Topological Problems for Set Theorists, II

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Problems should be "important", easy to state, hard to solve, apparently involve non-trivial set theory, and (preferably) have a pedigree.

Problem 1 (Arhangel'skiĭ)

Is it consistent that every Lindelöf T_2 space with points G_δ has cardinality $\leq 2^{\aleph_0}$?

Pedigree: Arhangel'skiĭ, Gorelic, Kočinac, Shelah, Stanley, Velleman, Hajnal, Juhász, Tall, Morgan, Knight, Scheepers, Usuba.
Methods: Countably closed forcing, morasses, large cardinal reflection, topological games.

Best partial results

Theorem 1 (Shelah-Stanley, Velleman82)
In L, there are counterexamples of size $\aleph_2$.

Theorem 2 (Gorelic)
By forcing, $2^{\aleph_1} > \aleph_2$ and there are counterexamples of size $2^{\aleph_1}$.

Theorem 3 (Tall-Usuba)
Lévy-collapse a weak compact and then add $\aleph_3$ Cohen subsets of ω_1. Then there are no counterexamples of size $\aleph_2$, even relaxing "points G_δ" to "pseudocharacter $\leq \aleph_1$".

Subproblems

a) Can higher gap morasses be used to obtain counterexamples of size $> \aleph_2$?

b) A Lindelöf space is **indestructible** if it is Lindelöf in every countably closed forcing extension. Does CH imply every Lindelöf T_2 space with points G_δ is indestructible? If so, collapsing a measurable to ω_2 will solve the problem, since there are no indestructible counterexamples in this model (Scheepers, Tall-Usuba). **Note:** X is indestructible iff ONE has no winning strategy in the ω_1-length Rothberger game (Scheepers-Tall).

Related problems

Problem 2

Is there a first countable Lindelöf T_1 space of size $> 2^{\aleph_0}$?

Pedigree: Arhangel'skiĭ, Tall, Koszmider.
Partial results: None except via Problem 1.

Problem 3

Is there a Lindelöf space X such that $L(X^2) > 2^{\aleph_0}$, where $L(Y) =$ least κ such that every open cover of X has a subcover of size $\leq \kappa$?

Pedigree: Same as Problem 1, plus Bagaria, Magidor, Brooke-Taylor.

Best partial results

Theorem 4 (Shelah)
In L, there is an example.

Theorem 5 (Bagaria-Magidor)
$L(X^2) <$ *first $\aleph_1$-strongly compact cardinal.*

Generalizing Souslin's Hypothesis

Problem 4 (Kurepa?)

Is it consistent that every linear order in which disjoint collections of open sets have size $\leq \aleph_1$ has a dense set of size $\leq \aleph_1$?

Theorem 6 (Laver-Shelah81)

Con(there is a weak compact) $\rightarrow$ *Con($2^{\aleph_1} > \aleph_2 + \aleph_2$-Souslin hypothesis)*.

What about with GCH?

What is the "right" generalization of "countable"?

Definition 7

A subspace is σ-**(closed)-discrete** if it can be partitioned into countably many (closed) discrete subspaces. A family $\mathcal{F}$ of subspaces is σ-**discrete** if $\mathcal{F} = \bigcup_{n<\omega} \mathcal{F}_n$, where $\forall n$, each point has a neighbourhood that meets at most one $F \in \mathcal{F}_n$.

Definition 8

A LOTS (linearly ordered topological space) satisfies the **(convex)** σ-**discrete chain condition** if each collection of (convex) open sets is σ-discrete.

Question 1 (Watson)

Is the existence of a LOTS satisfying the σ-discrete chain condition but not having a σ-closed-discrete dense set equivalent to the existence of a Souslin tree?

Theorem 9 (Qiao-Tall03)

Yes.

Definition 10

A LOTS satisfying the convex σ-discrete chain condition which does not have a σ-closed-discrete dense set is called a **generalized Souslin line**.

Problem 5

Is it consistent that there are no generalized Souslin lines? (Generalized Souslin's Hypothesis).

Definition 11

A space is **perfect** if closed sets are G_δ's. A space is **perfectly normal** if it is perfect and normal. A space is **non-Archimedean** if it has a base which is a tree under inclusion.

Definition 12

A **generalized Lusin line** is a LOTS without isolated points which does not have a σ-discrete dense subspace, but every nowhere dense subspace of it has such a subspace.

Theorem 13 (Qiao-Tall03)

The following are equivalent:

a) *there is a generalized Souslin line,*

b) *there is a generalized Lusin line,*

c) *there is a perfectly normal non-Archimedean space which is not metrizable,*

d) *there is a perfect LOTS which does not have a σ-closed-discrete dense subspace.*

Pedigree: Nyikos, Bennett, Lutzer, Tall, Qiao, Todorcevic.

Theorem 14 (Todorcevic81)

MA $+ \neg wKH$ implies there is no generalized Souslin line of weight $\leq \aleph_1$.

Theorem 15 (Qiao92,01)

MA $+ \neg CH$ does not imply there are no generalized Souslin lines.

Theorem 16 (Todorcevic92)

PFA does not imply there are no generalized Souslin lines. If there are no generalized Souslin lines, there are large cardinals in an inner model.

An irrational problem

Definition 17 (JT98)

Let $\langle X, \mathcal{T} \rangle$ be a space in an elementary submodel of some sufficiently large H_θ. Define X_M to be the space with universe $X \cap M$ and topology generated by $\{U \cap M : U \in \mathcal{T} \cap M\}$.

Theorem 18 (T00)

If X is regular without isolated points, and M is a countable elementary submodel of some sufficiently large H_θ, then $X_M \cong \mathbb{Q}$ (the rationals).

Theorem 19 (T00)

If $X_M \cong \mathbb{R}$, then $X = X_M$.

Problem 6 (T02)

If $X_M \cong \mathbb{R} - \mathbb{Q}$, does $X = X_M$?

Pedigree: Tall, Welch.

Theorem 20 (T02)

Yes if $|\mathbb{R} \cap M|$ is uncountable.

Corollary 21

Yes if $0^{\#}$ doesn't exist.

Theorem 22 (Welch02)

Yes if $2^{\aleph_0}$ is not a Jonsson cardinal.

Efimov's Problem

Problem 7 (Efimov69)

Does every compact T_2 space include either a copy of $\omega + 1$ or a copy of $\beta\mathbb{N}$ (the Stone-Čech compactification of ω)?

Pedigree: Efimov, Fedorčuk, Dow, Hart, Geschke, Shelah.

Lemma 23 (Šapirovskiĭ80)

For a compact T_2 space X, TFAE:

a) *X includes a copy of $\beta\mathbb{N}$,*

b) *X maps onto $^{\mathfrak{c}}[0,1]$,*

c) *some closed subset of X maps onto $^{\mathfrak{c}}2$,*

d) *there is a dyadic system $\{\langle F_{\alpha,0}, F_{\alpha,1}\rangle : \alpha < \mathfrak{c}\}$ of closed sets in X, i.e., $F_{\alpha,0} \cap F_{\alpha,1} = 0\ \forall\alpha$, $\bigcap_{\alpha\in\mathsf{dom}\,p} F_{\alpha,p(\alpha)} \neq 0$ for all $p \in \mathsf{Fn}(\mathfrak{c}, 2)$.*

Counterexamples:

- $\diamondsuit$ (Fedorčuk75)
- CH (Fedorčuk77)
- $\mathfrak{s} = \aleph_1 + 2^{\aleph_0} = 2^{\aleph_1}$ (Fedorčuk77')
- $\mathrm{cf}([\mathfrak{s}]^{\aleph_0}, \subseteq) = \mathfrak{s} + 2^{\aleph_0} < 2^{\aleph_1}$ (Dow05)
- $\mathfrak{b} = \mathfrak{c}$ (Dow-Shelah)

Survey: Hart07.

Related Problem: (Hrusak) Does every compact T_2 space include either a copy of $\omega + 1$ or $\omega_1 + 1$? Note that $\beta\mathbb{N}$ does include a copy of $\omega_1 + 1$.

"Katowice Problem" (Turzanski). Can $\mathcal{P}(\omega)/$ Finite be isomorphic to $\mathcal{P}(\omega_1)/$ Finite? Equivalently, $\omega^* \cong \omega_1^*$?
Pedigree: Turzanski, Comfort, Broverman, Weiss, Dow, Hart, Frankiewicz.

Theorem 24 (Hart)

If $\omega^* \cong \omega_1^*$, then there is an ω_1-scale, a strong Q-sequence, and a non-trivial autohomeomorphism of ω_1^*.

Definition 25 (Steprāns85)

A **strong Q-sequence** is an $\mathcal{A} \subseteq [\omega_1]^\omega$ such that if for each $A \in \mathcal{A}$, $f_A : A \to \{0, 1\}$, then $\exists f : \omega_1 \to \{0, 1\}$ such that for each A, $f|A = f_A$ for all but finitely many $n \in A$.

(Steprāns thinks there is a model of these 3 consequences.)

Problem 8 (Michael63)

Is there a Lindelöf X such that $X \times \mathbb{P}$ is not Lindelöf? (Such a space is called a Michael space).

Pedigree: Michael, Lawrence, Alster, van Douwen, Moore, Raghavan, Steprans.
Best Partial Result:

Theorem 26 (Moore99)

Yes if $\mathfrak{d} = \text{cov}(\mathcal{M})$.

Definition 27

X is **productively Lindelöf** if $X \times Y$ is Lindelöf for every Lindelöf Y. X is **powerfully Lindelöf** if X^ω is Lindelöf.

Problem 9 (Michael)

Is every productively Lindelöf space powerfully Lindelöf?

Pedigree: Michael, Alster, Tall, Burton.
Methods: CH, small cardinals, countably closed forcing, elementary submodels (X/M).

Best Partial Results

Definition 28

$\mathcal{G}$ is a k-**cover** of X if every compact subspace of X is included in a member of $\mathcal{G}$. X is **Alster** if every k-cover by G_δ's has a countable subcover.

Theorem 29 (Alster88)

CH implies productively Lindelöf spaces of weight $\leq \aleph_1$ are Alster. Alster spaces are powerfully Lindelöf.

Problem 10 (Alster88)

Does productively Lindelöf imply Alster? Converse is true.

Theorem 30 (Burton-Tall)

Yes if $L(X^\omega) \leq \aleph_1$.

Problem 11

If for every n, X^n is Lindelöf, is $L(X^\omega) \leq 2^{\aleph_0}$? What if X is productively Lindelöf?

Example 31 (Gorelic)

Con(CH $+\exists X, X^n$ Lindelöf $\forall n$, $L(X^\omega) = \aleph_2$).

Related problems, weaker hypotheses, weaker conclusions; what is true in ZFC?

$$
\begin{array}{ccc}
\text{CH} & \to & \text{(productively Lindelöf metrizable spaces are} \quad \sigma\text{-compact)} \\
\downarrow & & \downarrow \\
\mathfrak{d} = \aleph_1 & \to & \text{(productively Lindelöf spaces are} \quad \text{Hurewicz)} \\
\downarrow & & \downarrow \\
\mathfrak{b} = \aleph_1 & \to & \text{(productively Lindelöf spaces are} \quad \text{Menger)}
\end{array}
$$
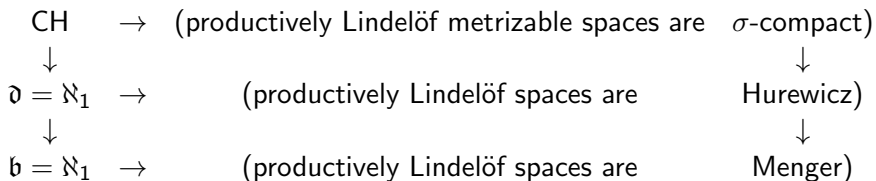
Recent Improvements

Theorem 32 (Repovš-Zdomskyy12)

$\exists$ *Michael space (this follows from* $\mathfrak{b} = \aleph_1$*)* $\rightarrow$ (*productively Lindelöf spaces are Menger*).

Theorem 33 (Repovš-Zdomskyy)

$\text{Add}(\mathcal{M}) = \mathfrak{d} \rightarrow$ (*productively Lindelöf spaces are Hurewicz*).

Theorem 34 (Zdomskyy)

$\mathfrak{u} = \aleph_1 \rightarrow$ (*productively Lindelöf spaces are Hurewicz*).

Theorem 35 (Brendle-Raghavan)

$\text{cov}(\mathcal{M}) = \mathfrak{c} < \aleph_\omega \rightarrow$ (*productively Lindelöf metrizable spaces are σ-compact*).

Theorem 36 (Miller-Tsaban-Zdomskyy)

$\mathfrak{d} = \aleph_1 \rightarrow$ (*productively Lindelöf metrizable spaces are productively Hurewicz*).

MAYBE ALL CONCLUSIONS TRUE IN ZFC!

Definition 37

Let X be a normal first countable space. Label an uncountable closed discrete subspace as $\{\alpha : \alpha < \omega_1\}$. For each $\alpha \in \omega_1$, consider $f : \omega_1 \to \omega$ as an assignment of basic open neighbourhoods to the points of the closed discrete subspace. $\mathcal{F} \subseteq {}^{\omega_1}\omega$ **witnesses normality** if for each $P : \omega_1 \to 2$, there exists an $f \in \mathcal{F}$ such that the neighbourhoods f assigns to $P^{-1}(0)$ are disjoint from those it assigns to $P^{-1}(1)$.

Problem 12

If $|\mathcal{F}| < 2^{\aleph_1}$ and $\mathcal{F}$ witnesses normality for ω_1, does there exist a single $g : \omega_1 \to \omega$ such that its assignment is disjoint?

Partial result (Tall81): Yes if $|\mathcal{F}| \leq \aleph_1$, or if $|\mathcal{F}| < 2^{\aleph_1}$ and GMA holds.

Problem 13

Is the box product of countably many copies of $\omega + 1$ normal?

Pedigree: Rudin, Kunen, van Douwen, Roitman, Lawrence, Williams.

Best partial results:

Theorem 38 (Roitman11)

$\mathfrak{b} = \mathfrak{d}$ *implies yes. (Previous known that $2^{\aleph_0} \leq \aleph_2$ implies yes; adding cofinally many Cohen reals implies yes.)*

Surveys: Roitman, Williams

Problem 14 (Hajnal-Juhász76)

Does every uncountable Lindelöf space have a Lindelöf subspace of size $\aleph_1$?

Partial result: Con(there is a counterexample) (Koszmider-Tall)
Also see Baumgartner-Tall02.
Related problem: Does every uncountable compact T_2 space have a Lindelöf subspace of size $\aleph_1$? (Hajnal-Juhász76).

Theorem 39 (Hajnal-Juhász76)

Yes if X has uncountabe tightness; CH implies yes if countably tight.

Problem 15 (Toronto Problem)

Is there an uncountable non-discrete T_2 space which is homeomorphic to each of its uncountable subspaces?

Pedigree: Steprāns, Weiss, Gruenhage, Moore.
Partial results: CH implies no, PFA implies no for regular spaces. See Steprans80 and Gruenhage-Moore00.

Definition 40

A space X is **Fréchet** if whenever $Y \subseteq X$ and $x \in \overline{Y}$, there is a sequence from Y converging to x.

Problem 16 (Malyhin)

Is there a non-metrizable, countable Fréchet group?

CH and $\mathfrak{p} > \omega_1$ both yield examples.
Survey: Moore-Todorcevic07.

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