STABILITY OF THE VOLUME-PRODUCT IN THE PLANE

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Let $K \subset \mathbb{R}^2$ be an 0-symmetric convex body, and K^* its polar body. Then we have $V(K) \cdot V(K^*) \geq 8$, with equality if and only if K is a parallelogram (V denotes volume). If $K \subset \mathbb{R}^2$ is a convex body, with $0 \in \operatorname{int} K$, then $V(K) \cdot V(K^*) \geq 27/4$, with equality if and only if K is a triangle and 0 is its centroid. These theorems are due to Mahler and Reisner, and to Mahler and Meyer, respectively. We give stability variants of these theorems. For this we use the Banach-Mazur distance, from parallelograms, or triangles, respectively. The stability variants are sharp, up to constant factors. Our key lemma is a stability estimate for the area product of two sectors of convex bodies polar to each other.

We prove that, for convex n-gons K, the product $V(K) \cdot V\left([(K - s(K)]^*\right)$ is maximal exactly for the affine regular n-gons (($s(K)$ is the Santaló point of K, i.e., the unique point $s \in \operatorname{int} K$, such that $V(K) \cdot V[(K - s)^*]$ is minimal). This is a sharpening of the Blaschke-Santaló inequality $V(K) \cdot V\left([(K - s(K)]^*\right) \leq \pi^2$, for $K \subset \mathbb{R}^2$ a convex body. Suppose that, for an 0-symmetric convex body K in the plane, the ellipse of minimal/maximal area containing/contained by K is the unit circle about 0. Then a sharpening of the Blaschke-Santaló inequality holds: even the arithmetic mean of the areas of K and $K^* = [(K - s(K)]^*$ is at most π. We give a stability version of the Blaschke-Santaló inequality in the plane, for the 0-symmetric case, by using as a measure of deviation from the ellipses the quotient of the areas of K, and any of the above two ellipses. This is sharp, up to a factor that is asymptotically 4. If K contains a regular n-gon inscribed to the unit circle about 0, and is contained in the polar regular n-gon, then for fixed area $V(K)$ we determine the exact maximum of $V(K^*)$.