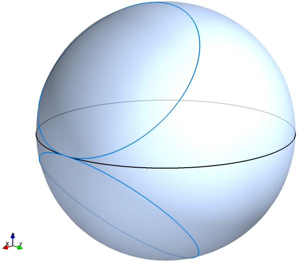
- $\bar{\sigma} = 0 \rightsquigarrow g(t)$ is horizontal geodesic, $q(t)$ is a geodesic.
- $[\bar{\sigma}, \bar{J}] = 0 \rightsquigarrow q(t)$ is a geodesic due to $D_t \dot{q} = \dot{\bar{J}}_Q(q) = ([\bar{\sigma}, \bar{J}])_Q(q) = 0$.
- $[\bar{J}, [\bar{J}, \bar{\sigma}]] = c\bar{\sigma}$, $[\bar{\sigma}, [\bar{\sigma}, \bar{J}]] = c\bar{J}$, for $c \in \mathbb{R}$.

Projections of non-horizontal geodesics (cont'd)

Consider $G = SO(3)$ and $Q = S^2$. A curve $\mathbf{x}(t) \in S^2$ is generated by a rotation vector $\boldsymbol{\Omega} = \bar{\mathbf{J}} + \bar{\boldsymbol{\sigma}}$. That is, $\dot{\mathbf{x}} = \boldsymbol{\Omega} \times \mathbf{x}$. The theorem is equivalent to $\boxed{\left(\|\bar{\boldsymbol{\sigma}}\|^2 - \|\bar{\mathbf{J}}\|^2\right) \bar{\mathbf{J}} \times \bar{\boldsymbol{\sigma}}.}$

- $\bar{\mathbf{J}} = 0$ or $\bar{\boldsymbol{\sigma}} = 0 \rightsquigarrow$ trivial projected curves $\mathbf{x}(t) = \mathbf{x}(0)$, or projections of horizontal geodesics.
- $\|\bar{\boldsymbol{\sigma}}\|^2 = \|\bar{\mathbf{J}}\|^2 \rightsquigarrow$ for given inital velocity $\dot{\mathbf{x}} = \mathbf{v}$, the projection $\mathbf{x}(t)$ describes a **constant speed** rotation of $\mathbf{x}(0)$ around the axis

$$\boldsymbol{\Omega} = \bar{\mathbf{J}} + \bar{\boldsymbol{\sigma}} = \mathbf{x} \times \dot{\mathbf{x}} \pm \|\dot{\mathbf{x}}\|\mathbf{x}.$$



Include all cubics: Finding the obstruction term

Goal: Find the obstruction for the projection of a cubic to be again a cubic. We will use second-order Lagrange–Poincaré reduction for this [Cendra, Marsden, Ratiu 2001], [Gay-Balmaz, Holm, Ratiu 2011]. We still assume that γ_G is bi-invariant, but we drop the condition $[\mathfrak{g}_q^\perp, \mathfrak{g}_q^\perp] \subset \mathfrak{g}_q$.

- Fix reference object $a \in Q$. Stabilizer G_a, with Lie algebra $\mathfrak{g}_a$. $\rightsquigarrow$ will reduce by G_a.
- Right-action of G_a,
$$\psi : G \times G_a \to G, \quad (g, h) = gh.$$
- Quotient space G/G_a is diffeomorphic to Q. Recall the projection map $\Pi : G \to Q$, $g \mapsto ga$.

$\rightsquigarrow$ Principal bundle $\boxed{(G, Q, G_a, \Pi, \psi).}$

- Introduce $\mathfrak{g}_a$-valued principal **connection** $\mathcal{A}$,
$$\boxed{\mathcal{A}} : TG \to \mathfrak{g}_a, \quad v_g \mapsto \mathcal{A}_g(v_g) := \mathrm{V}_a(g^{-1}v_g)$$
- **Adjoint bundle** is the associated vector bundle $\boxed{\tilde{\mathfrak{g}}_a} := (G \times \mathfrak{g}_a)/G_a$, where the quotient is taken wrt right-action of G_a on $G \times \mathfrak{g}_a$,
$$(G \times \mathfrak{g}_a) \times G_a \to G \times \mathfrak{g}_a, \quad (g, \xi, h) \mapsto (gh, \mathrm{Ad}_h^{-1} \xi).$$
- Induced **linear connection** with covariant derivative $\boxed{D_t^{\mathcal{A}}}$
- Need **map** $i : \tilde{\mathfrak{g}}_a \to \mathfrak{g}$, $[g, \xi] \mapsto \mathrm{Ad}_g \xi$, shorthand $\boxed{\sigma \mapsto \bar{\sigma}}$

Include all cubics: Finding the obstruction term (cont'd)

Start by reviewing the geodesic case.

- First-order Lagrange–Poincaré reduction makes use of the bundle diffeomorphism

$$\alpha^{(1)}_{\mathcal{A}} : TG/G_a \to TQ \times_Q \tilde{\mathfrak{g}}_a, \quad [g, \dot{g}] \mapsto (q, \dot{q}) \times [g, \mathcal{A}(\dot{g})].$$

- Reduced variables $\boxed{q, \dot{q}, \sigma.}$
- Geodesics on G arise as solutions to the kinetic energy action principle, where $L(g, \dot{g}) = \frac{1}{2}\|\dot{g}\|^2_G$. $\rightsquigarrow$ Reduced Lagrangian $\ell(q, \dot{q}, \sigma) = \frac{1}{2}\|\dot{q}\|^2_Q + \frac{1}{2}\|\sigma\|^2_{\tilde{\mathfrak{g}}_a}$.
- Lagrange–Poincaré equations describing geodesics are

$$\boxed{D_t \dot{q} = \nabla_{\dot{q}} \bar{\sigma}_Q, \quad D^{\mathcal{A}}_t \sigma = 0.}$$

$\rightsquigarrow$ this reveals the **obstruction term** for $q(t)$ to be a geodesic.

Include all cubics: Finding the obstruction term (cont'd)

Find the obstruction for a cubic to project to a cubic.

- Second-order Lagrange–Poincaré reduction uses bundle diffeomorphism $\alpha^{(2)}_{\mathcal{A}} : T^{(2)}G/G_a \to T^{(2)}Q \times_Q 2\widetilde{\mathfrak{g}}_a$,

$$[g, \dot g, \ddot g] \mapsto (q, \dot q, \ddot q) \times [g, \mathcal{A}(\dot g)] \oplus D^{\mathcal{A}}_t [g, \mathcal{A}(\dot g)] .$$

- Reduced variables $\boxed{q, \dot q, \ddot q, \sigma, \dot\sigma}$
- Lagrangian of Riemannian cubics is $L(g, \dot g, \ddot g) = \frac{1}{2} \left\| D_t \dot g \right\|^2_G$
  $\rightsquigarrow$ reduced Lagrangian

$$\ell(q, \dot q, \ddot q, \sigma, \dot\sigma) = \frac{1}{2} \| D_t \dot q - \nabla_{\dot q} \bar\sigma_Q \|^2_Q + \frac{1}{2} \| \dot\sigma \|^2_{\mathfrak{g}_a}$$

- The Lagrange–Poincaré equations are

$$\boxed{D^3_t \dot q + R\left(D_t \dot q, \dot q\right) \dot q} = D^2_t \nabla_{\dot q} \bar\sigma_Q - \nabla \bar\sigma^T_Q \cdot D_t V_q - \nabla (\partial_t \bar\sigma)^T_Q \cdot V_q$$
$$+ R\left(D_t \dot q, \bar\sigma_Q(q)\right) \dot q + R(\nabla_{\dot q} \bar\sigma_Q, \dot q - \bar\sigma_Q(q)) \dot q$$
$$+ \nabla_{\dot q} \left(i \left(\ddot\sigma + \mathrm{ad}^\dagger_\sigma \dot\sigma + i^T_q \partial_t \bar{\mathbf{J}}(V_q) \right) \right)_Q + F^T_\sigma \left(\mathcal{F}^\nabla \left(V^\flat_q, \dot q \right) \right)^\sharp$$
$$\left(D^{\mathcal{A}}_t + \mathrm{ad}^\dagger_\sigma \right) \left(\ddot\sigma + i^T_q \partial_t \bar{\mathbf{J}}(V_q) \right) = 0.$$

Thank you