Mok – "Galois Representations associated to cusp forms" 3/12/12

§ <u>Original approach</u> (Eichler-Shimura-Deligne) (only for modular curves)
- Eichler-Shimura congruence (also Taylor)

§ <u>Ihara, Langlands-Kottwitz</u>
- Advantage: works at least for all Shimura varieties associated to unitary & symplectic groups.
- brings out the group-theoretic structure in the problem.
• based on comparison of
① Lefshetz trace formula for Frobenius – Hecke correspondence (for modular curve)
② Arthur-Selberg trace formula for GL_2.

$G = GL_2/\mathbb{Q}$
$K = \prod_{v \text{ finite}} K_v$ open compact subgroup of $G(\mathbb{A}_f)$
so $K_v = GL_2(\mathbb{Z}_v)$ for almost all v.

Y_K = modular (Shimura) curve of level K take $K_\infty = SO_2(\mathbb{R})$, $\mathbf{Z}_{\mathbb{R}} = \mathbb{R}^\times$

its complex points are $Y_K(\mathbb{C}) = G(\mathbb{Q}) \backslash G(\mathbb{A}) / Z_{\mathbb{R}} K_\infty K$

$= \coprod_i \Gamma_i \backslash \mathcal{H}$ ← congruence subgroups (or conjugates of)

Shimura's theory of canonical model gives Y_K the structure of a variety$/\mathbb{Q}$
(as solution of a moduli problem: elliptic curve + level K structure)
(assuming K is small enough for moduli problem to be representable)

Fix a prime p. Assume $K = K^p K_p$ ($K_p = GL_2(\mathbb{Z}_p)$, i.e. "p does not divide the level" or modular curve has good reduction @ p)

Then Y_K has an integral, smooth model over $\mathbb{Z}_p$.

These also have compactification, again denoted $X_K/\mathbb{Q}$
$\mathcal{X}_K/\mathbb{Z}_p$ ← proper, smooth $/\mathbb{Z}_p$.

(of wt 2)
Let f be a cuspidal eigenform wrt Hecke algebra
- best to think of f as a cuspidal automorphic representation $GL_2(\mathbb{A})$
  recall $GL_2(\mathbb{A}) = \prod'_{\text{all } v} GL_2(\mathbb{Q}_v)$ (restricted direct product).
  Call it π.

Then $\pi = \bigotimes'_v \pi_v$ (restricted tensor product decomposition)
- π_v is an irreducible (admissible) rep'n of $GL_2(\mathbb{Q}_v)$.
- wt 2 → π_∞ is discrete series of lowest weight on $GL_2(\mathbb{R})$
- level N → for almost all v (those not ~~[illegible]~~ dividing level) π_v is unramified (can be completely described by Hecke eigenvalue of f at v)

e.g. if f of level $\Gamma_0(N)$, define $K_0(N) = \left\{ (g_v)_v : \begin{array}{l} g_v \in GL_2(\mathbb{Z}_v) \\ g_v \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod N\mathbb{Z}_v \end{array} \right\}$

then $\pi^{K_0(N)} \neq 0$.

So we have π, and choose $K = K^p K_p$, $K_p = GL_2(\mathbb{Z}_p)$ s.t. $\pi^K \neq 0$.
Write $\pi = \pi_\infty \otimes \pi_f$, π_f is defined over some number field L (usually called "Hecke field" generated by Hecke eigenvalues.)

The level K Hecke algebra
$\mathcal{H}_K$ = space of compactly-supported K-bi-invariant functions on $G(\mathbb{A}_f)$ (with values in L)
$= C_c^\infty\left(K \backslash G(\mathbb{A}_f) / K, L \right)$
$= \bigotimes'_{v \text{ finite}} \mathcal{H}_{K_v}$ where $\mathcal{H}_{K_v}$ is the local Hecke algebra of level K_v

$\mathcal{H}_{K_p}$ is the <u>spherical Hecke algebra</u> $= C_c^\infty\left(\underset{K_p}{GL_2(\mathbb{Z}_p)} \backslash GL_2(\mathbb{Q}_p) / \underset{K_p}{GL_2(\mathbb{Z}_p)}, L \right)$

$$T_p = K_p \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} K_p$$

$\mathcal{H}_K$ acts on π^K. It also acts on Y_K, X_K by Hecke correspondence.
We have not yet said anything about Galois action.
- We have the cohomology groups $H^i_{\text{Betti}}(X_K, \mathbb{Q})$ (w/o Galois action)
  choose a prime $\ell \neq p$ and a prime $\lambda | \ell$ in L: $H^i_{et}(X_{K/\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)$ or even better, $H^i_{et}(X_{K/\bar{\mathbb{Q}}}, L_\lambda)$

These have action by $G_{\mathbb{Q}}$ (since we've base changed to $\bar{\mathbb{Q}}$)

Furthermore, there is a commuting action $G_{\mathbb{Q}} \times \mathcal{H}_K$
on $H^i_{et}(X_{K/\overline{\mathbb{Q}}}, \mathcal{L}_\lambda)$ $(i=0,1,2)$

For π as before, define $W^i(\pi_f) := \mathrm{Hom}_{\mathcal{H}_K\text{-linear}}(\pi_f^K, H^i_{et}(X_{K/\overline{\mathbb{Q}}}, \mathcal{L}_\lambda))$
Rmk: since π is cuspidal, i.e. arises from a cusp form, $W^0 = 0 = W^2$

Think of these as virtual rep'n: $H^\bullet = H^0 - H^1 + H^2$
$W^\bullet = W^0 - W^1 - W^2$

K_p = maximal $\Rightarrow$ X_K has good reduction at p
$\Rightarrow$ $G_{\mathbb{Q}}$ on $H^\bullet$ & $W^\bullet$ are unramified at p.
So it suffices to compute $tr(\Phi_p, W^\bullet(\pi_f))$ where Φ_p is (any lift) of geometric Frobenius elt.

Can choose a Hecke function $f^\infty \in \mathcal{H}_K$
that picks out $\pi = \pi_\infty \otimes \pi_f$. (so acts as zero on all other aut rep'n)
i.e. f^∞ acts as zero on any automorphic $\tilde{\pi} = \pi_\infty \otimes \tilde{\pi}_f$ s.t. $\tilde{\pi}_f \neq \pi_f$.

(✱) Then $tr(\Phi_p, W^\bullet(\pi_f)) = tr(\Phi_p \times f^\infty, H^\bullet)$ } everything works in the good reduction setting

Frobenius correspondence to the special fibers of X_K

Frobenius Hecke correspondence

(We can choose f^∞ so that f_p = identity element of $\mathcal{H}_{K_p}$)

Use Lefschetz trace formula for correspondence, can compute this trace.

Formula of Kottwitz: (modulo some caveats)

$$(✱) = \sum_{\{\gamma\}} \tau(G_\gamma)\, O_{\gamma^{p\infty}}(f^{p\infty})\, O_{\gamma p}(\phi_p) \times \mathrm{vol}(\mathbb{R}^\times_{>0} \backslash I_{\gamma_\infty})$$

- γ ranges over the semisimple $G(\mathbb{Q})$-conjugacy classes of $G(\mathbb{Q}) = GL_2(\mathbb{Q})$

$(v \neq \infty, p)$
- $O_{\gamma_v}(f_v)$ = orbital integral of f_v
$= \int_{G_\gamma(\mathbb{Q}_v)\backslash G(\mathbb{Q}_v)} f_v(g_v^{-1}\gamma_v g_v)\, dg_v$ G_γ = centralizer of γ

$K_p = Gl_2(\mathbb{Z}_p)$

- At p, same orbital integral, but with respect to $\phi_p = K_p \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} K_p$

<u>Note</u>: γ is s.t. $\gamma_\infty \in Gl_2(\mathbb{R})$ is $\mathbb{R}$-elliptic

- τ – Tamagawa number
  $\tau(G_\gamma) = \mathrm{vol}\left(G_\gamma(\mathbb{Q}) \backslash G_\gamma(\mathbb{A}_\mathbb{Q}) \right)$
- think of I_{γ_∞} as G_{γ_∞} (almost always)
  (except when $\gamma_\infty = \mathrm{id}$, $I_{\gamma_\infty} = \mathbb{R}^\times SO_2(\mathbb{R})$)

<u>Caveats</u>: this formula of Kottwitz is really for the action of the Frobenius-Hecke correspondence $\Phi_p \times f^\infty$ on $H^i_{et,c}(Y_{K(\mathbb{Q})}, \mathcal{L}_\lambda)$.
(there is a 1-dim'l aut rep'n not seen by) $\longrightarrow$

Now we want to compare w/ Arthur-Selberg trace formula.

spectral side

$$\sum_{\{\pi\}} m(\pi)\, \mathrm{tr}\, \pi(f) = \sum_{\{\gamma\}} \tau(G_\gamma)\, O_\gamma(f) + \dots$$

geometric side

$f \in \mathcal{H}(G(\mathbb{A}))$

$\{\gamma\}$ $\leftarrow$ semisimple $G(\mathbb{Q})$-conj class

$O_\gamma(f) \to \prod O_{\gamma_v}(f_v)$

In order to compare w/ Kottwitz formula,
We take $f \in \mathcal{H}(G(\mathbb{A}))$ to be $f = f_\infty f^{\infty p} \phi_p$

$f_\infty = (-1) \times$ times a pseudo-coeff of the lowest weight discrete series of $Gl_2(\mathbb{R})$ $\longleftarrow$ sign change when $\gamma = 1$

With this choice of f_∞, $O_{\gamma_\infty}(f_\infty) = \begin{cases} \mathrm{vol}\left(\mathbb{R}^\times_{>0} \backslash I_{\gamma_\infty}\right) \\ 0 \text{ if } \gamma_\infty \text{ is not elliptic} \end{cases}$

$-\mathrm{tr}(\Phi_p, W^1(\pi_f))$

Conclusion: $\mathrm{tr}(\Phi_p, W^\cdot(\pi_f)) = \mathrm{tr}(\Phi_p \times f^\infty \mid H^\cdot)$

$= \sum_{\{\tilde{\pi}\}} m(\tilde{\pi})\, \mathrm{tr}(\tilde{\pi}(f))$ $\;= -m(\pi)\, \mathrm{tr}(\pi_p(\phi_p)) = \mathrm{tr}(\Phi_p \mid \mathcal{WR}(\pi_p))$

over all (discrete aut rep'n) $\quad$ where $f = f_\infty f^{\infty p} \phi_p$

Satake transform

WD rep'n

Furthermore, our choice of f^∞ picks out $\tilde{\pi} = \pi$ and $m(\pi) = 1$ by weak multiplicity.

unramified

$$\mathcal{L}(\pi_p): G_{\mathbb{Q}_p} \longrightarrow Gl_2(\mathbb{C}_\lambda)$$
$$\Phi_p \longmapsto \begin{pmatrix} \alpha_p & 0 \\ 0 & \beta_p \end{pmatrix}$$

(works for bad reduction used in [HT])