Part 2 "Galois Deformation" 3/13/12

Take $N=2$, $K=\mathbb{Q}$: $\bar{\rho}: G_{\mathbb{Q},S} \longrightarrow GL_2(k)$ $\#k<\infty$

$R^{\square}$, R (exists if $\mathrm{End}(\bar{\rho})=k$)

choose $\ell \in S$

$\bar{\rho}|_{D_\ell} = \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$

Q: Find deformation ρ of $\bar{\rho}$ s.t. $\rho|_{D_\ell} = \begin{pmatrix} \tilde{\chi}_1 & \tilde{*} \\ 0 & \tilde{\chi}_2 \end{pmatrix}$

ρ:

For $R^{\square}$, this is easy

For R, this is possible if $\mathrm{End}\begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix} = k$ $\chi_1 \neq \chi_2$, $* \neq 0$

(ℓ-distinguished)

For a field K, let $H^*(K,M) := H^*(G_K, M)$ where $M = G_K$-module

To understand tangent spaces of deformation rings, want to understand $H^1(\mathbb{Q}, \mathrm{Ad}\,\bar\rho)$.

$$\begin{tikzcd}
H^1(\mathbb{Q}, M) \arrow[rr] & & \bigoplus_v H^1(\mathbb{Q}_v, M) & & \\
H^1(G_{\mathbb{Q},S}, M) \arrow[urr] \arrow[rr, dotted] \arrow[rrrr, bend right, "0"'] & & \bigoplus_{v\notin S} H^1(\mathbb{Q}_v, M) \arrow[rr] & & \bigoplus_{v\notin S} H^1(I_v, M)
\end{tikzcd}$$

$$1 \longrightarrow I_v \longrightarrow G_{\mathbb{Q}_v} \longrightarrow G_{\ell} \longrightarrow 1$$

Understand $H^1(K,M)$, K = local field $[K:\mathbb{Q}_p]<\infty$

§ Mirror theorems and Duality

(K local)

Thm (Tate): M = finite G_K-module, $M^* = \mathrm{Hom}(M, \varinjlim \mu_N)$

There is a perfect pairing $H^i(K,M) \times H^{2-i}(K,M^*) \longrightarrow \mathbb{Q}/\mathbb{Z}$

In particular, $H^i(K,M)=0$ if $i \neq 0,1,2$

Pf in a special case: K, $L = K(\zeta_p)$

$$\mathrm{Gal}(L/K) \longrightarrow (\mathbb{Z}/p\mathbb{Z})^\times$$

There is a natural character $\chi : (\mathbb{Z}/p\mathbb{Z})^\times \longrightarrow \mathrm{Aut}(\mathbb{F}_p)$

$$a \longmapsto \text{"mult. by } a\text{"}$$

Any character of $(\mathbb{Z}/p\mathbb{Z})^\times = \chi^i$ for some i. $(\chi^i)^* = \chi^{1-i}$

TO: $H^1(K,\chi^i)$ is dual to $H^1(K,\chi^{1-i})$

$$|M| = p$$

Inflation-restriction

$$1 \longrightarrow \underset{\overset{\shortparallel}{0}}{H^1(\mathrm{Gal}(L/K), \chi^i)} \longrightarrow H^1(K,\chi^i) \longrightarrow H^1(L,\chi^i)^{\mathrm{Gal}(L/K)} \longrightarrow \underset{\overset{\shortparallel}{0}}{H^2(\mathrm{Gal}(L/K), \chi^i)}$$

Note: $H^1(L, \chi^i) = H^1(L, \mathbb{F}_p)$

$$H^1(L,\chi^i)^{Gal(L/K)} = H^1(L,\mathbb{F}_p)^{Gal(L/K) = \chi^{-i}} = Hom(G_L, \mathbb{F}_p)^{G=\chi^{-i}}$$

$$(CFT \Rightarrow \quad L^\times \longrightarrow G_L^{ab} \text{ has dense image}) = Hom(L^\times/(L^\times)^p, \mathbb{F}_p)^{G=\chi^{-i}}$$
$$= Hom((L^\times/(L^\times)^p)^{\chi^i}, \mathbb{F}_p)$$

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duality? Kummer theory (cohomologically):

$$1 \longrightarrow \mu_p \longrightarrow \bar{E}^\times \xrightarrow{p^{th}} \bar{E}^\times \longrightarrow 1 \qquad M \mapsto M^{G_E}$$

$$1 \longrightarrow E^\times/E^{\times p} \longrightarrow H^1(E,\chi) \longrightarrow \underset{\overset{\shortparallel}{0 \text{ by HT90}}}{H^1(E, \bar{E}^\times)}$$

Above, we saw, $H^1(K,\chi^i) = Hom((L^\times/(L^\times)^p)^{\chi^i}, \mathbb{F}_p)$

Now, $H^1(K,\chi^{1-i}) = H^1(L,\chi^{1-i})^G = H^1(L,\chi)^{G=\chi^i} = (L^\times/(L^\times)^p)^{G=\chi^i}$ $\square$

K-local

$$1 \longrightarrow H^1(G_k, M^I) \longrightarrow H^1(K,M) \longrightarrow H^1(I,M)$$

Prop: We have a pairing $H^1(K,M) \times H^1(K,M^*)$. Now if we take a submodule, $H^1(G_k, M^I)^\perp = H^1(G_k, (M^*)^I)$ $\qquad p \nmid |M|$

Back to global: $H^1_{\mathcal{L}}(\mathbb{Q}, M) \subseteq H^1(\mathbb{Q}, M)$ is defined as $\{[c] \in H^1 : c_v \in \mathcal{L}_v\}$ $\qquad \mathcal{L} = \{\mathcal{L}_v,\ v \text{ finite},\ \mathcal{L}_v \subseteq H^1(\mathbb{Q}_v, M)\}$ for all but finitely many v, $\mathcal{L}_v = H^1(\mathbb{F}_p, M^{I_v})$

Define $H^1_{\mathcal{L}^*}(\mathbb{Q}, M^*)$ $\qquad \mathcal{L}_v^* = \mathcal{L}_v^\perp$

Summary: CFT $\Rightarrow$ $H^1_{\mathcal{L}}(\mathbb{Q},M)$, $H^1_{\mathcal{L}^*}(\mathbb{Q},M^*)$ are finite

There is a nice formula relating: $|H^1_{\mathcal{L}}(\mathbb{Q},M)|$ and $|H^1_{\mathcal{L}^*}(\mathbb{Q},M^*)|$

Thm (Mirror Theorem) M = finite

$$\frac{|H^1_{\mathcal{L}}(\mathbb{Q}, M)|}{|H^1_{\mathcal{L}^*}(\mathbb{Q}, M^*)|} = \frac{\# H^0(\mathbb{Q}, M)}{\# H^0(\mathbb{Q}, M^*)} \underbrace{\prod_v \frac{\# \mathcal{L}_v}{\# H^0(\mathbb{Q}_v, M)}}_{\text{can check that local factors } = 1 \text{ for all but finitely many prime.}}$$

Reason for name:

Ex: $K = \mathbb{Q}$, $L = \mathbb{Q}(\zeta_p)$ $M = \chi^i$ $\chi : \mathrm{Gal}(L/\mathbb{Q}) \longrightarrow (\mathbb{Z}/p\mathbb{Z})^\times$

$i \neq 0, 1 \mod p-1$ $(p \neq 2)$

$\mathcal{L}_v = H^1(\mathbb{F}_v, \chi^i)$ if $v \neq p$
$\mathcal{L}_p$ = unramified $(=0)$ $\Big\}$ Then $H^1_{\mathcal{L}}(\mathbb{Q}, \chi^i)$ are extensions $\begin{pmatrix} 1 & * \\ 0 & \chi^i \end{pmatrix}$

Dual: $\mathcal{L}^*_v$ = unramified if $v \neq p$
$\mathcal{L}_p$ = no restriction $\Big\}$ Then $H^1_{\mathcal{L}^*}(\mathbb{Q}, \chi^{1-i})$ are ext'ns $\begin{pmatrix} 1 & * \\ & \chi^{1-i} \end{pmatrix}$

Deduce that $\dfrac{|H^1_{\mathcal{L}}(\chi^i)|}{|H^1_{\mathcal{L}^*}(\chi^{1-i})|} = \begin{cases} 1 & i \text{ is odd} \\ 1/p & i \text{ is even} \end{cases}$

Scholze reflection theorem

Thm (Shimura, Deligne, et al)
Let f be a cuspidal (normalized) eigenform of level N. Let $T_n f = a_n f$ and $S_n f = \chi(n) f n^{k-2}$. Choose a field L containing all $a_n, \chi(n)$. Let λ be a prime of L above ℓ. Then there exists

$\exists\ \rho_{f,\lambda} : G_{\mathbb{Q}} \longrightarrow GL_2(L_\lambda)$ • if $p \nmid \ell N$, then $\mathrm{char}(\rho_{f,\lambda}(\mathrm{Frob}_p))$ is $X^2 - a_p X + \chi(p) p^{k-1}$.

More generally, $f \longmapsto \pi = \pi_\infty \otimes \pi_f$ automorphic rep'n $(\pi_f = \otimes' \pi_{f,p})$

$\rho_{f,\lambda}\big|_{D_p} \xleftrightarrow[\text{via LL}]{\text{correspond}} \pi_{f,p}$ (characterized by α, β roots of char poly / Satake)

for $p \neq \ell$

If $\pi_{f,p}^{\Gamma_0(p)} \neq 0$, $\pi_{f,p}^{GL_2 \mathbb{Z}_p} = 0$, then $\rho_{f,\lambda}\big|_{D_p} \cong \begin{pmatrix} \varepsilon & * \\ 0 & 1 \end{pmatrix} \psi$ where ε = cyclotomic character

nonzero

Pf? restrict to f wt 2.

$f \longmapsto H^0(X, \Omega^1) \curvearrowleft \mathbb{T} = \mathbb{Z}_p[T_1, T_2, \ldots T_n \ldots S_n]$

f has a q-expansion, can be viewed as an element of $\mathbb{C}[[q]]$
if you know the eigenvalues $\Longrightarrow$ know q-expansion $f = \sum a_n q^n$

$$\mathbb{T} \circlearrowleft \underset{f \text{ gives 1-dim}}{H^0(X, \Omega^1)} \longrightarrow \underset{2\text{-dim}}{H^1_{dR}(X)} - \underset{2\text{-dim}}{H^1_{et}(X, \hat{\mathbb{Q}}_\ell)} \supseteq V \circlearrowleft G_{\mathbb{Q}}$$

K_λ

still have action $\mathbb{T} \curvearrowright V$ T_p acting as a_p
(along w/ our $G_{\mathbb{Q}}$-action)

$$H^1_{et}(X_{\bar{\mathbb{Q}}}, \mathcal{L}_\lambda) \longrightarrow H^1_{et}(X_{\bar{\mathbb{F}}_p}, \mathcal{L}_\lambda) \supseteq V \quad (2\text{-dim'l})$$
$G_{\mathbb{F}_p} \circlearrowleft$

Recall $T_p = F + F^*$

σ - Frob elt.

$$(\sigma - F^*)(\sigma - F) = 0$$
$$\sigma^2 - T_p \sigma + p = 0$$

$T_p \curvearrowright V$
acts as a_p

WTS: $\mathrm{char}(\sigma) = X^2 - T_p X + p = 0$
have minpoly(σ) divides $\sigma^2 - T_p\sigma + p$

What is $\det(\mathrm{Frob}_p)$? $V \times V \longrightarrow$ 1-dim
can get that $= p$

<u>References for various talks (of Calegari)</u>

I: Katz Paper in Antwerp "modular forms and modular schemes"
LNM349

III, II: Mazur has several papers (Boston volume on FLT)
Gouvea, Park City Volume.

IV: Boston Volume on FLT (Cornell, Silverman, Stevens)