# Cardiac Electrophysiology on the Moving Heart

## (... or a short story on shape analysis)

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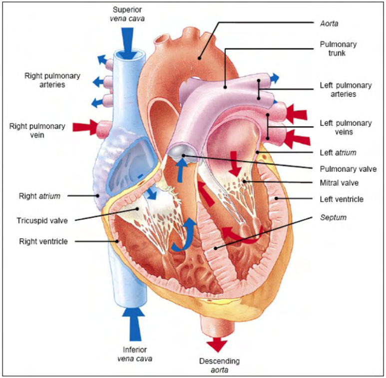
The Legacy of Jerry Marsden

The Fields Institute July 12, 2012

# Outline

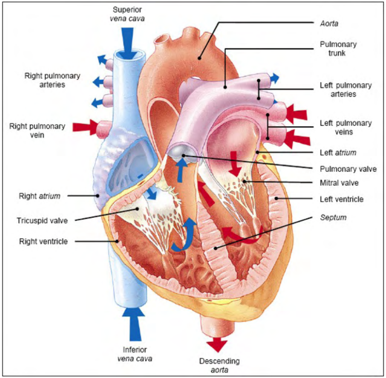
# Cardiovascular diseases




Heart diseases are today responsible for the 28% of deaths in western countries (Cancer is 30%).

- Heart failure is responsible of 400,000 deaths per year in Europe
  - fatigue
  - insufficient ventricular contraction
- Sudden Death affects 1 in 10,000 per year in developed countries
  - electrical disorder
  - ventricular fibrillation

# Cardiovascular diseases




Heart diseases are today responsible for the 28% of deaths in western countries (Cancer is 30%).

## Individualized model for cardiac electrophysiology

- Diagnosis.
- Efficacy of defibrillation in infarcted hearts.
- Ablation location for correcting arrhythmias.

# Outline

# Model: Images + Electrophysiology, but...

**Where is the mechanic model?**

- The Heart Mechanics involves the detailed knowledge of elastic constants at different levels inside the myocardium.
- No-negligible importance of internal fibers.

**and ...**

# Model: Images + Electrophysiology, but...

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- The Heart Mechanics involves the detailed knowledge of elastic constants at different levels inside the myocardium.
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**and ...**

- Can we infer more from a study mainly based on medical images?
- Is it possible to infer good/bad behaviour from motion itself?. Idea: Given a geometry, the wave is going to move like this, contracting the muscle like this, but is actually doing this...
- The solution requires less modeling.

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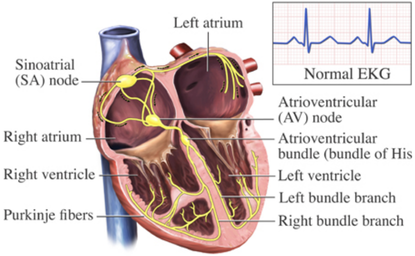
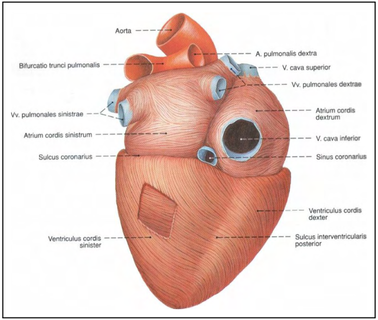
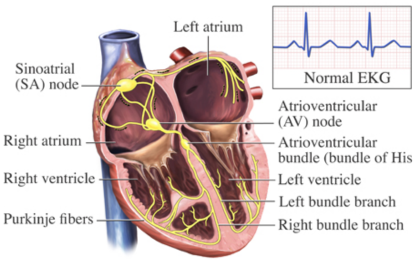
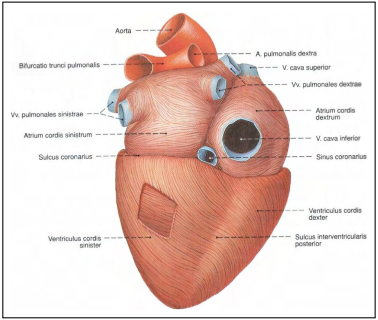



**Maxwell's equations, but...**

1. Electrical field strengths are not too high ⇒ biological tissue is assumed to behave linear with regard to its electrical properties.
2. Cardiac electrical activity is reflected by low frequency components only (≤ several kHz) ⇒ derivatives with respect to time can be neglected ('quasi-static' approximation of Maxwell's equations)




**Maxwell:**

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$J = -DE$$

*and...*

$$\frac{\partial B}{\partial t} = 0 \Rightarrow E = -\nabla u$$

$\longrightarrow$

*Monodomain Model:*

1) $\dfrac{du}{dt} = c_1 f(u,w) + \nabla \cdot (D \nabla u)$

2) $\dfrac{dw}{dt} = \epsilon(u - \gamma w)$

3) $f(u,w) = c_1 u(u-\alpha)(u-1) + c_2 u w$

*FitzHugh-Nagumo Model*

$$I_{ion} = f(u,w)$$

$$\frac{dw}{dt} = \epsilon(u - \gamma w)$$

# Outline

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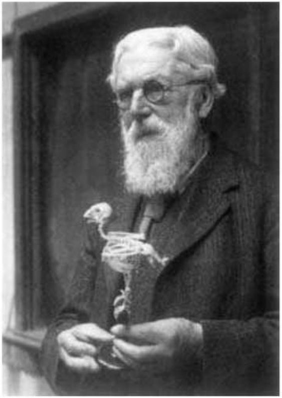
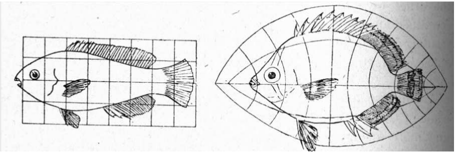
## The Method of coordinates: On Growth and Form (1917)

*" In a very large part of morphology, our essential task lies in the comparison of related forms rather than in the precise definition of each; and the deformation of a complicated figure may be a phenomenon easy of comprehension, though the figure itself may have to be left unanalyzed and undefined. …This method is the Method of Coordinates, on which is based the Theory of Transformations."*



D'ARCY THOMPSON (1860-1948)



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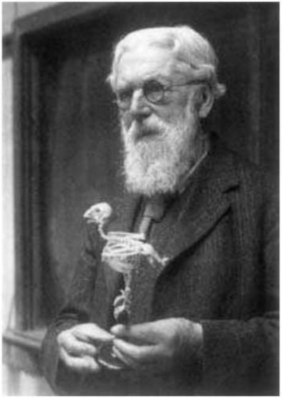

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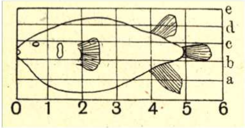


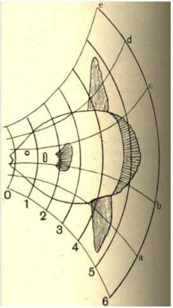

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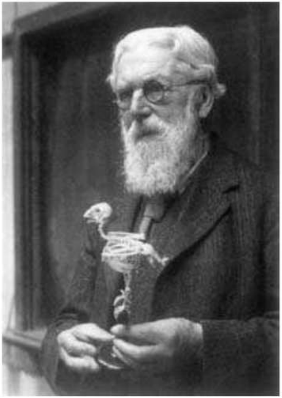

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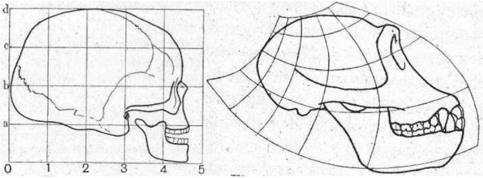
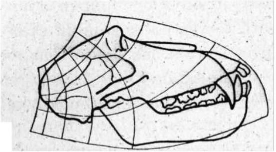
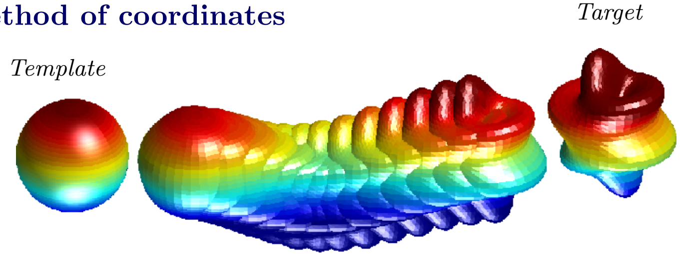


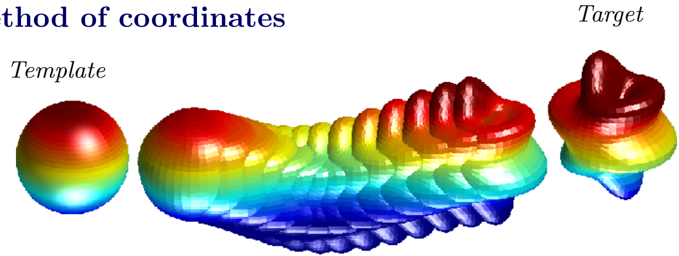



## Our Method of coordinates




$$E(\varphi \cdot Temp, Targ) = \underbrace{d_{\mathcal{V}}(id, \varphi)^2}_{Regulariz.} + \underbrace{U(\varphi \cdot Temp, Targ)}_{Dissimilarity}$$

# Our Method of coordinates




$$E(\varphi \cdot Temp, Targ) = \underbrace{d_{\mathcal{V}}(id, \varphi)^2}_{Regulariz.} + \underbrace{U(\varphi \cdot Temp, Targ)}_{Dissimilarity}$$

**or**

$$\widetilde{E}(v) = \int_0^1 \|v\|_{\mathcal{V}}^2 \, dt + \lambda U(\varphi(1))$$

# Our Method of coordinates


... or $\left(\mu(t)|\, w\right) := \left(\rho_0|\, [D\varphi(t,x)]^{-1}\, w\right)_x$

$$E(\mu(t)) = \sum_{i=1}^{d} \int_0^1 \left(\mu(t)|\, \left(\mu(t)|\, K(\varphi(t,x),\varphi(t,y))e_i\right)_x e_i\right)_y dt + \lambda U(\varphi(1))$$

$$\frac{d\varphi(t,y)}{dt} = \sum_{i=1}^{d} \left(\mu(t)|\, K^i\left(\varphi(t,x),\varphi(t,y)\right)\right)_x e_i$$

$$\left(\frac{d\mu(t)}{dt}\middle|\, w\right) = -\sum_{i=1}^{d} \left(\mu(t)|\, \left(\mu(t)|\, D_2K^i\left(\varphi(t,z),\varphi(t,x)\right).w\right)_z e_i\right)_x$$

# Our Method of coordinates


For points...

$$E(\boldsymbol{x}, \boldsymbol{\alpha}) = \frac{1}{2} \sum_{i,j=1}^{N} \alpha_i(t)^T \alpha_j(t) \, \gamma(x_i(t), x_j(t))$$

$$\frac{dx_s(t)}{dt} = \sum_{k=1}^{N} \gamma\left(x_k(t), x_s(t)\right) \alpha_k(t)$$

$$\frac{d\alpha_s(t)}{dt} = -\sum_{k=1}^{N} \left\{ \alpha_s(t)^T \alpha_k(t) \right\} \nabla_2 \gamma\left(x_k(t), x_s(t)\right)$$

# Gradient Descent on the initial momentum

Energy on the initial momentum: $E(\rho_0) = (\, \rho_0 |\, K\rho_0) + \lambda U(\varphi(1))$
Variation on the initial momentum: $\rho_0 \to \rho_0 + \delta\rho_0$

$$\delta E(\rho_0) = 2\,(\,\delta\rho_0|\,K\rho_0) + \lambda\left(\frac{\delta U}{\delta\varphi}(\varphi(1))\middle|\,\delta\varphi(1)\right) \tag{1}$$

1) From *EPDiff* ... Linearized model:

$$\boxed{\partial_t \begin{Bmatrix}\delta\varphi\\ \delta\mu\end{Bmatrix} = \mathcal{J}_{\varphi,\mu}\begin{Bmatrix}\delta\varphi\\ \delta\mu\end{Bmatrix}} \qquad \delta\varphi(0) = 0, \delta\mu(0) = \delta\rho_0$$

2) Adjoint system: $\xi_\varphi := \delta\varphi^*$, $\xi_\mu := \delta\mu^*$

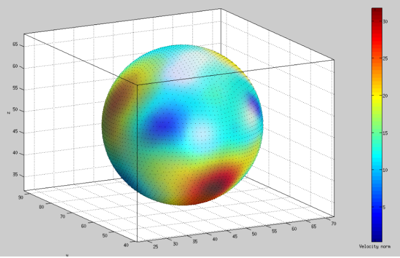
$$\boxed{\partial_t \begin{Bmatrix}\xi_\varphi\\ \xi_\mu\end{Bmatrix} = -\mathcal{J}^*_{\varphi,\mu}\begin{Bmatrix}\xi_\varphi\\ \xi_\mu\end{Bmatrix}} \qquad \xi_\varphi(1) = \frac{\delta U}{\delta\phi}(\varphi(1)), \xi_\mu(1) = 0$$

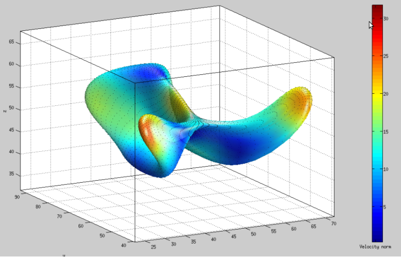
3) So (1) becomes

$$\delta E(\rho_0) = (\,\delta\rho_0|\,2K\rho_0 + \lambda\xi_\mu(0))$$

$\Rightarrow$ $\boxed{\nabla E(\rho_0) = 2\rho_0 + \lambda K^{-1}\xi_\mu(0)}$

# … Gradient Descent on the initial momentum

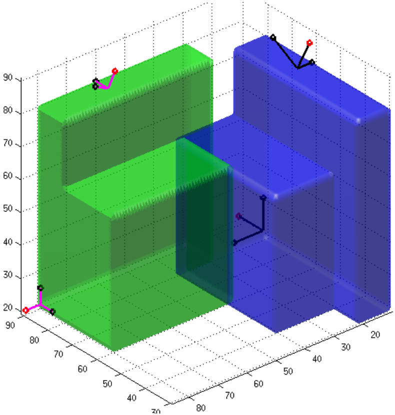


# Outline

# Tracking the in-vivo fibers: Piecewise Affine + Diffeomorphic

1. **Affine Registration:**
   - Translations + Rotations + Scale.
   - Dissimilarity measure: Normalized correlation.

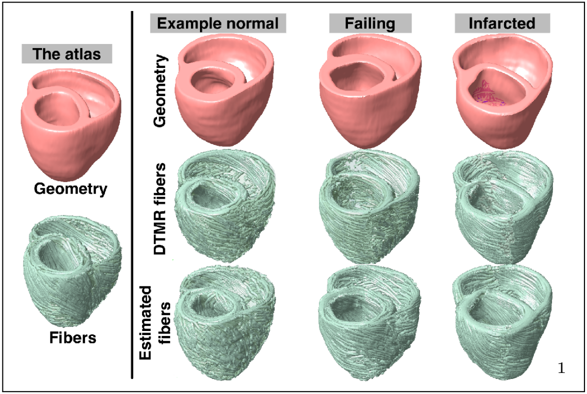



- Pipeline: Performs an affine registration, including scaling.
- Follows a PPD procedure to affine register the tensors.
- It makes easier (faster, more accurate registration) for LDDMM to perform the nonlinear registration once the affine deformed template is much closer to the target.
- It allows the user to input the obtained deformed tensors into DtiStudio.

2. **Nonlinear Registration:** Diffeomorphic matching.

# Tracking the in-vivo fibers: Piecewise Affine + Diffeomorphic

3. **Results:** Tensor Translation + Rotation + Nonlinear Deformation.



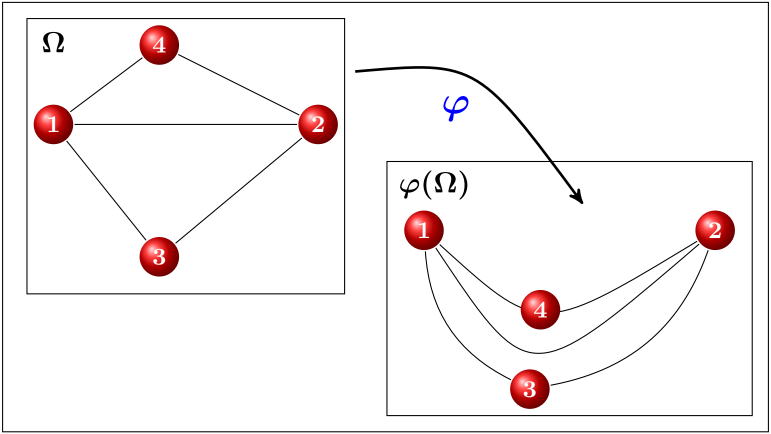

[1] Vadakkumpadan, Trayanova, N., et al., *“Image-based models of cardiac structure in health and disease”*, Wiley Interdisciplinary Reviews: Systems Biology and Medicine, Vol. 2, 2010. (No Scale, Closest Neighbor Interpolation, Reorienting Vectors not complete tensors)

# Outline

# Mesh or ... Meshless Method?

- Finite element methods have been extensively used for the spatial discretization of the myocardium.
- Complicated meshing procedures and element-based interpolation functions often result in algorithms which are either easy to implement, but numerically inaccurate, or accurate but labor-intensive
- The meshfree platform is more adaptive to different cardiac geometries and thus beneficial to individualized analysis.

AND...



Ω
4
1
2
3
φ
φ(Ω)
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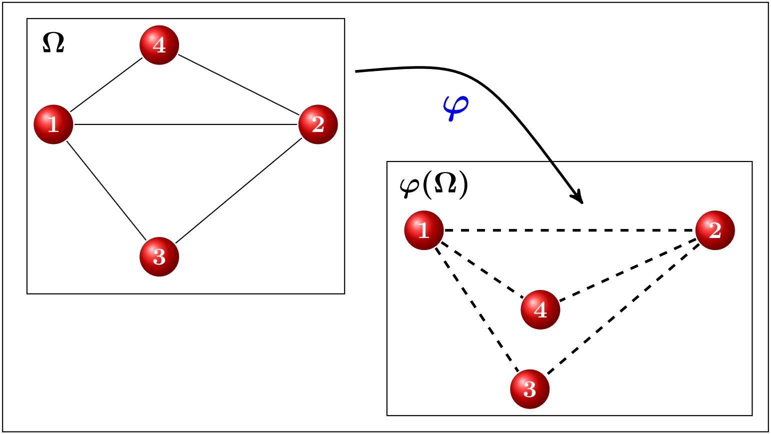
2
4
3



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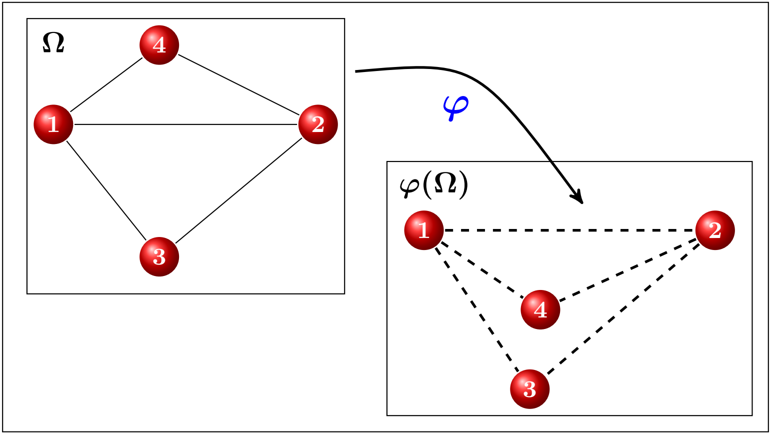


Particle Methods

- Complicated volume meshing procedures are excluded.
- No re-meshing is needed for improving spatial accuracy when deformation occurs.

# Moving Least Squares (MLS) Approximation

1. $\{x_1(t), \ldots, x_N(t)\}$ nodes (particles) in $\Omega \subset \mathbb{R}^3$
2. $\mathbf{p}^T(x) = [p_1(x), \ldots, p_m(x)]$ polynomial basis.

## Governing equation

*Given:*

- *locations $x_i(t)$, for $i = 1, \ldots, N$,*
- *values $u(x_i, t)$, for $i = 1, \ldots, N$*

*Solve the associated ODE system*

$$\frac{d}{dt}\begin{Bmatrix} u \\ w \end{Bmatrix} = \mathbf{\Phi}(u, w)$$

*and obtain*

- *new locations $x_i(t + \Delta t)$, for $i = 1, \ldots, N$,*
- *new values $u(x_i, t + \Delta t)$, for $i = 1, \ldots, N$*

## Moving Least Squares

- *Approximate the solution by:*

$$u(x) = \sum_{k=1}^{m} p_k(x) a_k(x)$$

- *Minimizing the functional*

$$\mathcal{J} = \sum_{i=1}^{N} w(x - x_i) \left[\mathbf{p}^T(x_i)\mathbf{a}(x) - u_i\right]^2$$

*($w(x - x_i)$ weighing function with compact support)*

- *Solve the MLS problem*

$$A(x)\mathbf{a}(x) = B(x)\mathbf{u}$$

# MLS Approximation: Monodomain Model

## 1. Monodomain model

*u - membrane potential*
*w - recovery variable.*

$$\frac{\partial u}{\partial t} = c_1 f(u,w) + \nabla \cdot (D \nabla u)$$
$$\frac{\partial w}{\partial t} = \epsilon(u - \gamma w)$$
$$f(u,w) = c_1 u(u-\alpha)(u-1) - c_2 uw$$

## 2. Weak formulation

$\phi$ *- regular test function*

$$\int_\Omega \phi \frac{\partial u}{\partial t} = \int_\Omega \phi\, c_1 f(u,w) - \int_\Omega \nabla \phi^T (D \nabla u)$$
$$\int_\Omega \phi \frac{\partial v}{\partial t} = \int_\Omega \phi\, \epsilon(u - \gamma w)$$

## 3. Meshfree approximation

$\Phi = [\phi_1(x), \dots, \phi_N(x)]$ *- shape function*

$$u \sim \Phi \mathbf{u} \qquad w \sim \Phi \mathbf{w}$$
$$\left[\int_\Omega \Phi^T \Phi\right] \frac{\partial \mathbf{u}}{\partial t} = \left[\int_\Omega \Phi^T \Phi\right] f(\mathbf{u}, \mathbf{w}) - \left[\int_\Omega \nabla \Phi^T D \nabla \Phi\right] \mathbf{u}$$
$$\left[\int_\Omega \Phi^T \Phi\right] \frac{\partial \mathbf{w}}{\partial t} = \left[\int_\Omega \Phi^T \Phi\right] \epsilon(\mathbf{u} - \gamma \mathbf{w})$$

## 4. ODE system

$$\frac{\partial \mathbf{u}}{\partial t} = f(\mathbf{u}, \mathbf{w}) + M^{-1} K\, \mathbf{u}$$
$$\frac{\partial \mathbf{w}}{\partial t} = \epsilon(\mathbf{u} - \gamma \mathbf{w})$$
$$f(\mathbf{u}, \mathbf{w}) = c_1 \mathbf{u} \circ (\mathbf{u} - \alpha) \circ (\mathbf{u} - 1) - c_2 \mathbf{u} \circ \mathbf{w}$$

*where*

$$M = \int_\Omega \Phi^T \Phi \qquad K = \int_\Omega \nabla \Phi^T D \nabla \Phi$$

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**W**

**R**

**O**

**N**

**G!**

**!**

**!**

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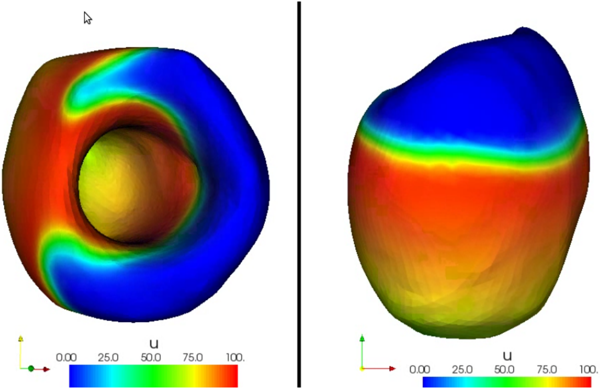
## 3. Meshfree Approximation

$$M \frac{\partial \mathbf{u}}{\partial t} + \left[ \int_\Omega \Phi^T \left[ (J\Phi)\dot{\mathbf{x}} \right]^T \right] \mathbf{u} = M f(\mathbf{u}, \mathbf{w}) - K \mathbf{u}$$
$$M \frac{\partial \mathbf{w}}{\partial t} + \left[ \int_\Omega \Phi^T \left[ (J\Phi)\dot{\mathbf{x}} \right]^T \right] \mathbf{w} = M \epsilon(\mathbf{u} - \gamma \mathbf{w})$$

## ...Some preliminary results... fixed heart



## Thanks!