Mok – "GL_2" 3/14/12

We think of GL_2 as the unit group of M_2 over a (local or number) field.
let $G_0 = GL_2 = (M_2/F)^\times$
We want to consider the inner forms of G_0, which are given by unit groups of quaternion algebras.

- B/F – quaternion algebra $/F$, $G = G/F$, $G(F) = B^\times$
- If F is local field (either F p-adic or $\mathbb{R}$) then $\exists!$ (up to isomorphism) quaternion algebra B/F, e.g. for $F=\mathbb{R}$, $B = \mathbb{H}$ (Hamiltonian quaternions)
- if F is a number field, B/F quaternion algebra $/F$, then for every place v of F, let $B_v := B \otimes F_v = \begin{cases} M_2(F_v) & \text{– } B \text{ "splits" at } v \\ \text{or} & \\ \text{quaternion algebra}/F_v & \text{– } B \text{ "ramifies" at } v \end{cases}$

Write $\mathrm{Ram}(B)$ = set of places v of F s.t. B_v ramifies.
Then ① $\#\mathrm{Ram}(B)$ is even & $\mathrm{Ram}(B)$ uniquely determines B (up to isom)
② Conversely, if S is a finite set of places of F of even cardinality, then $\exists\, B/F$ s.t. $\mathrm{Ram}(B) = S$.
(if $S = \emptyset$, $B = M_2(F)$)

(If we have $h: G_{0/\bar{F}} \xrightarrow{\sim} G_{/\bar{F}}$, $\sigma \in \mathrm{Gal}(\bar{F}/F)$, then $({}^\sigma h)^{-1} \circ h \in \mathrm{Inn}(G_0/\bar{F})$ is an inner form of G_0.)

Take B/F, quaternion algebra $/F$ and $G(F) = B^\times$
If F is local, then $G(F)/F^\times$ is compact (e.g. $F=\mathbb{R}$, $\mathbb{H}^\times/\mathbb{R}^\times \cong SO_2/\{\pm1\}$)
Consequence: All the irred. admissible rep'n of $G(F)$ are finite-dimensional.
(e.g. $F = \mathbb{R}$, $G = \mathbb{H}^\times$ then $\mathbb{H}^\times \hookrightarrow G_0(\mathbb{C}) \circlearrowright \mathrm{Sym}^{k-2}\mathbb{C}^2$ $k \geq 2$)

to get Shimura variety, add maximal compact of Arch places

If F is global, then consider $G(F)\backslash G(\mathbb{A}_F)/\cancel{Z(\mathbb{R})}\,(F\otimes_{\mathbb{Q}}\mathbb{R})^\times = B^\times\backslash(B\otimes_F \mathbb{A}_F)^\times/(F\otimes_{\mathbb{Q}}\mathbb{R})^\times K_\infty$

is compact (as long as B is nonsplit)
e.g.: F totally real i.e. $B \otimes_{\mathbb{Q}} \mathbb{R} = B \otimes_F (F \otimes \mathbb{R}) \cong \prod M_2(\mathbb{R})^d \times \prod (\mathbb{H}^\times)^e$
In this case, the Shimura variety defined by $G = B^\times$ is d-dimensional (called a Quaternionic Shimura Variety) When $d=1$, called Shimura curve

When $d=0$, the double quotient is a finite set (both compact + discrete)

<u>Note</u>: Modular forms on ~~[illegible]~~ such G are especially well-adapted for doing the Taylor-Wiles' argument.

§ <u>Transfer of (automorphic) representations</u>
<u>(Jacquet-Langlands correspondence)</u>

First assume F is local, $G_0 = Gl_2/F$ ⟵ called <u>quasi-split</u>
(it has a Borel subgroup ...)

$G = B^\times$ (non-quasi-split), B quaternion $/F$.

(G & G_0 have the same L-group, expect that there is a close relation between automorphic on G & G_0)

<u>JL-correspondence</u>: there is an injection:

$$\left\{\begin{matrix}\text{irred. admissible}\\ \text{rep'n of } G(F)=B^\times\end{matrix}\right\} \xhookrightarrow{JL} \left\{\begin{matrix}\text{irred admissible}\\ \text{rep'n of } Gl_2(F)\end{matrix}\right\}$$

<u>Furthermore</u>, the image consists <u>exactly</u> of the irreducible, admissible rep'n of $Gl_2(F)$ that are <u>discrete series</u>.

We have a local character identity between π on $B^\times$ & $JL(\pi)$ on $Gl_2(F)$
i.e. for some associated elements $t' \in B^\times$, $t \in Gl_2(F)$, $tr(\pi)(t') = -tr(JL(\pi))(t)$

Now assume F is global. Then there is an injection

$$\left\{\begin{matrix}\text{automorphic representation}\\ \text{on } B^\times, \text{ i.e. on } G(\mathbb{A}_F)\\ G \text{``=''} B^\times\end{matrix}\right\} \setminus \left\{\begin{matrix}\text{1-dim'l}\\ \text{rep'n}\end{matrix}\right\} \xhookrightarrow{JL} \left\{\begin{matrix}\text{automorphic rep'n}\\ \text{of } Gl_2(\mathbb{A}_F)\end{matrix}\right\}$$

s.t. ① $\forall v$ of F, and π on LHS, $(JL(\pi))_v = JL(\pi_v)$.
w/ $v \in Ram(B)$ ↑ local JL

② $JL(\pi)$ is cuspidal

③ Image(JL) consists exactly of those $\tilde{\pi} \in$ RHS s.t. $\forall v \in Ram(B)$
$\tilde{\pi}_v \in$ Image(JL_v), i.e. $\tilde{\pi}_v$ is a discrete series rep'n of $Gl_2(F_v)$.

Pf by comparison of trace formulas for $B^\times$ & Gl_2/F.
(done first by Jacquet & Langlands)

comparison of geometric side (of trace formulas) require the transfer of conjugacy classes from $B^\times$ to $Gl_2(F)$.
(can be done explicitly in this case)

γ - conjugacy class in $B^\times$

$\rightsquigarrow char(\gamma) = X^2 - \underset{\text{reduced}}{Trd}(\gamma) X + Nrd(\gamma)$ has coefficient in F

We can find γ_0 conjugacy class in $Gl_2(F)$ s.t. $char(\gamma) = char(\gamma_0)$.
e.g. can take $\gamma_0 = Gl_2(F)$-conjugacy class of $\begin{pmatrix} 0 & -1 \\ Nrd\gamma_0 & trd\gamma_0 \end{pmatrix}$
Can compare the geometric side of the 2 trace formula. (knowing equality of spectral side).

§ <u>Results on Galois representation</u> (F=global)

Assume $F = \begin{cases} \text{totally real} \\ \text{CM} \end{cases}$ Recall $G_0 = Gl_2/F$

For totally real, automorphic forms on Gl_2/F
- Hilbert modular forms

f a Hilbert modular form of vector weight $(k_{v_1}, \dots, k_{v_d})$ where $d = [F:\mathbb{Q}]$, $v_i: F \hookrightarrow \mathbb{R}$, $k_{v_i} \geq 2$. s.t. the k_{v_i} are of the same parity.

Then if f is a cuspidal eigenform, (ℓ any prime) then
$\rho_f: Gal(\bar{F}/F) \longrightarrow Gl_2(\bar{\mathbb{Q}}_\ell)$ exists
(due to Carayol, Blasius-Rogawski, Taylor)

- In most cases, ρ_f can be found in the cohomology of Shimura curve, (Carayol) or Picard modular surface (Blasius-Rogawski)

Taylor used a congruence method similar to Deligne-Serre in wt 1.

For F CM, let F_0 be the index 2 totally real subfield $[F:F_0] = 2$
Let f be a cuspidal eigenform on $Gl_2(\mathbb{A}_F)$ of vector weight $(k_{v_1}, \dots, k_{v_d})$ where $d = \frac{1}{2}[F:\mathbb{Q}]$, v_i are pairs of (conjugate) embeddings $F \hookrightarrow \mathbb{C}$.
$k_{v_i} \geq 2$, same parity

<u>Note</u>: $Gl_2(\mathbb{C})$ does not have discrete series.
Assume central character ω of f is invariant under $Gal(F/F_0)$
Then $\rho_f: Gal(\bar{F}/F) \longrightarrow Gl_2(\bar{\mathbb{Q}}_\ell)$ exists.

This situation is done in a rather indirect way, have to use Langlands functoriality to move to a different group ($GSp(4)$)

(F imag quadratic done by Harris-Soudry-Taylor, Taylor
F CM (Mok))

For $F \neq$ totally real or CM ??

(or to get rid of invariance of central character condition)

<u>Generalizations: Gl_2</u>

Obvious generalization: Gl_n

(Langlands philosophy: any automorphic rep'n on any reductive group can ultimately be studied from auto. rep. on Gl_n)

$F = \mathbb{Q}$, $G = Gl_n/\mathbb{Q}$, However $n > 2$, G does not have a Shimura variety

However some other groups do:

$$G = GSp_{2g}/\mathbb{Q} = \{ g \in Gl_{2g} : {}^t g J g = \underset{\mathbb{Q}^\times}{\underset{\uparrow}{c(g)}} J \} \text{ where } J = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$$

$\downarrow$

Siegel variety of dim: $\frac{g(g+1)}{2}$

Another direction is unitary group ($K/\mathbb{Q}$ imaginary quadratic)

$GU(1,1) = Gl_2(\mathbb{Q}) \cdot K^\times$

give unitary Shimura variety.

<u>Note</u>: endoscopy phenomenon in those other groups