Instability in Hamiltonian systems and Arnold diffusion

based on Joint works with P. Bernard and V. Kaloshin

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Outline

Origin of the study: the solar system

- $$\text{Solar system} = \prod(\text{Sun-planet}) + (\text{interactions}).$$

  This is an example of nearly integrable systems.

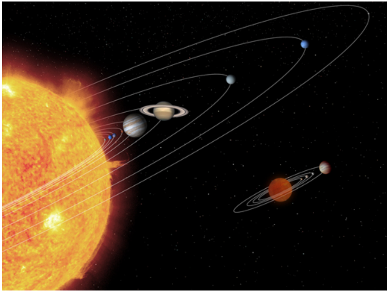

Figure: Solar system

Nearly integrable systems

- Action-angle coordinates:

$$H_\epsilon(\theta, p) = H_0(p) + \epsilon H_1(\theta, p), \theta \in \mathbb{T}^m, p \in \mathbb{R}^m.$$

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$$\begin{cases} \dot{\theta} = \partial_p H_0(p) + O(\epsilon) \\ \dot{p} = O(\epsilon) \end{cases}.$$

- The energy surface $\{H_\epsilon = E\}$ is invariant. There is a reduction to a time-periodic system

$$H_\epsilon(\theta, p, t) = H_0(p) + \epsilon H_1(\theta, p, t), \theta \in \mathbb{T}^n, p \in \mathbb{R}^n,$$

where $n = m - 1$. The system has $n\frac{1}{2}$ degrees of freedom.

Non-integrability

Theorem (Poincaré)

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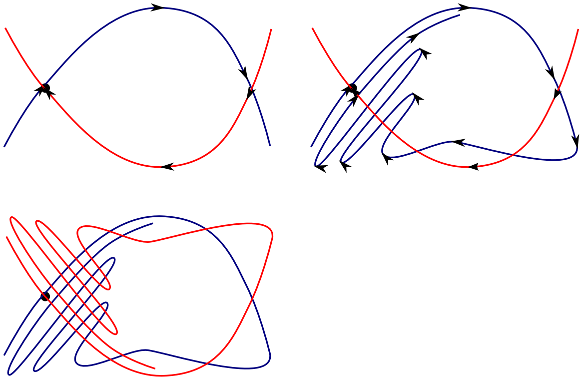

Figure: Homoclinic tangles

Nearly integrable systems are not ergodic

Question

Ergodic hypothesis?

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Theorem (Kolmogorov-Arnold-Moser)

For a nearly integrable system with m degrees of freedom, a nearly full measure set of the phase space is filled with $m-$dimensional invariant tori. Each invariant torus is $\sqrt{\epsilon}-$close (and diffeomorphic) to

$$\mathbb{T}^m \times \{p = p_0\}.$$

(The p variable is stable).

KAM tori (picture)

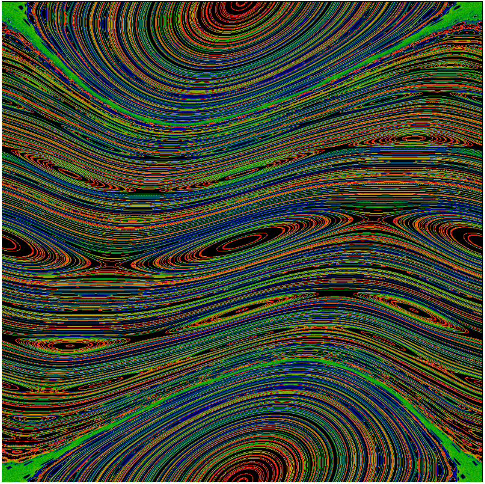

Figure: KAM tori for the standard map

Resonances and KAM

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- KAM theorem applies to "very non-resonant" vectors.

Quasi-ergodic hypothesis

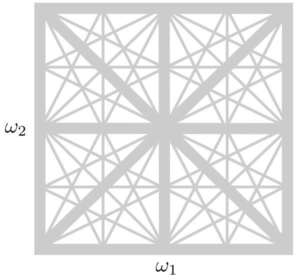

Figure: Resonances

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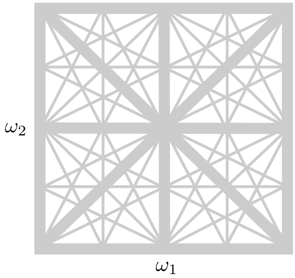

Figure: Resonances

Question

Is there a dense orbit (for a generic system)?

Arnold diffusion

Conjecture (Arnold 1963)

For a “typical” nearly integrable system, there is topological instability when the KAM tori do not divide the phase space ($n \geq 2$).

Main results

Theorem

For a typical $H_\epsilon = H_0 + \epsilon H_1$ with $n \geq 2$, there exists an orbit $(\theta_\epsilon(t), p_\epsilon(t))$ and $T_\epsilon > 0$ such that

$$\|p_\epsilon(T_\epsilon) - p_\epsilon(0)\| > l(H_1) > 0.$$

(Bernard, Kaloshin, Z, preprint)

Main results, cont.

Theorem

For a given $\gamma > 0$, for a typical H_ϵ with $n = 2$, there exists a $\gamma-$dense orbit.

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Remark

This theorem was announced by J. Mather (2003). We provide an alternative approach.
This theorem does not imply existence of a dense orbit. As $\gamma \to 0$, the parameter $\epsilon \to 0$.

Path of diffusion

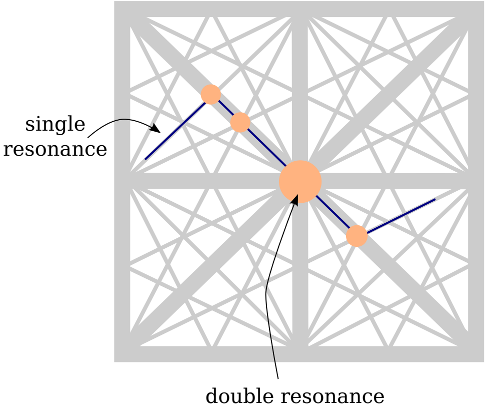

Figure: Diffusion path

Diffusion picture

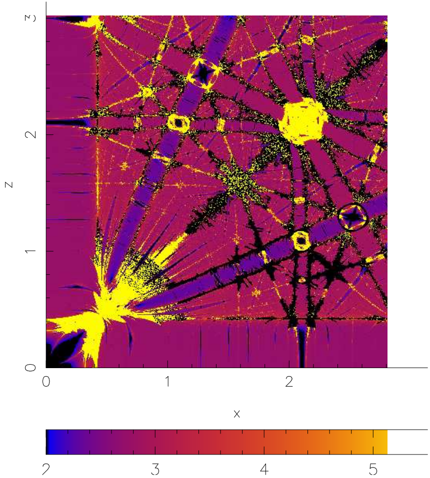

Figure: Numeric simulation by Guzzo, Lega and Froeschlé

The Arnold mechanism

- System:

$$H(\theta_1, \theta_2, p_1, p_2, t) = \frac{1}{2}p^2 + \epsilon(\cos\theta_1 - 1) - \epsilon\mu(\cos\theta_1 - 1) f(\theta_2, t).$$

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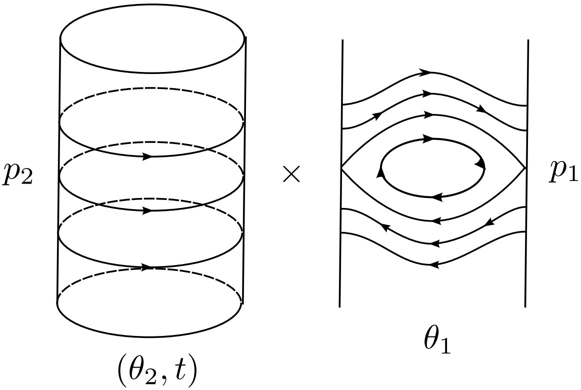
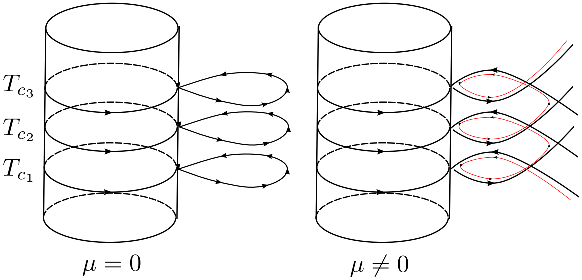


Arnold mechanism: picture



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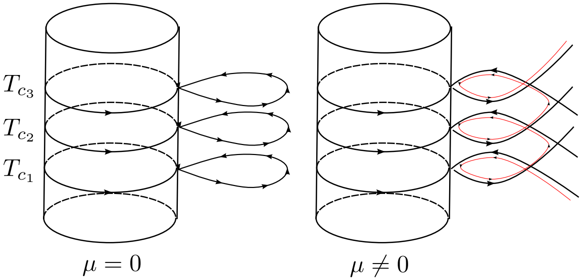

Figure: Arnold mechanism

Diffusion orbit follows the invariant cylinder $\{p_1 = \theta_1 = 0\}$, p_1 stays close to 0, p_2 slowly increases.

Shadowing a transition chain

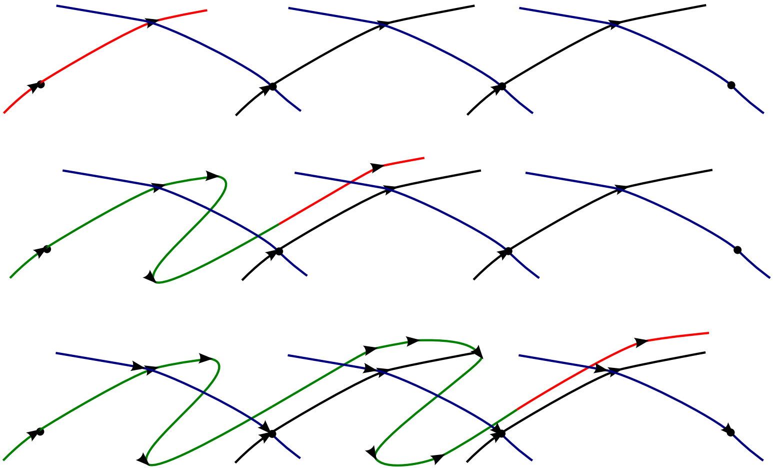

Figure: Lambda lemma

Mather mechanism

Theorem (Mather 1991)

The only obstruction to diffusion in a $1\frac{1}{2}$ degrees of freedom system is the existence of invariant tori.

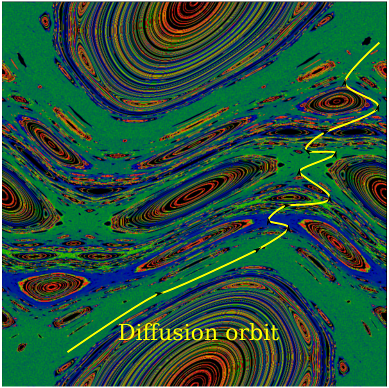

Figure: Mather mechanism

NHIC near a single resonance

- Near the single resonance $(1, 0, 0) \cdot (\omega_1, \omega_2, 1) = 0$, the system takes the normal form

$$H_0(\theta, p, t) = H_0(p) + \epsilon Z(\theta_1, p) + O(\epsilon\delta).$$

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- (Bernard, Kaloshin, Z) There exists a normally hyperbolic invariant cylinder along a single resonance, away from double resonances.

Diffusion along a single resonance

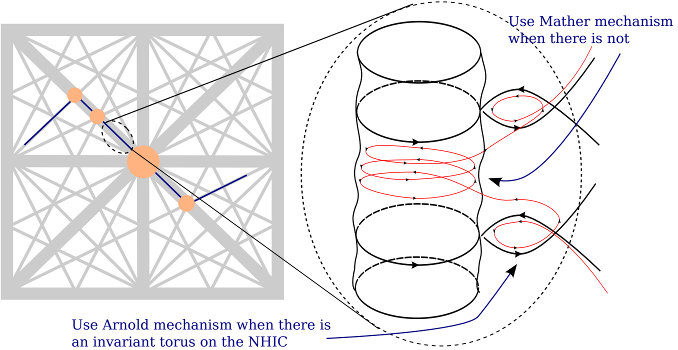

Figure: Diffusion along a single resonance

Bernard, Cheng-Yan, Bernard-Kaloshin-Z.

The role of the double resonance

Photo by ruffin_ready at flickr.

The slow mechanical system

- The system near a double resonance can be rescaled into the following form:

$$H_\epsilon^s(\theta, I, \tau) = const + K(I) - U(\theta) + \sqrt{\epsilon}P(\theta, I, \tau),$$

$$\theta \in \mathbb{T}^2, I \in \mathbb{R}^2, \tau \in \sqrt{\epsilon}\mathbb{T}.$$

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- Generically, the system $H^s = K(I) - U(\theta)$ admits a four-dimensional saddle at $I = 0$, $\theta = \theta_0$, where $U(\theta_0) = \min U$. We assume all its eigenvalues are distinct.
- Generically, the system H_ϵ still admits an NHIC near double resonance, attached to the NHIC from the single resonance, but it may be destroyed near the saddle.

Non-simple cylinder

For the slow system H^s, it is possible that the cylinder pinches at the saddle. This picture will be destroyed by small perturbation.

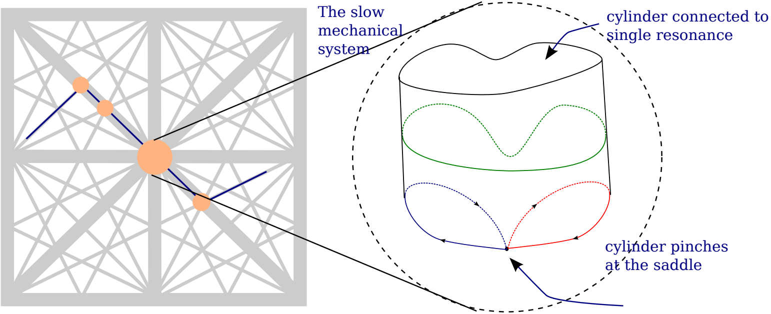
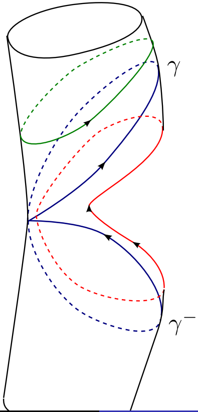


Figure: Non-simple cylinder

Simple cylinders

Let γ be a homoclinic orbit to the saddle for the slow system. Let γ^- be the time reversal of γ. Then there exists a normally hyperbolic invariant manifold containing both γ and γ^-. This cylinder persists under perturbation. (Shil'nikov, Shil'nikov Tureav, Bolotin-Rabinowitz)



Simple cylinders associated to a non-simple one

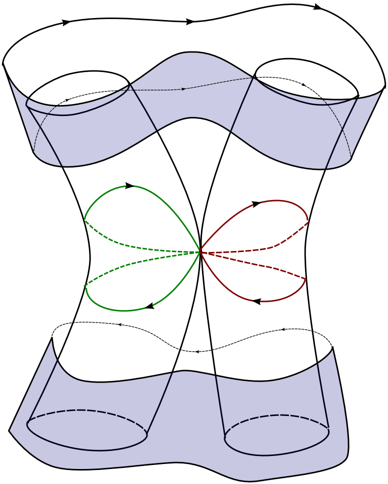


Figure: Kissing property

Diffusion across a double resonance

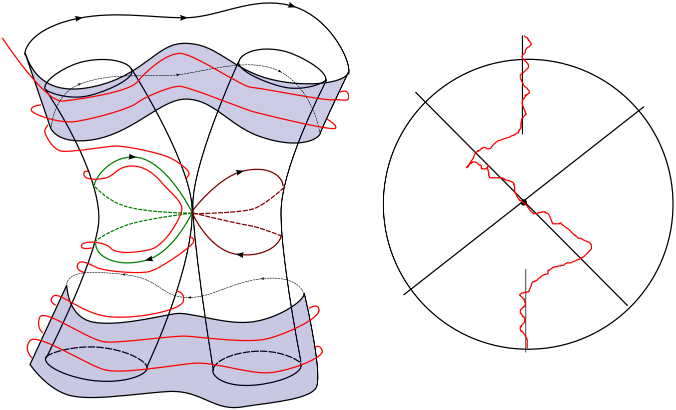
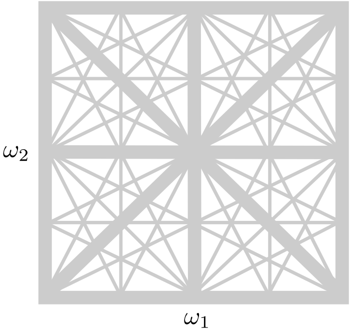


Figure: Diffusion using a simple cylinder

Dense orbit?

Conjecture (M. Herman)

Does there exist an example of nearly integrable Hamiltonian system, such that there exists a dense orbit?



Property of a positive measure set of orbits

Conjecture (Féjoz-Guàdia-Kaloshin-Roldán)

For the a priori unstable version of the Arnold example

$$H_\epsilon = \frac{1}{2}p^2 + (\cos\theta_1 - 1) + \epsilon f(\theta, p, t),$$

with $\theta \in \mathbb{T}^2, p \in \mathbb{R}^2, t \in \mathbb{T}$. Then there exists $c > 0$, $C > 0$ such that

$$Leb\left\{(\theta(0), p(0)) : \sup_{0 \le T \le C|\ln\epsilon|/\epsilon} \|p(0) - p(T)\| > 1\right\} > c.$$

has positive measure.