Dirac structures

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Geometry, mechanics and dynamics: the legacy of J. Marsden
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Outline:

1. Mechanics and constraints (Dirac's theory)
2. "Degenerate" symplectic geometry: two viewpoints
3. Origins of Dirac structures
4. Properties of Dirac manifolds
5. Recent developments and applications

1. Phase spaces and constraints

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Global structure behind "presymplectic foliations"?

2. Two viewpoints to symplectic geometry

nondegenerate $\omega \in \Omega^2(M)$	nondegenerate $\pi \in \Gamma(\wedge^2 TM)$
$d\omega = 0$	$[\pi, \pi] = 0$
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Going degenerate: presymplectic and Poisson geometries...

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Non-skew bracket: $[\![(X,\alpha),(Y,\beta)]\!] = ([X,Y], \mathcal{L}_X\beta - i_Y d\alpha)$.

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$$M = \mathbb{R}^3, \quad \text{coordinates } (x, y, z)$$

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For $z \neq 0$, this is graph of $\pi = \frac{1}{z}\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$:

$$\{x, y\} = \frac{1}{z}, \quad \{x, z\} = 0, \quad \{y, z\} = 0.$$

singular Poisson versus smooth Dirac ...

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- Presymplectic foliation

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Quotient Poisson manifolds...

Dirac structures = “pre-Poisson”

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$\diamond$ Transversality condition:

Enough that $L \cap TC^\circ$ has constant rank.

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◇ Moment level sets

$J : M \to \mathfrak{g}^*$ Poisson map (=moment map), $C = J^{-1}(0) \hookrightarrow M$
Transversality ok e.g. if 0 is regular value, $\mathfrak{g}$-action free.
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Dirac geometry = intrinsic geometry of constraints...

5. Recent developments and applications

- Courant algebroids, twist by closed 3-forms
- Lie algebroids/groupoids, equivariant cohomology
- Generalized symmetries and moment maps (e.g. G-valued ...)
- Spinors and generalized complex geometry
- Supergeometric viewpoint

Back to mechanics:

- Lagrangian systems with constraints (nonholonomic), implicit Hamiltonian systems (e.g. electric circuits); generalizations to field theory (multi-Dirac)...
- Geometry of nonholonomic brackets...

among others...

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Then

- Dirac structures: modified integrability conditions, but similar properties...
- Twisted Poisson structure: $\frac{1}{2}[\pi,\pi] = \pi^\sharp(\phi)$

The Cartan-Dirac structure on Lie groups

G Lie group, $\langle \cdot, \cdot \rangle_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ Ad-invariant.

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This is ϕ_G-integrable, where $\phi_G \in \Omega^3(M)$ is the Cartan 3-form.

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Singular foliation: Conjugacy classes

Leafwise 2-form (G.H.J.W. '97):

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Compare with Lie-Poisson on $\mathfrak{g}^*$...

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After all, everything is a Lagrangian submanifold...