Rational Self Maps of $K3$ Surfaces and Calabi-Yau Manifolds

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Outline

Rational Self Maps

Question

Let X be a projective Calabi-Yau (CY) manifold over $\mathbb{C}$. Does X admit a rational self map $\phi : X \dashrightarrow X$ of degree $\deg\phi > 1$?

Fibrations of Abelian Varieties

Let $\pi : X \dashrightarrow B$ be a dominant rational map, where $X_b = \overline{\pi^{-1}(b)}$ is an abelian variety for $b \in X$ general. There are rational maps $\phi : X \dashrightarrow X$ induced by $\mathrm{End}(X_b)$:

- Fixing an ample divisor L on X, there is a multi-section $C \subset X/B$ cut out by general members of $|L|$.
- Let $n = \deg(C/B)$. There is a rational map $\phi : X \dashrightarrow X$ sending a point $p \in X_b = \pi^{-1}(b)$ to $C - (n-1)p$. Clearly, $\deg \phi = n^2$.
- We can make n arbitrarily large by choosing L sufficiently ample.
- The same construction works for X_b birational to a finite quotient of an abelian variety.

Potential Density of Rational Points on $K3$

- There are rational self maps of arbitrarily high degrees for an elliptic or Kummer $K3$ surface.
- (Bogomolov-Tschinkel, Amerik-Campana) Let X be a $K3$ surface over a number field k. Suppose that there is a nontrivial rational self map $\phi : X \dashrightarrow X$ over a finite extension $k' \to k$ of k. By iterating ϕ, one can produce many k'-rational points on X. Under suitable conditions, these k'-rational points are Zariski dense in X.
- This works for elliptic and Kummer $K3$'s. For an elliptic $K3$ surface $X/\mathbb{P}^1$, it suffices to find a suitable multi-section.

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Density of Rational Curves on $K3$

- Let $\phi : X \dashrightarrow X$ be a dominant rational self map of X. Then $\phi(C)$ is rational for every rational curve $C \subset X$. For an elliptic $K3$ surface $X/\mathbb{P}^1$,

$$\bigcup_n \phi^n(C)$$

  is dense on X in the analytic topology under suitable conditions.
- Elliptic $K3$'s are dense in the moduli space of $K3$ surfaces.

Theorem (Chen-Lewis)

On a very general projective $K3$ surface X,

$$\bigcup_{C\subset X \text{ rational curve}} C$$

is dense in the analytic topology.

Bogomolov-Hassett-Tschinkel, Li-Liedtke

Rational curves are Zariski dense on almost "every" $K3$ surface.

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Conjecture

The union of rational curves is dense in the *analytic* topology on *every* $K3$ surface.

Complex Hyperbolicity

Question

Does there exist a dominant meromorphic map $f : \mathbb{C}^n \dashrightarrow X$ for a CY manifold X of dimension n?

Cantat

If there is a dominant rational self map $\phi : X \dashrightarrow X$, $\lim_{m\to\infty} \phi^m : \mathbb{C}^n \dashrightarrow X$ is dominant under certain dilating conditions.

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Elliptic and Kummer $K3$'s are holomorphically dominable by $\mathbb{C}^2$.

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Every $K3$ surface is holomorphically dominable by $\mathbb{C}^2$.

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The Kobayashi-Royden infinitesimal metric vanishes everywhere on every $K3$ surface.

$$||\mathbf{v}||_{KR} = \inf\left\{\frac{1}{R} : \exists f : \{|z| < R\} \to X \text{ holomorphic and } f(0) = p,\ f_*\frac{\partial}{\partial z} = \mathbf{v}\right\} \text{ for } \mathbf{v} \in T_{X,p}$$

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Theorem (Chen-Lewis)

On a very general K3 surface X,

$$||v_p||_{KR} = 0$$

for a dense set of points $(p, v_p) \in \mathbb{P}T_X$ in the analytic topology.

Main Theorem

Theorem

A very general projective K3 surface X does not admit rational self maps $\phi : X \dashrightarrow X$ *of degree* $\deg \phi > 1$.

Theorem (Dedieu)

If there is a rational self map $\phi : X \dashrightarrow X$ *of* $\deg(\phi) > 1$ *for a generic K3 surface, then the Severi varieties* $V_{d,g,X}$ *are reducible for* $d >> g$.

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- Let

$$\begin{aligned} \mathrm{Aut}(X) &= \{\text{Automorphisms of } X\} \\ \mathrm{Bir}(X) &= \{\text{Birational self maps of} X\} \\ \mathrm{Rat}(X) &= \{\text{Dominant rational self maps of} X\}. \end{aligned}$$

- For a very general $K3$ surface X,

$$\mathrm{Rat}(X) = \mathrm{Bir}(X) = \mathrm{Aut}(X).$$

Generalization

Rational Self Maps of CY Complete Intersections

For a very general complete intersection $X \subset \mathbb{P}^n$ of type $(d_1, d_2, ..., d_r)$ with $d_1 + d_2 + ... + d_r \geq n + 1$ and $\dim X \geq 2$,

$$\mathrm{Rat}(X) = \mathrm{Bir}(X) = \mathrm{Aut}(X).$$

Corollary (Voisin?)

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Birational Geometry

Let $X \subset \mathbb{P}^n$ be a very general hypersurface of degree d and $\dim X \geq 3$:

- When $d > n$, X is of CY or general type. Then

$$\mathrm{Rat}(X) = \mathrm{Bir}(X) = \mathrm{Aut}(X).$$

- (Iskovskih-Manin, Pukhlikov, ...) When $d = n$, $-K_X$ is ample and $\mathrm{Pic}(X) = \mathbb{Z}K_X$, i.e., X is primitive Fano. Then X is birationally super rigid and hence

$$\mathrm{Bir}(X) = \mathrm{Aut}(X).$$

- When $d < n$, $\mathrm{Bir}(X) = ?$.

Some Trivial Remarks

- Let $\phi \in \mathrm{Rat}(X)$ and let

$$\begin{array}{ccc} Y & \xrightarrow{\varphi} & X \\ {\scriptstyle f}\downarrow & \nearrow_{\phi} & \\ X & & \end{array}$$

  be a resolution of ϕ, where $f : Y \to X$ is a projective birational morphism.

- Suppose that $K_X = \mathcal{O}_X$. Then

$$K_Y = f^* K_X + \sum \mu_i E_i = \sum \mu_i E_i = \varphi^* K_X + \sum \mu_i E_i$$

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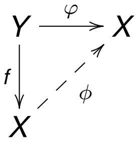

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- $a(E_i, X) + 1$ (log discrepancy of E_i) is the ramification index of E_i under φ if $\varphi_* E_i \neq 0$.
- If ϕ is regular, ϕ is unramified.
- If $\pi_1(X) = 0$ and ϕ is regular, then $\phi \in \mathrm{Aut}(X)$.
- If X is an abelian variety, $\varphi_* E_i = 0$ and hence ϕ is regular.
- If X is a $K3$ surface and $\phi \in \mathrm{Bir}(X)$, $\varphi_* E_i = 0$ and hence $\mathrm{Bir}(X) = \mathrm{Aut}(X)$.
- $\phi^* H^{1,1}_{\mathrm{alg}}(X) = H^{1,1}_{\mathrm{alg}}(X)$ and $\phi^* H^{1,1}_{\mathrm{trans}}(X) = H^{1,1}_{\mathrm{trans}}(X)$.
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Question

Does there exist a rational self map $\phi : X \dashrightarrow X$ of $\deg \phi > 1$ for a $K3$ surface X which is neither elliptic nor Kummer?

Degeneration

Consider the hypersurfaces in $\mathbb{P}^n$ of degree $n+1$ ($n \geq 3$).

- (Kulikov Type II) Let $W \subset \mathbb{P}^n \times \Delta$ be a pencil of hypersurfaces of degree $n+1$ with $W_0 = S_1 \cup S_2$, where $\deg S_1 = 1$ and $\deg S_2 = n$.
- S_1 and S_2 meet transversely along D, where D is a hypersurface in $\mathbb{P}^{n-1}$ of degree n.
- W has rational double points ($xy = tz$) along $\Lambda = D \cap W_t$.
- When $n = 3$, Λ consists of 12 points. Note that $12 = 20 - h^{1,1}(S_1) - h^{1,1}(S_2)$.
- Λ is the vanishing locus of the T^1 class of W_0 in $T^1(W_0) = N_{D/S_1} \otimes N_{D/S_2}$.
- We resolve the singularities of W by blowing up W along S_1. Let X be the resulting n-fold and $X_0 = R_1 \cup R_2$. Then R_1 is the blowup of S_1 along Λ and $R_2 \cong S_2$.

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$$\phi : X \otimes \overline{\mathbb{C}((t))} \dashrightarrow X \otimes \overline{\mathbb{C}((t))}$$

  of $\deg \phi > 1$.
- Otherwise, we have a rational map $\phi : X/\Delta \dashrightarrow X/\Delta$ of $\deg \phi > 1$ after a base change of degree m.
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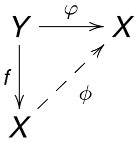

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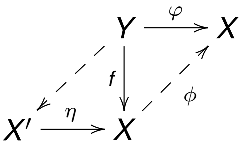

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- $\varphi_* E \neq 0$ and $E \subset Y_0 \Rightarrow E = Q_k$ for some k.
- $\varphi_*(Q_0 + Q_1 + ... + Q_m) = (\deg \phi)(R_0 + R_1)$.
- $Q_k \overset{\sim}{\dashrightarrow} D \times \mathbb{P}^1$ for $0 < k < m$.
- $\deg \phi = 1$ if and only if $\varphi_* Q_k = 0$ for all $0 < k < m$.
- Consider the case $n = 4$, i.e., $X_t \subset \mathbb{P}^4$ a quintic 3-fold.
- D is a very general quartic $K3$ surface. Then $\mathrm{Rat}(D) = \mathrm{Bir}(D) = \mathrm{Aut}(D) = \{1\}$ by induction.
- Suppose that $\varphi : Q_k \to R_1$ is dominant for some $0 < k < m$.
- Then there exists $i : C \hookrightarrow D$ such that $\dim C \leq 1$ and $i_* : \mathrm{CH}_0(C) \to \mathrm{CH}_0(D)$ is surjective.

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Theorem (Mumford, Roĭtman, Bloch-Srinivas)

Let X be a smooth projective variety of dimension n. If there exists $i : Y \hookrightarrow X$ such that $\dim Y < n$ and

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is surjective, then $h^{n,0}(X) = 0$.

Conclusion

$\deg \phi = 1 \Rightarrow \mathrm{Rat}(X) = \mathrm{Bir}(X)$ for a very general quintic 3-fold $X \subset \mathbb{P}^4$.

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