Higher-order covariant Euler-Poincaré reduction

(Work in progress)

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CONTENTS

REFERENCES

- M.C.L., P.L. García, T. Ratiu, *Euler-Poincaré reduction on principal fiber bundles*, Lett. Math. Phys. **58** (2001) 167–180.
- M.C.L., J. E. Marsden, *Some remarks on Lagrangian and Poisson reduction for Field Theories*, J. Geom. Phys. **48** (2003) 52–83.
- F. Gay-Balmaz, D. Holm, D. Meier, T. Ratiu, F. Vialard, *Invariant higher-order variational problems*, Comm. Math. Phys. **309** (2012), 413–458.
- S. Kouranbaeva, S. Shkoller, *A variational approach to second-order multisymplectic field theory*, J. Geom. Phys. **35** (2000), 333–366.
- J. Muñoz Masqué, *Poincaré-Cartan Forms in Higher Order Variational Calculus on Fibred Manifolds*, Rev. Iberoamericana Mat. **1** (1985), 85–126.

I.- INTRODUCTION and FIRST ORDER

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- In Mechanics ($L : TQ \to \mathbb{R}$) the first main example is **Euler-Poincaré**

$$L : TG \to \mathbb{R} \quad \rightsquigarrow \quad l : (TG)/G = \mathfrak{g} \to \mathbb{R}.$$

with equations

$$\frac{d}{dt}\frac{\delta l}{\delta \sigma} = \pm \mathrm{ad}^*_\sigma \frac{\delta l}{\delta \sigma}.$$

or

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or

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- Further results: Lagrange-Poincaré, by stages, semidirect,...
- There is also the Hamiltonian picture of this formulation.

In Field Theories, the equivalent result takes place in **principal bundles** $\pi : P \to M$. The *objects* are (local) sections and the Lagrangian is defined in the phase bundle

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Geometry of covariant Euler-Poincaré:

- The BUNDLE:

$$P \to M \quad \rightsquigarrow \quad (J^1 P)/G = C \to M$$

  the bundle of connections (affine bundle whose sections are connections). Then $l : C \to \mathbb{R}$.

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  the bundle of connections (affine bundle whose sections are connections). Then $l : C \to \mathbb{R}$.

- The VARIATIONS:

$$\begin{array}{ccccc} s & \in & \Gamma(P) & \rightsquigarrow & \delta s \text{ free} \\ \sigma & \in & \Gamma(C) & \rightsquigarrow & \delta\sigma \text{ gauge} \end{array}$$

  Gauge vector fields in $P \to M$ (G-invariant and vertical vector fields) induce vector fields in $C \to M$. They can be seen as sections of the adjoint bundle $\tilde{\mathfrak{g}} \to M$.

Theorem Given a local section $s : U \to P$ of π and the section $\sigma : U \to C$, $\sigma = [j^1 s]$, the following are equivalent:

1.-s satisfies the Euler-Lagrange equations for L,

2.-the variational principle

$$\delta \int_M L(j^1_x s) dx = 0$$

holds, for arbitrary variations with compact support,

3.-the Euler-Poincaré equations hold:

$$\operatorname{div}^\sigma \frac{\delta l}{\delta \sigma} = 0,$$

4.-the variational principle

$$\delta \int_M l(\sigma(x)) dx = 0$$

holds, using variations of the form

$$\delta\sigma = \nabla^\sigma \eta$$

where $\eta : U \to \tilde{\mathfrak{g}}$ is an arbitrary section of the adjoint bundle.

- Remark:

$$\operatorname{div}^{\sigma} \frac{\delta l}{\delta \sigma} = 0 \qquad \Longleftrightarrow \qquad \left(\operatorname{div}^{\mathcal{H}} + \operatorname{ad}^{*}_{\sigma^{\mathcal{H}}}\right) \frac{\delta l}{\delta \sigma} = 0.$$

- Remark:

$$\operatorname{div}^{\sigma} \frac{\delta l}{\delta \sigma} = 0 \qquad \Longleftrightarrow \qquad \left(\operatorname{div}^{\mathcal{H}} + \operatorname{ad}^{*}_{\sigma^{\mathcal{H}}}\right) \frac{\delta l}{\delta \sigma} = 0.$$

- Remark: If $\sigma = [j^1 s]$, then σ is flat ($\operatorname{Curv}(\sigma) = 0$) and the integral leaves of σ are the solutions s.

$$\mathcal{EL}(L)(s) = 0 \overset{loc}{\Longleftrightarrow} \begin{cases} \mathcal{EP}(l)(\sigma) = 0 \\ \operatorname{Curv}(\sigma) = 0 \end{cases}$$

- There are some topological "issues" (defects, phases,...)

II.- HIGHER-ORDER. THE SPACE

Higher order variational problems are found in important situations

- In Mechanics

$$L : T^{(k)}Q \to \mathbb{R}$$

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What about **reduction** in this context?

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What about **reduction** in this context?

In Mechanics, the basic instance is higher-order Euler-Poincaré

$$L : T^{(k)}G \to \mathbb{R} \quad \rightsquigarrow \quad l : (TG)/G = \oplus^k \mathfrak{g} \to \mathbb{R}.$$

with equations

$$\left(\frac{d}{dt} \mp \mathrm{ad}^*_\sigma\right)\left(\sum_{j=0}^{k-1}(-1)^j \frac{d^k}{dt^k}\frac{\delta l}{\delta \sigma^{(j)}}\right) = 0.$$

- We now consider a fiber bundle

$$E \longrightarrow M$$

a Lagrangian $L : J^k E \longrightarrow \mathbb{R}$ and the action

$$\int_M L(j^k s)\mathbf{v} = 0, \qquad s \text{ section.}$$

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A section s is critical (solution of the variational problem) iff the **Euler-Lagrange equations** are satisfied

$$\mathcal{E}\mathcal{L}_k(\mathcal{L})(s) = 0.$$

Locally, for a fiber coordinate system (x^i, y^α) in E, we have

$$\mathcal{E}\mathcal{L}_k(L)(s) = \sum_{j=0}^{k} (-1)^j \frac{d^k}{dx^{i_1} \cdots dx^{i_j}} \left(\frac{\partial L}{\partial y^\mu_{i_1 \ldots i_j}} \circ j^{2k} s \right) \otimes dy^\mu$$

with (x^i, y^μ) fibred coordinates ($\mathbf{v} = dx^1 \wedge \ldots \wedge dx^n$).

- A covariant definition of $\mathcal{E}\mathcal{L}_k(L)$ as a fiber map from $J^{2k}E \longrightarrow V^*E$ requires the introduction of a connection ∇ in M.

We now assume that the configuration bundle is a G-principal bundle $P \to M$ and have a Lagrangian

$$L : J^k P \to \mathbb{R}$$

invariant with respect to the action of G

$$j^k_x s \cdot g = j^k_x (s \cdot g).$$

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$$(J^k P)/G =?$$

- In Mechanics

$$(TG)/G \simeq \mathfrak{g} \quad \rightsquigarrow \quad (T^{(k)}G)/G \simeq \oplus^k \mathfrak{g} = T^{(k-1)}\mathfrak{g}$$

$$(J^1P)/G \simeq C \quad \rightsquigarrow \quad (J^kP)/G \simeq ????$$

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Proposition 1 We have diagram of bundles

$$\begin{array}{ccc} J^{k-1}(J^1P) & \longrightarrow & J^{k-1}C \\ \downarrow & & \downarrow \\ J^1P & \longrightarrow & C \end{array}$$

Then, for $k > 1$, we consider

$$J^kP \hookrightarrow J^{k-1}(J^1P)$$

then

$$(J^kP)/G = C^{k-1} \hookrightarrow J^{k-1}C.$$

Proposition 2 If we consider the mapping

$$\Omega\colon J^1C \longrightarrow \wedge^2 T^*M \otimes \tilde{\mathfrak{g}}$$
$$j^1_x\sigma \mapsto \Omega^\sigma_x = \mathrm{Curv}(\sigma)_x$$

and its prolongation

$$\begin{array}{rcl} j^{r-1}\Omega\colon J^rC & \longrightarrow & J^{r-1}(\wedge^2 T^*M \otimes \tilde{\mathfrak{g}}) \\ j^r_x\sigma & \longmapsto & j^{r-1}_x\Omega^\sigma \end{array}$$

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In other words

$$\begin{array}{rcl} (J^kP)/G & = & C^{k-1} = \ker j^{k-2}\Omega \\ & = & \{j^{k-1}_x\sigma \in J^{k-1}C : j^{k-2}_x\Omega^\sigma = 0\}. \end{array}$$

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$$\begin{aligned} (J^kP)/G &= C^{k-1} = \ker j^{k-2}\Omega \\ &= \{j^{k-1}_x\sigma \in J^{k-1}C : j^{k-2}_x\Omega^\sigma = 0\}. \end{aligned}$$

Example $k = 2$

$$(J^2P)/G = C^1 = \{j^1_x\sigma \in J^1C : \Omega^\sigma_x = 0\}.$$

Proposition 3 The bundle

$$J^k P \longrightarrow J^{k-1} P$$

is an affine subbundle of

$$J^{k-1}(J^1 P) \longrightarrow J^{k-1} P$$

(this last bundle is the prolongation of $J^1 P \longrightarrow P$.)

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- Locally, if (x^i, y^α) are fiber coordinates in $P \to M$, we consider the induced coordinates $(x^i, y^\alpha, y^\alpha_j)$ in $J^1 P$. Then the coordinates in $J^{k-1}(J^1 P)$ are

  $$(x^i, y^\alpha, y^\alpha_j; y^\alpha_{;i_1 \ldots i_s}, y^\alpha_{j;i_1 \ldots, i_s}), \qquad 1 \leq s \leq k-1,$$

  so that $J^k P \subset J^{k-1}(J^1 P)$ is given by

  $$\begin{aligned} y^\alpha_j &= y^\alpha_{;j} \\ y^\alpha_{i_1;j i_2 \ldots i_s} &= y^\alpha_{j;i_1 i_2 \ldots i_s} \qquad 1 \leq s \leq k-1 \end{aligned}$$

Proposition 4 The bundle

$$\begin{array}{ccc} C^{k-1} & \hookrightarrow & J^{k-1}C \\ \downarrow & & \downarrow \\ M & = & M \end{array}$$

is an affine subbundle.

III.- HIGHER ORDER. THE REDUCTION

Let $L : J^k P \to \mathbb{R}$ a G-invariant Lagrangian and

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- The VARIATIONS. For a (local) section s of $P \to M$ and any δs, consider the gauge vector field X (identified with a section η of $\tilde{\mathfrak{g}} \to M$) such that

  $$X|_s = \delta s$$

  We have the prolongations:

  $$j^1 X \in \mathfrak{X}(J^1 P), \qquad j^k X \in \mathfrak{X}(J^k P)$$
  $$j^{k-1}(j^1 X) \in \mathfrak{X}(J^{k-1}(J^1 P)).$$

Lemma We have that

$$j^{k-1}(j^1 X) = j^k X \text{ on } J^k P \subset J^{k-1}(J^1 P).$$

Then, as $j^1 X$ projects to $\delta\sigma = \nabla^\sigma \eta$:

$j^{k-1}(j^1 X)$ along $j^k s$ projects to $j^{k-1}(\delta\sigma) = j^{k-1}(\nabla^\sigma \eta)$ along $j^{k-1}\sigma$

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Proposition Gauge transformations send flat connections to flat connections. Then, gauge vector fields (the jet prolongation) are tangent to the subbundle $C^{k-1} \subset J^{k-1}C$.

- The VARIATIONAL PRINCIPLE

  How do we study the reduction of the variational principle?

  1) Using Lagrange multipliers $C^{k-1} \subset J^{k-1}C$.

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- For a variation

$$\begin{aligned} \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \int_M \bar{l}(j^{k-1}\sigma_\varepsilon)\mathbf{v} &= \int_M \langle \mathcal{EL}_{k-1}(\bar{l})(\sigma), \nabla^\sigma \eta \rangle \mathbf{v} \\ &= \int_M \langle \operatorname{div}^\sigma \mathcal{EL}_{k-1}(\bar{l}), \eta \rangle \mathbf{v} \end{aligned}$$

where the variation is admissible, i.e., $j^{k-1}\sigma_\varepsilon \in C^{k-1}$, $\forall x \in M$.

- The **higher-order Euler-Poincaré** equations are

$$\mathcal{EP}_{k-1}(\bar{l})(\sigma) = \operatorname{div}^\sigma[\mathcal{EL}_{k-1}(\bar{l})(\sigma)] = 0,$$

for a section σ such that $j_x^{k-1}\sigma \in C^{k-1}$, that is, **for σ flat**.

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- We recover the equivalence

$$\mathcal{EL}_k(L)(s) = 0 \overset{loc}{\Longleftrightarrow} \begin{cases} \mathrm{div}^\sigma[\mathcal{EL}_{k-1}(\bar{l})(\sigma)] = 0 \\ \mathrm{Curv}(\sigma) = 0 \end{cases}$$

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The complete result:

Theorem Given a local section $s : U \to P$ of π and the section $\sigma : U \to C$, $\sigma = [j^1 s]$, the following are equivalent:

1.-s satisfies the Euler-Lagrange equations $\mathcal{E}\mathcal{L}_k(L)(s) = 0$,

2.-the variational principle

$$\delta \int_M L(j^k_x s)dx = 0$$

holds, for arbitrary variations with compact support,

3.-the Euler-Poincaré equations hold, for any extensión $\bar{l}$

$$\mathcal{E}\mathcal{P}_{k-1}(\bar{l})(\sigma) = \operatorname{div}^{\sigma}[\mathcal{E}\mathcal{L}_{k-1}(\bar{l})(\sigma)] = 0,$$

4.-the variational principle

$$\delta \int_M \bar{l}(j^{k-1}\sigma(x))dx = 0$$

holds, using variations of the form

$$\delta\sigma = \nabla^{\sigma}\eta$$

where $\eta : U \to \tilde{\mathfrak{g}}$ is an arbitrary section of the adjoint bundle.

Remark

For $M = \mathbb{R}$ (Mechanics) we recover the existing result

$$\left(\frac{d}{dt} + \mathrm{ad}^*_\sigma \right) \left(\sum_{j=0}^{k-1} (-1)^j \frac{d^k}{dt^k} \frac{\delta l}{\delta \sigma^{(j)}} \right) = 0.$$

without any condition about the curvature of σ.

IV.- NOETHER CONSERVATION LAW

- Let Θ be a Poincaré-Cartan n-form of $L : J^k P \longrightarrow \mathbb{R}$. Then for critical sections

$$(j^{2k-1}s)^* i_Y d\Theta = 0, \qquad \forall Y \in \mathfrak{X}(J^{2k-1}P).$$

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$$(j^{2k-1}s)^* i_Y d\Theta = 0, \qquad \forall Y \in \mathfrak{X}(J^{2k-1}P).$$

- In addition

$$0 = \pounds_{j^{2k-1}B^*}\Theta = i_{j^{2k-1}B^*}d\Theta + d i_{j^{2k-1}B^*}\Theta,$$

for any $B \in \mathfrak{g}$, so that

$$d\left((j^{2k-1}s)^* i_{j^{2k-1}B^*}\Theta\right) = 0,$$

which is **Noether Theorem**.

Proposition The form

$$J = i.\Theta$$

is a $\mathfrak{g}^*$-valued $(n-1)$-form in $J^{2k-1}P$ that projects to a $\tilde{\mathfrak{g}}^*$-valued $(n-1)$-form in $(J^{2k-1}P)/G$. Using the volume form $\mathbf{v}$ we have a section $\mathcal{J}$ of $TM \otimes \tilde{\mathfrak{g}}^*$ in $(J^{2k-1}P)/G$.

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Proposition For any section σ, we have that

$$\mathcal{E}\mathcal{L}_{k-1}(\bar{l})(\sigma) = (j^{2k-2}\sigma)^* \mathcal{J}$$

The equation (for σ flat)

$$\mathrm{div}^\sigma [\mathcal{E}\mathcal{L}_{k-1}(\bar{l})(\sigma)] = 0$$

is equivalent to

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- The Euler-Poincaré equation is equivalent to the higher order Noether conservation law.

V.- EXAMPLE

- We consider $M = \mathbb{R} \times \mathbb{R}$ with points $x = (t, x)$ and $P = M \times G$. The group G is endowed with an invariant Riemannian norm $\|\cdot\|$.

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- The Lagrangian

$$L : J^2 P \to \mathbb{R}$$

$$j^2_{(t,x)} s \mapsto \left\| \frac{\nabla s_t}{\partial t} \right\|^2 + \left\| \frac{\nabla s_x}{\partial x} \right\|^2 + \lambda \left(\|s_t\|^2 + \|s_x\|^2 \right)$$

where

$$s_t = \frac{\partial s}{\partial t}, \qquad s_x = \frac{\partial s}{\partial x},$$

and $\lambda \in \mathbb{R}$.

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$$j^2_{(t,x)} s \mapsto \left\| \frac{\nabla s_t}{\partial t} \right\|^2 + \left\| \frac{\nabla s_x}{\partial x} \right\|^2 + \lambda \left(\|s_t\|^2 + \|s_x\|^2 \right)$$
where
$$s_t = \frac{\partial s}{\partial t}, \qquad s_x = \frac{\partial s}{\partial x},$$
and $\lambda \in \mathbb{R}$.
- This is a "model" of a two dimensional cubic spline in a group ($\lambda = 0$) or two dimensional elastica ($\lambda > 0$).
- For $M = \mathbb{R} \times S^1$, the "model" describes the evolution of a closed curve. (?)

- L is G-invariant and the (extended) reduced Lagrangian is

$$\bar{l} : J^1 C \to \mathbb{R}$$

$$\bar{l}(j^1_x \sigma) = \left\| \frac{\partial \sigma}{\partial t} + \mathrm{ad}^*_{\sigma_1} \sigma \right\|^2 + \left\| \frac{\partial \sigma}{\partial x} + \mathrm{ad}^*_{\sigma_2} \sigma \right\|^2 + \lambda \left\| \sigma \right\|^2$$

where $\sigma = \sigma_1 dt + \sigma_2 dx$ is a section of $C = T^* \mathbb{R}^2 \otimes \mathfrak{g}$, and the norm in $\mathfrak{g}$ is given by the invariant metric.

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- The reduced equations are a bit involved, but if the metric in G is bi-invariant, we get $(\lambda = 0)$

$$(\mathrm{div} + \mathrm{ad}^*_\sigma)(\partial_{tt}\sigma^\flat + \partial_{xx}\sigma^\flat) = 0$$

  together with

$$d\sigma + [\sigma, \sigma] = 0.$$

FUTURE WORK

- Extension to other bundles and actions (higher-order Lagrange-Poincaré)
- Higher order semidirect product reduction
- The Hamiltonian picture (Lie-Poisson)
- Study of the problem of Lagrange
- Reduction under other symmetries (diffeomorphisms...)

REFERENCES

- M.C.L., P.L. García, T. Ratiu, *Euler-Poincaré reduction on principal fiber bundles*, Lett. Math. Phys. **58** (2001) 167–180.
- M.C.L., J. E. Marsden, *Some remarks on Lagrangian and Poisson reduction for Field Theories*, J. Geom. Phys. **48** (2003) 52–83.
- F. Gay-Balmaz, D. Holm, D. Meier, T. Ratiu, F. Vialard, *Invariant higher-order variational problems*, Comm. Math. Phys. **309** (2012), 413–458.
- S. Kouranbaeva, S. Shkoller, *A variational approach to second-order multisymplectic field theory*, J. Geom. Phys. **35** (2000), 333–366.
- J. Muñoz Masqué, *Poincaré-Cartan Forms in Higher Order Variational Calculus on Fibred Manifolds*, Rev. Iberoamericana Mat. **1** (1985), 85–126.

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