Optimal control of underactuated mechanical systems with symmetries

FOCUS PROGRAM ON GEOMETRY, MECHANICS AND DYNAMICS: the Legacy of Jerry Marsden.

ICMAT Instituto de Ciencias Matemáticas

CONSEJO SUPERIOR DE INVESTIGACIONES CIENTÍFICAS

CSIC

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Outline

1 Introduction and motivation
- Euler-Lagrange equations on trivial principal bundles
- Optimal control problem

2 Second-order Euler-Lagrange equations on trivial principal bundles

3 Optimal control of a homogeneous ball on a rotating plate

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Hamilton's principle

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The motion of the mechanical system is described by applying the Hamilton's principle,

$$\delta \int_0^T L(q(t), \dot{q}(t), \xi(t), g(t))dt = 0$$

for all variations $\delta q(t)$ where $\delta q(0) = \delta q(T) = 0$, $q(t) \in M$ and $\delta \xi$ verifying $\delta \xi(t) = \dot{\eta}(t) - [\xi(t), \eta(t)]$, where $\eta(t)$ is an arbitrary curve on the Lie algebra with $\eta(0) = \eta(T) = 0$ and $\eta = (\delta g)g^{-1}$.

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- $L : TM \times \mathfrak{g} \times G \to \mathbb{R}$.

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Euler-Lagrange equations on trivial principal bundles

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$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}, \quad \xi = \dot{g}g^{-1}$$
$$\frac{d}{dt}\left(\frac{\delta L}{\delta \xi}\right) = -\mathrm{ad}^*_{\xi}\left(\frac{\delta L}{\delta \xi}\right) + r^*_g \frac{\delta L}{\delta g},$$

If the Lagrangian L is right-invariant the above equations are written as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}$$
$$\frac{d}{dt}\left(\frac{\delta L}{\delta \xi}\right) = -\mathrm{ad}^*_{\xi}\left(\frac{\delta L}{\delta \xi}\right), \quad \xi = \dot{g}g^{-1}.$$

Euler-Lagrange equations on trivial principal bundles for systems with constraints

We suppose that the system is subject to some constraints equations, $\Phi^{\alpha}: TM \times \mathfrak{g} \times G \to \mathbb{R}$ with $\alpha = 1, \ldots, m \leq n$. The equations of motion for this kind of systems are given using the Lagrange multipliers theorem defining the extended Lagrangian $\widetilde{L} = L + \lambda_{\alpha}\Phi^{\alpha}$, $\alpha = 1, \ldots, m$.

Euler-Lagrange equations on trivial principal bundles for systems with constraints

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{d}{dt}\left(\lambda_{\alpha}\frac{\partial \Phi^{\alpha}}{\partial \dot{q}}\right) &= \frac{\partial L}{\partial q} + \lambda_{\alpha}\frac{\partial \Phi^{\alpha}}{\partial q}, \quad \xi = \dot{g}g^{-1} \\
\frac{d}{dt}\left(\frac{\delta L}{\delta \xi}\right) + \lambda_{\alpha}\frac{d}{dt}\left(\frac{\delta \Phi^{\alpha}}{\delta \xi}\right) &= -\mathrm{ad}^{*}_{\xi}\left(\frac{\delta L}{\delta \xi}\right) - \mathrm{ad}^{*}_{\xi}\left(\lambda_{\alpha}\frac{\delta \Phi^{\alpha}}{\delta \xi}\right) + r^{*}_{g}\frac{\delta L}{\delta g}, \\
&\quad + r^{*}_{g}(\lambda_{\alpha}\frac{\delta \Phi^{\alpha}}{\delta g}) + \dot{\lambda}_{\alpha}\frac{\delta \Phi^{\alpha}}{\delta \xi} \\
\Phi^{\alpha}(q, \dot{q}, g, \xi) &= 0
\end{aligned}
$$

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$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{d}{dt}\left(\lambda_\alpha \frac{\partial \Phi^\alpha}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q} + \lambda_\alpha \frac{\partial \Phi^\alpha}{\partial q}$$

$$\frac{d}{dt}\left(\frac{\delta L}{\delta \xi}\right) + \frac{d}{dt}\left(\lambda_\alpha \frac{\delta \Phi^\alpha}{\delta \xi}\right) = -\mathrm{ad}^*_\xi\left(\frac{\delta L}{\delta \xi}\right) - \mathrm{ad}^*_\xi\left(\lambda_\alpha \frac{\delta \Phi^\alpha}{\delta \xi}\right),$$

$$\xi = \dot{g}g^{-1}, \quad \Phi^\alpha(q, \dot{q}, g, \xi) = 0$$

Optimal control problem

• The first step for studying control systems with symmetries is to take as a configuration manifold a trivial principal bundle $Q = M \times G$.

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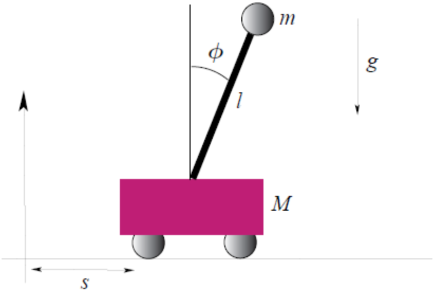
• In what follows we assume that all the control systems are controllable, that is, for any two points x_0 and x_f in the configuration space Q, there exists an admissible control $u(t) \in U \subset \mathbb{R}^r$ defined on some interval $[0, T]$ such that the system with initial condition x_0 reaches the point x_f in time T.
• A control system is called underactuated if the number of control inputs is less than the dimension of the configuration space.

Underactuated systems



Controlled Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^A}\right) - \frac{\partial L}{\partial q^A} = u_a \mu_A^a(q),$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \xi}\right) + ad_\xi^*\left(\frac{\partial L}{\partial \xi}\right) = u_a \eta^a(q),$$

where we denote by $\mathcal{B}^a = \{(\mu^a, \eta^a)\}$, $\mu^a(q) \in T_q^*M$, $\eta^a(q) \in \mathfrak{g}^*$, $a = 1, \ldots, r$; and $A = 1, \ldots, n$. Here, we are assuming that $\{(\mu^a, \eta^a)\}$ are independent elements of $\Gamma(T^*M \times \mathfrak{g}^*)$ and u_a are admissible controls. Taking this into account, the optimal control problem can be formulated as...

Optimal control problem:

Finding trajectories $(q(t), \xi(t), u(t))$ of the state variables and the inputs satisfying the control equations, subject to initial conditions $(q(0), \dot{q}(0), \xi(0))$ and final conditions $(q(T), \dot{q}(T), \xi(T))$, and, moreover, extremizing the functional

$$\mathcal{J}(q, \dot{q}, \xi, u) = \int_0^T C(q(t), \dot{q}(t), \xi(t), u(t))\, dt.$$

We can reformulate this optimal control problem as a higher-order order variational problem subject to higher-order constraints by the following procedure: complete $\mathcal{B}^a$ to a basis $\{\mathcal{B}^a, \mathcal{B}^\alpha\}$ of $\Gamma(T^*M \times \mathfrak{g}^*)$. Take its dual basis $\{\mathcal{B}_a, \mathcal{B}_\alpha\}$ on $\Gamma(TM \times \mathfrak{g})$. This basis induces coordinates $(q^A, \dot{q}, \xi^a, \xi^\alpha)$ on $TM \times \mathfrak{g}$. If we denote by $\mathcal{B}_a = \{(X_a, \chi_a)\} \in \Gamma(TM \times \mathfrak{g})$ (resp. $\mathcal{B}_\alpha = \{(X_\alpha, \chi_\alpha)\} \in \Gamma(TM \times \mathfrak{g})$), controlled equations are rewritten as

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Controlled Euler Lagrange equations

$$\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a}\right) - \frac{\partial L}{\partial q^a}\right) X_a(q) + \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \xi}\right) + \left(ad^*_\xi \frac{\partial L}{\partial \xi}\right)\right) \chi_a(q) = u_a,$$
$$\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^\alpha}\right) - \frac{\partial L}{\partial q^\alpha}\right) X_\alpha(q) + \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \xi}\right) + \left(ad^*_\xi \frac{\partial L}{\partial \xi}\right)\right) \chi_\alpha(q) = 0.$$

The proposed optimal control problem is equivalent to a variational problem with second order constraints, where we define the Lagrangian $\widetilde{L}: T^{(2)}M \times \mathfrak{g}^2 \to \mathbb{R}$ given, in the selected coordinates, by

$$\widetilde{L}(q^A, \dot{q}^A, \ddot{q}^A, \xi^i, \dot{\xi}^i) = C\left(q^A, \dot{q}^A, \xi^i, F_a(q^A, \dot{q}^A, \ddot{q}^A, \xi^i, \dot{\xi}^i)\right),$$

where C is the cost function and

$$\begin{aligned} F_a(q^A, \dot{q}^A, \ddot{q}^A, \xi^i, \dot{\xi}^i) &= \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a}\right) - \frac{\partial L}{\partial q^a}\right) X_a(q) \\ &+ \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \xi}\right) + \left(ad_{\xi}^{*}\frac{\partial L}{\partial \xi}\right)\right)\chi_a(q). \end{aligned}$$

subjected to the second-order constraints:

$$\Phi^{\alpha}(q^A, \dot{q}^A, \ddot{q}^A, \xi^i, \dot{\xi}^i) = \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^{\alpha}}\right) - \frac{\partial L}{\partial q^{\alpha}}\right) X_{\alpha}(q) + \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \xi}\right) + \left(ad^*_{\xi}\frac{\partial L}{\partial \xi}\right)\right)\chi_{\alpha}(q).$$

• A.M. Bloch. Nonholonomic Mechanics and Control. Interdisciplinary Applied Mathematics Series, 24, Springer-Verlag, New York (2003).

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Now we derive the Euler-Lagrange equations for Lagrangians defined on $T^{(2)}Q \simeq T^{(2)}M \times 2\mathfrak{g} \times G$.

- The Lie algebra structure of $2\mathfrak{g}$ is given by

$[(\xi_1, \eta_1), (\xi_2, \eta_2)] = ([\xi_1, \xi_2], [\xi_1, \eta_2] - [\xi_2, \eta_1]) \in 2\mathfrak{g}$.

Hamilton's principle

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Hamilton's principle

Finding the critical curves of the action defined by

$$\mathcal{A}(c(t)) = \int_0^T L(q, \dot{q}, \ddot{q}, g, \xi, \dot{\xi})dt$$

among all the curves $c(t) \in \mathcal{C}^{(2)}(T^{(2)}M \times 2\mathfrak{g} \times G)$ satisfying the boundary conditions for arbitrary variations $\delta c = (\delta q, \delta q^{(1)}, \delta q^{(2)}, \delta g, \delta\xi, \delta\dot{\xi})$, where $\delta q = \frac{d}{d\epsilon}\big|_{\epsilon=0} q_\epsilon$, $\delta q^{(l)} = \frac{d^l}{dt^l}\delta q$, for $l = 1, 2$; and $\delta g = \frac{d}{d\epsilon}\big|_{\epsilon=0} g_\epsilon$.
The corresponding variations $\delta\xi$ induced by δg are given by $\delta\xi = \dot{\eta} - [\xi, \eta]$ where $\eta := \delta g g^{-1} \in \mathfrak{g}$ $(\delta g = \eta g)$.

Second-order Euler-Lagrange equations on trivial principal bundles

Second-order Euler-Lagrange equations on trivial principal bundles

$$\left(-\frac{d}{dt} - ad^*_\xi\right)\left(\frac{\delta L}{\delta \xi} - \frac{d}{dt}\frac{\delta L}{\delta \dot{\xi}}\right) = -r^*_g \frac{\partial L}{\partial g}, \quad \dot{g} = \xi g, \tag{1a}$$

$$\frac{d}{dt}\left(\frac{d}{dt}\frac{\partial L}{\partial \ddot{q}} - \frac{\partial L}{\partial \dot{q}}\right) = -\frac{\partial L}{\partial q}, \tag{1b}$$

which splits into a M part (1a) and a G part (1b).

Higher-order Euler-Lagrange equations on trivial principal bundles for systems with constraints

The equations of motion given by the higher-order variational principle for $L : T^{(k)}M \times k\mathfrak{g} \times G \to \mathbb{R}$ with higher-order constraints $\Phi^{\alpha} : T^{(k)}M \times k\mathfrak{g} \times G \to \mathbb{R}$ reads:

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Higher-order Euler-Lagrange equations on trivial principal bundles for systems with constraints

$$
\begin{aligned}
0 &= \sum_{l=0}^{k}(-1)^l \frac{d^l}{dt^l}\left(\frac{\partial L}{\partial q^{(l)i}} - \lambda_\alpha \frac{\partial \Phi^\alpha}{\partial q^{(l)i}}\right), \\
0 &= \left(\frac{d}{dt} + ad^*_\xi\right)\sum_{l=0}^{k-1}(-1)^l \frac{d^l}{dt^l}\left(\frac{\partial L}{\partial \xi^{(l)}} - \lambda_\alpha \frac{\partial \Phi^\alpha}{\partial \xi^{(l)}}\right) - r_g^*\left(\frac{\partial L}{\partial g} - \lambda_\alpha \frac{\partial \Phi^\alpha}{\partial g}\right), \\
0 &= \Phi^\alpha(c(t)), \\
\dot{g} &= \xi g,
\end{aligned}
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Optimal control of a homogeneous ball on a rotating plate

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• Koon, Wang-Sang; Marsden, Jerrold E. *Optimal control for holonomic and nonholonomic mechanical systems with symmetry and Lagrangian reduction.* SIAM J. Control Optim. 35 (1997), no. 3, 901929.

• Bloch, Anthony; Krishnaprasad, P. S; Marsden, Jerrold E; Murray, Richard M. *Nonholonomic mechanical systems with symmetry.* Arch. Rational Mech. Anal. 136 (1996), no. 1, 2199.

• Bloch, A. M.J. Baillieul, P. Crouch and J. Marsden. *Nonholonomic mechanics and control.* Interdisciplinary Applied Mathematics, 24. Systems and Control. Springer-Verlag, New York, 2003.

Optimal Control of an homogeneous ball on a rotating plate

A (homogeneous) ball of radius $r > 0$, mass m and inertia mk^2 about any axis rolls without sliding on a horizontal table which rotates with angular velocity Ω about a vertical axis x_3 through one of its points. Apart from the constant gravitational force, no other external forces are assumed to act on the sphere.

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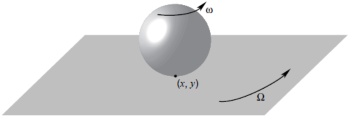
- (x, y) denotes the position of the point of contact of the sphere with the table.
- $Q = \mathbb{R}^2 \times SO(3)$ where may be parametrized Q by (x, y, g), $g \in SO(3)$, all measured with respect to the inertial frame.
- Let $\omega = (\omega_x, \omega_y, \omega_z)$ be the angular velocity vector of the sphere measured also with respect to the inertial frame.



Optimal Control of an homogeneous ball on a rotating plate

• The potential energy is constant, so we may put $V = 0$.
The nonholonomic constraints are

$$\begin{aligned} \dot{x} + \frac{r}{2} Tr(\dot{g} g^T E_2) &= -\Omega(t) y, \\ \dot{y} - \frac{r}{2} Tr(\dot{g} g^T E_1) &= \Omega(t) x, \end{aligned}$$

where $\{E_1, E_2, E_3\}$ is the standard basis of $\mathfrak{so}(3)$.
The matrix $\dot{g} g^T$ is skew-symmetric therefore we may write

$$\dot{g} g^T = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

where $(\omega_1, \omega_2, \omega_3)$ represents the angular velocity vector of the sphere measured with respect to the inertial frame.

Optimal Control of an homogeneous ball on a rotating plate

Then, we may rewrite the constraints in the usual form:

$$\begin{aligned} \dot{x} - r\omega_2 &= -\Omega(t)y, \\ \dot{y} + r\omega_1 &= \Omega(t)x. \end{aligned}$$

Optimal Control of an homogeneous ball on a rotating plate

Then, we may rewrite the constraints in the usual form:

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In addition, since we do not consider external forces the Lagrangian of the system corresponds with the kinetic energy

$$K(x, y, \dot{x}, \dot{y}, \omega_1, \omega_2, \omega_3) = \frac{1}{2}(m\dot{x}^2 + m\dot{y}^2 + mk^2(\omega_1^2 + \omega_2^2 + \omega_3^2)).$$

Optimal Control of an homogeneous ball on a rotating plate

•$Q = \mathbb{R}^2 \times SO(3)$ is the total space of a trivial principal $SO(3)$-bundle over $\mathbb{R}^2$

• the bundle projection $\phi : Q \to M = \mathbb{R}^2$ is just the canonical projection on the first factor.

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Therefore, we may consider the corresponding quotient bundle $E = TQ/SO(3)$ over $M = \mathbb{R}^2$.

• $TSO(3) \simeq \mathfrak{so}(3) \times SO(3)$ by using right translation.

• The tangent action of $SO(3)$ on $T(SO(3)) \cong \mathfrak{so}(3) \times SO(3)$ is the trivial action

$$(\mathfrak{so}(3) \times SO(3)) \times SO(3) \to \mathfrak{so}(3) \times SO(3), \quad ((\omega, g), h) \mapsto (\omega, gh).$$

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Thus, the quotient bundle $TQ/SO(3)$ is isomorphic to the product manifold $T\mathbb{R}^2 \times \mathfrak{so}(3)$, and the vector bundle projection is $\tau_{\mathbb{R}^2} \circ pr_1$, where $pr_1 : T\mathbb{R}^2 \times \mathfrak{so}(3) \to T\mathbb{R}^2$ and $\tau_{\mathbb{R}^2} : T\mathbb{R}^2 \to \mathbb{R}^2$ are the canonical projections.

Optimal Control of an homogeneous ball on a rotating plate

A section of $E = TQ/SO(3) \cong T\mathbb{R}^2 \times \mathfrak{so}(3) \to \mathbb{R}^2$ is a pair (X, f), where X is a vector field on $\mathbb{R}^2$ and $f : \mathbb{R}^2 \to \mathfrak{so}(3)$ is a smooth map. Therefore, a global basis of sections of $T\mathbb{R}^2 \times \mathfrak{so}(3) \to \mathbb{R}^2$ is

$$e_1 = (\frac{\partial}{\partial x}, 0), \quad e_2 = (\frac{\partial}{\partial y}, 0),$$
$$e_3 = (0, E_1), \quad e_4 = (0, E_2), \quad e_5 = (0, E_3).$$

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There exists a one-to-one correspondence between the space $\Gamma(E = TQ/SO(3))$ and the G-invariant vector fields on Q. If $[\![\cdot,\cdot]\!]$ is the Lie bracket on the space $\Gamma(E = TQ/SO(3))$, then the only non-zero fundamental Lie brackets are

$$[\![e_4, e_3]\!] = e_5, \quad [\![e_5, e_4]\!] = e_3, \quad [\![e_3, e_5]\!] = e_4.$$

Optimal Control of an homogeneous ball on a rotating plate

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Moreover, it follows that the Lagrangian function $L = K$ and the constraints are $SO(3)$-invariant. Consequently, L induces a Lagrangian function L' on $E = TQ/SO(3) \simeq T\mathbb{R}^2 \times \mathfrak{so}(3)$.

Optimal Control of an homogeneous ball on a rotating plate

We have a constrained system on $E = TQ/SO(3) \simeq T\mathbb{R}^2 \times \mathfrak{so}(3)$ and note that in this case the constraints are nonholonomic and affine in the velocities. The constraints define an affine subbundle of the vector bundle $E \simeq T\mathbb{R}^2 \times \mathfrak{so}(3) \to \mathbb{R}^2$ which is modeled over the vector subbundle $\mathcal{D}$ generated by the sections

$$\mathcal{D} = span\{e_5; re_1 + e_4; re_2 - e_3\}$$

Optimal Control of an homogeneous ball on a rotating plate

We have a constrained system on $E = TQ/SO(3) \simeq T\mathbb{R}^2 \times \mathfrak{so}(3)$ and note that in this case the constraints are nonholonomic and affine in the velocities. The constraints define an affine subbundle of the vector bundle $E \simeq T\mathbb{R}^2 \times \mathfrak{so}(3) \to \mathbb{R}^2$ which is modeled over the vector subbundle $\mathcal{D}$ generated by the sections

$$\mathcal{D} = span\{e_5; re_1 + e_4; re_2 - e_3\}$$

After some computations the equations of motion for this constrained system are precisely

$$\left.\begin{array}{rcl} \dot{x} - r\omega_2 & = & -\Omega(t)y, \\ \dot{y} + r\omega_1 & = & \Omega(t)x, \\ \dot{\omega}_3 & = & 0, \end{array}\right\}$$

together with

$$\ddot{x} + \frac{k^2\Omega(t)}{r^2 + k^2}\dot{y} = 0, \quad \ddot{y} - \frac{k^2\Omega(t)}{r^2 + k^2}\dot{x} = 0$$

Optimal Control of an homogeneous ball on a rotating plate

Assume full control over the motion of the center of the ball (the shape variables). The controlled system can be written as,

$$\begin{aligned} \ddot{x} + \frac{k^2\Omega(t)}{r^2+k^2}\dot{y} &= u_1, \\ \ddot{y} - \frac{k^2\Omega(t)}{r^2+k^2}\dot{x} &= u_2 \end{aligned}$$

subject to

$$\left.\begin{aligned} \omega_2 - \frac{1}{r}\dot{x} &= \frac{\Omega(t)y}{r}, \\ \omega_1 + \frac{1}{r}\dot{y} &= \frac{\Omega(t)x}{r}, \\ \dot{\omega}_3 &= 0. \end{aligned}\right\}$$

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\end{aligned}
\right\}
$$

Now, consider the cost function

$$
\mathcal{C} = \frac{1}{2}\left(u_1^2 + u_2^2\right) \ ,
$$

Plate ball problem

Given $q_0, q_f \in \mathbb{R}^2$, $\dot{q}_0 \in T_{q_0}\mathbb{R}^2, \dot{q}_f \in T_{q_f}\mathbb{R}^2$, $q = (x, y) \in \mathbb{R}^2$, $\omega_0, \omega_f \in \mathfrak{so}(3)$ find an optimal control curve $(q(t), \omega(t), u(t))$ on the reduced space that steer the system from q_0, ω_0 to q_f, ω_f minimizing

$$\int_0^1 \frac{1}{2}\left(u_1^2 + u_2^2\right) dt,$$

We define the second order Lagrangian $\widetilde{L}: T^{(2)}\mathbb{R}^2 \times 2\mathfrak{so}(3) \to \mathbb{R}$ given by

$$\widetilde{L}(x, y, \dot{x}, \dot{y}, \ddot{x}, \ddot{y}, \omega_1, \omega_2, \omega_3, \dot{\omega}_1, \dot{\omega}_2, \dot{\omega}_3) =$$
$$\frac{1}{2}\left(\ddot{x} + \frac{k^2\Omega(t)}{r^2 + k^2}\dot{y}\right)^2 + \frac{1}{2}\left(\ddot{y} - \frac{k^2\Omega(t)}{r^2 + k^2}\dot{x}\right)^2$$

subject to second-order constraints $\Phi^\alpha : T^{(2)}\mathbb{R}^2 \times 2\mathfrak{so}(3) \to \mathbb{R}$, $\alpha = 1, 2, 3$.

$$\begin{aligned}
\Phi^1 &= \omega_1 + \frac{1}{r}\dot{y} - \frac{\Omega(t)x}{r}, \\
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\end{aligned}$$

The optimal control problem is prescribed by solving the following system of 4-order differential equations (ODEs).

Plate ball problem

$$
\begin{aligned}
0 &= \lambda_1\frac{\Omega(t)}{r}+\frac{\dot\lambda_2}{r}+x^{(iv)}+\frac{k^2\Omega''(t)\dot y}{r^2+k^2}+\frac{2k^2\Omega'(t)\ddot y}{r^2+k^2}+\frac{2k^2\Omega(t)\dddot y}{r^2+k^2},\\
&+ \frac{k^2\Omega'(t)\dddot y}{r^2+k^2}-\frac{k^4\Omega^2(t)\ddot x}{(r^2+k^2)^2}-\frac{2k^4\Omega'(t)\Omega(t)\dot x}{(r^2+k^2)^2}\\
0 &= \lambda_2\frac{\Omega(t)}{r}+\frac{\dot\lambda_1}{r}+y^{(iv)}-\frac{k^2\Omega''(t)\dot x}{r^2+k^2}-\frac{3k^2\Omega'(t)\ddot x}{r^2+k^2}-\frac{2k^2\Omega(t)\dddot x}{r^2+k^2},\\
&- \frac{k^4\Omega^2(t)\ddot y}{(r^2+y^2)^2}-\frac{2k^4\Omega(t)\Omega'(t)\dot y}{(r^2+k^2)^2},\\
0 &= \dot\lambda_1-\lambda_2\omega_3+\lambda_3\omega_2,\quad 0=\dot\lambda_2+\lambda_1\omega_3-\lambda_3\omega_1,\\
0 &= \dot\lambda_3-\lambda_1\omega_2+\lambda_2\omega_1,\quad 0=\omega_2-\frac{1}{r}\dot x-\frac{\Omega(t)y}{r},\\
0 &= \omega_1+\frac{1}{r}\dot y-\frac{\Omega(t)x}{r},\quad 0=\dot\omega_3.
\end{aligned}
$$

Geometric formalism for optimal control problem of underactuated mechanical systems

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⋆ The Skinner and Rusk formulation is a simultaneous mixed formulation of the mechanics between the Lagrangian and Hamiltonian formalism.
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⋆ Solve a higher-order problem with higher-order constraints is equivalent to solve a presymplectic Hamiltonian problem.

⋆ With the Skinner-Rusk formalism we solve a presymplectic Hamiltonian problem and therefore we solve the optimal control problem for underactuated mechanical systems.

¡¡Thank you!!