On the value of Borcherds Φ-function

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Contents

Dedekind η-function

Let $\mathfrak{H} = \{x + iy \in \mathbb{C};\ y > 0\}$ be the complex upper half-plane.

Definition (Dedekind η-function)

Dedekind η-function is the holomorphic function on $\mathfrak{H}$ defined as

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n>0} (1 - q^n), \qquad q := e^{2\pi i \tau}.$$

Fact

$\eta(\tau)^{24}$ is a modular form for $SL_2(\mathbb{Z})$ of weight 12 vanishing at $+i\infty$, i.e.,

- $\eta(\frac{a\tau+b}{c\tau+d})^{24} = (c\tau + d)^{12} \eta(\tau)^{24}$ *for all* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$.
- $\lim_{\Im\tau \to +\infty} \eta(\tau) = 0$.

These properties characterize the Dedekind η-function up to a constant.

Determinant of Laplacian: an analytic counter part

For $\tau \in \mathfrak{H}$, let E_τ be the elliptic curve

$$E_\tau := \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z},$$

which is equipped with the flat Kähler metric of normalized volume 1

$$g_\tau = dz \otimes d\bar{z}/\Im\tau.$$

The Laplacian of (E_τ, g_τ) is the differential operator defined as

$$\Box_\tau := -\Im\tau \frac{\partial^2}{\partial z \partial \bar{z}} = -\frac{\Im\tau}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

Definition (Zeta Function of $\Box_\tau$)

$$\zeta_\tau(s) := \sum_{\lambda \in \sigma(\Box_\tau)\setminus\{0\}} \lambda^{-s} = \sum_{(m,n)\neq(0,0)} \left(\frac{\pi^2 |m\tau + n|^2}{\Im\tau}\right)^{-s}$$

Fact (classical)

$\zeta_\tau(s)$ converges absolutely when $\Re s > 1$ and extends to a meromorphic function on $\mathbb{C}$. Moreover, $\zeta_\tau(s)$ is holomorphic at $s = 0$.

By the identity for finite dimensional non-degenerate, Hermitian matrices

$$\log\det H = -\left.\frac{d}{ds}\right|_{s=0} \operatorname{Tr} H^{-s} = -\zeta_H'(0),$$

the value $\exp(-\zeta_\tau'(0))$ is called the (regularized) determinant of $\Box_\tau$.

Theorem (Kronecker's limit formula)

The Petersson norm of the Dedekind η-function is the determinant of $\Box_\tau$

$$\begin{aligned}\exp(-\zeta_\tau'(0)) &= 4\Im\tau\left|e^{2\pi i\tau}\prod_{n>0}(1-e^{2\pi in\tau})^{24}\right|^{1/6}\\ &= 4\|\eta(\tau)\|^4.\end{aligned}$$

Discriminant of elliptic curve: an algebraic counter part

Recall that every elliptic curve is expressed as the complete intersection

$$E = E_A := \left\{ [x] \in \mathbb{P}^3; \begin{array}{ll} f_1(x) = a_{11}x_1^2 + a_{12}x_2^2 + a_{13}x_3^2 + a_{14}x_4^2 & = 0 \\ f_2(x) = a_{21}x_1^2 + a_{22}x_2^2 + a_{23}x_3^2 + a_{24}x_4^2 & = 0 \end{array} \right\}$$

where $A = (a_{ij}) = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4) \in M_{2,4}(\mathbb{C})$. We define

$$\Delta_{ij}(A) := \det(\mathbf{a}_i, \mathbf{a}_j), \qquad i < j.$$

Theorem (classical result due essentially to Jacobi?)

$$2^8 \|\eta(E_A)\|^{24} = \prod_{1 \leq i < j \leq 4} |\Delta_{ij}(A)|^2 \cdot \left(\frac{2\sqrt{-1}}{\pi^2} \int_{E_A} \alpha_A \wedge \bar{\alpha}_A \right)^6$$

Here $\alpha_A \in H^0(E_A, \Omega^1)$ *is defined as the residue of* f_1, f_2, *i.e.,* $\alpha_A := \Xi|_{E_A}$,

$$df_1 \wedge df_2 \wedge \Xi = \sum_{i=1}^{4} (-1)^{i-1} x_i dx_1 \wedge dx_{i-1} \wedge dx_{i+1} \wedge dx_4$$

Remark: another elliptic curve associated to $A \in M_{2,4}(\mathbb{C})$

For $A = (a_{ij}) \in M_{2,4}(\mathbb{C})$, one can associate another elliptic curve:

$$C_A := \{(x,y) \in \mathbb{C}^2;\ y^2 = 4(a_{11}x+a_{21})(a_{12}x+a_{22})(a_{13}x+a_{23})(a_{14}x+a_{24})\}$$

Then $C_A \cong E_A$ and

$$2^8\|\eta(C_A)\|^{24} = \prod_{1\leq i<j\leq 4} |\Delta_{ij}(A)|^2 \cdot \left(\frac{\sqrt{-1}}{2\pi^2}\int_{C_A}\frac{dx}{y}\wedge\overline{\frac{dx}{y}}\right)^6.$$

Goal of talk

We generalize the trinity of Dedekind η-function to Borcherds Φ-function by making the following replacements:

- elliptic curve $\longrightarrow$ Enriques surface
- determinant of Laplacain $\longrightarrow$ analytic torsion
- discriminant $\longrightarrow$ resultant of a certain system of polynomials

The Borcherds Φ-function

As a by-product of his construction of the fake monster Lie (super)algebra, Borcherds introduced a nice automorphic form on the moduli space of Enriques surfaces.

To explain Borcherds Φ-function, we fix some notation.

- Let
$$\mathbb{L} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \mathbb{E}_8(-1)$$
be an even unimodular Lorentzian lattice of rank 10 with signature $(1,9)$, where $\mathbb{E}_8(-1)$ is the negative-definite E_8-lattice.
- Let
$$\mathcal{C}_{\mathbb{L}} = \mathcal{C}_{\mathbb{L}}^{+} \amalg -\mathcal{C}_{\mathbb{L}}^{+} := \{x \in \mathbb{L} \otimes \mathbb{R};\ x^2 > 0\}$$
be the positive cone of $\mathbb{L}$.
- We consider the tube domain of $\mathbb{L} \otimes \mathbb{C}$
$$\mathbb{L} \otimes \mathbb{R} + i\,\mathcal{C}_{\mathbb{L}}^{+} \cong \text{ symmetric bounded domain of type IV of dim 10}$$

Definition (Borcherds '96, '98: Borcherds Φ-function of rank 10)

The Borcherds Φ-function is the formal Fourier series on $\mathbb{L}\otimes\mathbb{R}+i\,\mathcal{C}_{\mathbb{L}}^{+}$

$$\prod_{\lambda\in\mathbb{L},\,\langle\lambda,\mathcal{W}\rangle>0}\left(\frac{1-e^{2\pi i\langle\lambda,z\rangle}}{1+e^{2\pi i\langle\lambda,z\rangle}}\right)^{c(\lambda^2/2)}=\sum_{\lambda\in\mathbb{L}\cap\overline{\mathcal{C}_{\mathbb{L}}^{+}},\,\lambda^2=0,\,\text{primitive}}\frac{\eta(\langle\lambda,z\rangle)^{16}}{\eta(2\langle\lambda,z\rangle)^{8}}$$

where $\mathcal{W}$ is a Weyl chamber of $\mathbb{L}$ and $\{c(n)\}$ is defined by the series

$$\sum_{n\in\mathbb{Z}}c(n)\,q^n:=\eta(\tau)^{16}\eta(2\tau)^{-8}.$$

Theorem (Borcherds '96)

The Borcherds Φ-function converges absolutely when $\Im z\gg 0$ and extends to an automorphic form on $\mathbb{L}\otimes\mathbb{R}+i\,\mathcal{C}_{\mathbb{L}}^{+}$ for an arithmetic subgroup of $\Gamma\subset\mathrm{Aut}(\mathbb{L}\otimes\mathbb{R}+i\,\mathcal{C}_{\mathbb{L}}^{+})_0\cong SO(2,10;\mathbb{R})_0$ of weight 4.

$K3$ and Enriques surfaces

Definition ($K3$ surface)

A compact connected complex surface X is a $K3$ surface $\iff$

- $H^1(X, \mathcal{O}_X) = 0$
- $K_X = \Omega^2_X \cong \mathcal{O}_X$.

Definition (Enriques surface)

A compact connected complex surface Y is an Enriques surface $\iff$

- $H^1(Y, \mathcal{O}_Y) = 0$,
- $K_Y \not\cong \mathcal{O}_Y$,
- $K_Y^{\otimes 2} \cong \mathcal{O}_Y$.

Fact

An Enriques surface is the quotient of a $K3$ surface by a free involution.

N.B. A $K3$ surface can cover many distinct Enriques surfaces.

The moduli space of Enriques surfaces

Theorem (Horikawa '78)

Recall that $\Gamma \subset \mathrm{Aut}(\mathbb{L} \otimes \mathbb{R} + i\,\mathcal{C}_{\mathbb{L}}^{+})_0 \cong SO(2,10;\mathbb{R})_0$ *is the arithmetic subgroup, for which* Φ *is an automorphic form. Then there is a divisor* $\mathcal{D} \subset \mathbb{L} \otimes \mathbb{R} + i\,\mathcal{C}_{\mathbb{L}}^{+}$ *such that the period mapping induces an isomorphism*

$$\textit{Moduli space of Enriques surfaces} \cong \frac{(\mathbb{L} \otimes \mathbb{R} + i\,\mathcal{C}_{\mathbb{L}}^{+}) - \mathcal{D}}{\Gamma}.$$

Theorem (Borcherds '96)

The Borcherds Φ*-function vanishes exactly on* $\mathcal{D}$ *of order* 1.

Definition (Petersson norm)

The Petersson norm of Φ is the C^∞ function on the moduli of Enriques surfaces

$$\|\Phi(z)\|^2 := \langle \Im z, \Im z \rangle^4 |\Phi(z)|^2.$$

Analytic torsion: an analytic counter part of Φ

Definition (Ray-Singer '73)

Let (M, h_M) be a compact connected Kähler manifold.
Let $\square_q = (\bar{\partial} + \bar{\partial}^*)^2$ be the Laplacian acting on $(0, q)$-forms on M.
Let $\zeta_q(s)$ be the zeta function of $\square_q$:

$$\zeta_q(s) := \sum_{\lambda \in \sigma(\square_q) \setminus \{0\}} \lambda^{-s} \dim E(\lambda, \square_q),$$

where $E(\lambda, \square_q)$ is the eigenspace of $\square_q$ with respect to the eigenvalue λ.
The analytic torsion of (M, h_M) is the real number

$$\tau(M) := \exp[-\sum_{q \geq 0} (-1)^q q\, \zeta_q'(0)].$$

When $\dim M = 1$, $\tau(M)$ is essentially the determinant of Laplacian that appeared in the formula for $\|\eta(\tau)\|$.

Φ-function as the analytic torsion of Enriques surface

Recall that the Petersson norm of the Dedekind η-function at $z \in \mathfrak{H}$ coincides with the analytic torsion of the flat elliptic curve E_z with normalized volume 1, up to a constant (Kronecker's limit formula)

$$\tau(E_z) = \|\eta(z)\|^{-4}/4.$$

Theorem (Y. '04)

- *For an Enriques surface Y, its analytic torsion $\tau(Y)$ with respect to a Ricci-flat Kähler metric of normalized volume 1, is independent of the choice of such a Kähler metric. Namely, $\tau(Y)$ is an invariant of Y.*
- *There is a constant $C \neq 0$ such that for every Enriques surface Y equipped with a Ricci-flat Kähler metric of normalized volume 1,*

$$\tau(Y) = C\,\|\Phi(Y)\|^{-1/4}.$$

N.B. In fact, $C = 2^a\pi$ with some $a \in \mathbb{Q}$.

Generic Enriques surfaces

Let $f_1(x), g_1(x), h_1(x) \in \mathbb{C}[x_1, x_2, x_3]$ and $f_2(x), g_2(x), h_2(x) \in \mathbb{C}[x_4, x_5, x_6]$ be homogeneous polynomials of degree 2. We define $f, g, h \in \mathbb{C}[x_1, \ldots, x_6]$

$$f := f_1 + f_2, \qquad g = g_1 + g_2, \qquad h = h_1 + h_2.$$

For generic quadrics f_1, g_1, h_1, f_2, g_2, h_2, we get a $K3$ surface in $\mathbb{P}^5$

$$X_{(f,g,h)} := \{[x] \in \mathbb{P}^5;\, f(x) = g(x) = h(x) = 0\},$$

which is preserved by the involution

$$\iota(x_1 : x_2 : x_3 : x_4 : x_5 : x_6) = (x_1 : x_2 : x_3 : -x_4 : -x_5 : -x_6).$$

Then ι has no fixed points on $X_{(f,g,h)}$ for generic f, g, h.

Fact

A generic Enriques surface is expressed as the quotient

$$Y_{(f,g,h)} := X_{(f,g,h)}/\iota$$

An algebraic expression of $\|\Phi\|$

Theorem (Kawaguchi-Mukai-Y. '11)

For all generic Enriques surface $Y_{(f,g,h)}$ defined by the quadric polynomials $f = f_1 + f_2$, $g = g_1 + g_2$, $h = h_1 + h_2 \in \mathbb{C}[x_1, \ldots, x_6]$, one has

$$\|\Phi(Y_{(f,g,h)})\|^2 = |R(f_1, g_1, h_1)R(f_2, g_2, h_2)| \left(\frac{2}{\pi^4} \int_{X_{(f,g,h)}} \alpha \wedge \overline{\alpha} \right)^4 .$$

Here $R(f_1, g_1, h_1)$ and $R(f_2, g_2, h_2)$ are the resultants of the three quadric polynomials of three variables,
and $\alpha \in H^0(X_{(f,g,h)}, \Omega^2)$ is defined as the residue of f, g, h:

$$\alpha := \Xi|_{X_{(f,g,h)}},$$

$$\sum_{i=1}^{6} (-1)^i x_i dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_6 = df \wedge dg \wedge dh \wedge \Xi.$$

A Thomae type formula for the Borcherds Φ-function

Corollary

Let $\mathbf{v}, \mathbf{v}' \in H^2(X_{(f,g,h)}, \mathbb{Z})$ *be anti-ι-invariant, primitive, isotropic vectors with* $\langle \mathbf{v}, \mathbf{v}' \rangle = 1$ *and let* $\mathbf{v}^\vee \in H_2(X_{(f,g,h)}, \mathbb{Z})$ *be the Poincaré dual of* $\mathbf{v}$*. Under the identification of lattices* $(\mathbb{Z}\mathbf{v} + \mathbb{Z}\mathbf{v}')^\perp \cong \binom{0\,2}{2\,0} \oplus \mathbb{E}_8(-2) =: L$*,*

$$z_{(f,g,h),\mathbf{v},\mathbf{v}'} := \frac{\alpha - \langle \alpha, \mathbf{v}' \rangle \mathbf{v} - \langle \alpha, \mathbf{v} \rangle \mathbf{v}'}{\langle \alpha, \mathbf{v} \rangle} \in L \otimes \mathbb{R} + i\,\mathcal{C}_L^+$$

is regarded as the period of $Y_{(f,g,h)}$*. Then, by a suitable choice of cocycles* $\{\mathbf{v}, \mathbf{v}'\}$*, one has*

$$\Phi\left(z_{(f,g,h),\mathbf{v},\mathbf{v}'}\right)^2 = R(f_1, g_1, h_1) R(f_2, g_2, h_2) \left(\frac{2}{\pi^2} \int_{\mathbf{v}^\vee} \alpha\right)^8,$$

where α *is the residue of* $f, g, h \in \mathbb{C}[x_1, \ldots, x_6]$ *as before.*

An expression of $\|\Phi\|$ —— special case

For a 3×6-complex matrix $A \in M_{3,6}(\mathbb{C})$, we define

$$X_A := \left\{ [x] \in \mathbb{P}^5;\ \begin{array}{ll} a_{11}x_1^2 + a_{12}x_2^2 + a_{13}x_3^2 + a_{14}x_4^2 + a_{15}x_5^2 + a_{16}x_6^2 & = 0 \\ a_{21}x_1^2 + a_{22}x_2^2 + a_{23}x_3^2 + a_{24}x_4^2 + a_{25}x_5^2 + a_{26}x_6^2 & = 0 \\ a_{31}x_1^2 + a_{32}x_2^2 + a_{33}x_3^2 + a_{34}x_4^2 + a_{35}x_5^2 + a_{36}x_6^2 & = 0 \end{array} \right\}$$

on which acts the involution

$$\iota\colon \mathbb{P}^5 \ni (x_1 : x_2 : x_3 : x_4 : x_5 : x_6) \to (x_1 : x_2 : x_3 : -x_4 : -x_5 : -x_6) \in \mathbb{P}^5.$$

For $A = (\mathbf{a}_1, \dots, \mathbf{a}_6) \in M(3, 6; \mathbb{C})$ and $i < j < k$, set

$$\Delta_{ijk}(A) = \det(\mathbf{a}_i, \mathbf{a}_j, \mathbf{a}_k).$$

$A \in M(3, 6; \mathbb{C})$ is said to be generic if $\prod_{i<j<k} \Delta_{ijk}(A) \neq 0$.

Fact

For a generic $A \in M_{3,6}(\mathbb{C})$, X_A is a K3 surface and ι is a free involution. Hence

$$Y_A := X_A/\iota$$

is an Enriques surface.

Corollary

For every generic $A \in M_{3,6}(\mathbb{C})$,

$$\|\Phi(Y_A)\|^2 = |\Delta_{123}(A)|^4 |\Delta_{456}(A)|^4 \left(\frac{2}{\pi^4} \int_{X_A} \alpha_A \wedge \overline{\alpha}_A \right)^4 .$$

As before, $\alpha_A \in H^0(X_A, \Omega^2)$ is defined as $\alpha_A := \Xi|_{X_A}$, where

$$\sum_{i=1}^{6} (-1)^{i-1} x_i dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_6 = \left(\prod_{i=1}^{3} \sum_{j=1}^{6} a_{ij} x_j dx_j\right) \wedge \Xi$$

Corollary

Let $A \in M_{3,6}(\mathbb{C})$ be generic. For a partition

$$\{i,j,k\} \cup \{l,m,n\} = \{1,2,3,4,5,6\},$$

define an involution $\iota_{\binom{ijk}{lmn}}$ on $\mathbb{P}^5$ by

$$\iota_{\binom{ijk}{lmn}}(x_i,x_j,x_k,x_l,x_m,x_n) = (x_i,x_j,x_k,-x_l,-x_m,-x_n).$$

Then $\iota_{\binom{ijk}{lmn}}$ is a free involution on X_A called a switch*. Moreover,*

$$\|\Phi(X_A/\iota_{\binom{ijk}{lmn}})\|^2 = |\Delta_{ijk}(A)|^4 |\Delta_{lmn}(A)|^4 \left(\frac{2}{\pi^4} \int_{X_A} \alpha_A \wedge \overline{\alpha}_A \right)^4.$$

In particular, if $A \in M_{3,6}(K)$ with $K \subset \mathbb{C}$, then

$$\frac{\|\Phi(X_A/\iota_{\binom{ijk}{lmn}})\|^2}{\|\Phi(X_A/\iota_{\binom{i'j'k'}{l'm'n'}})\|^2} = \frac{|\Delta_{ijk}(A)|^4 |\Delta_{lmn}(A)|^4}{|\Delta_{i'j'k'}(A)|^4 |\Delta_{l'm'n'}(A)|^4} \in K.$$

Another $K3$ surface associated to $A \in M_{3,6}(\mathbb{C})$

To a generic $A \in M_{3,6}(\mathbb{C})$, one can associate another $K3$ surface

$$Z_A := \{((x_1 : x_2 : x_3), y) \in \mathcal{O}_{\mathbb{P}^2}(3);\ y^2 = \prod_{i=1}^{6}(a_{1i}x_1 + a_{2i}x_2 + a_{3i}x_3)\}$$

which is identified with its minimal resolution.
We define a holomorphic 2-form η_A on Z_A by

$$\eta_A := \frac{x_1 dx_2 \wedge dx_3 - x_2 dx_1 \wedge dx_3 + x_3 dx_1 \wedge dx_2}{y}.$$

Fact (Matsumoto-Sasaki-Yoshida)

There are 6 independent transcendental 2-cycles $\{\gamma_{ij}\}_{1 \leq i < j \leq 4}$ on Z_A and 16 independent algebraic 2-cycles on Z_A, which form a basis of $H_2(Z_A, \mathbb{Z})$.

Fact (Matsumoto-Sasaki-Yoshida)

Define the period of Z_A as the matrix

$$\Omega_A := \frac{1}{\eta_{34}(A)} \begin{pmatrix} \eta_{14}(A) & -\frac{\eta_{13}(A)-\sqrt{-1}\eta_{24}(A)}{1+\sqrt{-1}} \\ -\frac{\eta_{13}(A)+\sqrt{-1}\eta_{24}(A)}{1-\sqrt{-1}} & -\eta_{23}(A) \end{pmatrix}, \quad \eta_{ij}(A) = \int_{\gamma_{ij}} \eta_A$$

By a suitable choice of the basis $\{\gamma_{ij}\}_{1\leq i<j\leq 4}$ of $H_2(Z_A, \mathbb{Z})$, one has

$$\Omega_A \in \mathbb{D} := \{T \in M_{2,2}(\mathbb{C});\ (T - {}^t\overline{T})/2i > 0\} \cong \text{SBD of type IV of dim 4.}$$

Definition (Freitag theta function)

Write $\mathbf{e}(x) := \exp(2\pi i x)$. For $\Omega \in \mathbb{D}$ and $a = (a_1, a_2), b = (b_1, b_2) \in \mathbb{Z}[i]^2$,

$$\Theta_{\frac{a}{1+i}, \frac{b}{1+i}}(\Omega) :=$$

$$\sum_{n\in\mathbb{Z}[i]^2} \mathbf{e}\left[\frac{1}{2}\left(n + \frac{a}{1+i}\right)\Omega\,{}^t\overline{\left(n + \frac{a}{1+i}\right)} + \Re\left(n + \frac{a}{1+i}\right){}^t\overline{\left(\frac{b}{1+i}\right)}\right]$$

Borcherds Φ-function and theta function

Theorem (Kawaguchi-Mukai-Y. '11)

For a generic $A = (A_1, A_2) \in M_{3,6}(\mathbb{C})$ *with* $A_1, A_2 \in M_{3,3}(\mathbb{C})$, *define*

$$A^\vee := ({}^t A_1^{-1}, {}^t A_2^{-1}).$$

Then

$$\left\| \Phi(X_A / \iota_{\binom{ijk}{lmn}}) \right\| = \det\left(\frac{\Omega_A - {}^t\overline{\Omega_A}}{2\sqrt{-1}} \right)^2 \left| \Theta_{\binom{ijk}{lmn}}(\Omega_{A^\vee}) \right|^4 =: \left\| \Theta_{\binom{ijk}{lmn}}(Z_{A^\vee}) \right\|^4$$

under the identification $\Theta_{\binom{pqr}{stu}}(\Omega) := \Theta_{\frac{a}{1+i}, \frac{b}{1+i}}(\Omega)$, $\binom{a}{b} = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \leftrightarrow \binom{pqr}{stu}$,

$$\begin{matrix} \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} & \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} i & i \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} i & i \\ i & i \end{pmatrix} & \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ i & i \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix} & \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \binom{123}{456} & \binom{124}{356} & \binom{125}{346} & \binom{126}{345} & \binom{134}{256} & \binom{135}{246} & \binom{136}{245} & \binom{145}{236} & \binom{146}{235} & \binom{156}{234} \end{matrix}$$

The case of Jacobian Kummer surfaces

For $\lambda = (\lambda_1, \dots, \lambda_6)$ with $\lambda_i \neq \lambda_j$ $(i \neq j)$, define a genus 2 curve

$$C_\lambda := \{(x,y) \in \mathbb{C}^2;\ y^2 = \prod_{i=1}^{6}(x - \lambda_i)\}.$$

Define holomorphic differentials $\omega_1 := dx/y$, $\omega_2 := xdx/y$ on C_λ.
Let $\{A_1, A_2, B_1, B_2\}$ be a certain symplectic basis of $H_1(C_\lambda, \mathbb{Z})$ and set

$$T_\lambda := \begin{pmatrix} \int_{B_1}\omega_1 & \int_{B_2}\omega_1 \\ \int_{B_1}\omega_2 & \int_{B_2}\omega_2 \end{pmatrix}^{-1} \begin{pmatrix} \int_{A_1}\omega_1 & \int_{A_2}\omega_1 \\ \int_{A_1}\omega_2 & \int_{A_2}\omega_2 \end{pmatrix} \in \mathfrak{S}_2.$$

Fact (classical)

The Kummer surface $K(C_\lambda)$ of $\mathrm{Jac}(C_\lambda)$ is expressed as follows:

$$K(C_\lambda) \cong X_A, \qquad A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 & \lambda_5^2 & \lambda_6^2 \end{pmatrix} \in M_{3,6}(\mathbb{C}).$$

Fact (classical)

$A \in M_{3,6}(\mathbb{C})$ is associated to a Jacobian Kummer surface, i.e., $K(C_\lambda) = X_A$ if and only if A is self-dual in the sense that

$$X_A \cong X_{A^\vee}.$$

By this fact and the previous theorem, we get the following.

Theorem (Kawaguchi-Mukai-Y. '11)

If the partition $\binom{pqr}{stu}$ corresponds to the even characteristic (a, b), then

$$\left\| \Phi(K(C_\lambda)/\iota_{\binom{pqr}{stu}}) \right\| = (\det \Im T_\lambda)^2 \left| \theta_{\Re(\frac{a}{1+i}),\Re(\frac{b}{1+i})}(T_\lambda)\, \theta_{\Im(\frac{a}{1+i}),\Im(\frac{b}{1+i})}(T_\lambda) \right|^4 .$$

Here $\theta_{\alpha,\beta}(T)$, $(\alpha,\beta) = (\alpha_1,\alpha_2,\beta_1,\beta_2)$, is the Riemann theta constant

$$\theta_{\alpha,\beta}(T) := \sum_{n\in\mathbb{Z}^2} \mathbf{e}\left[\frac{1}{2}(n+\alpha)T\,{}^t\overline{(n+\alpha)} + (n+\alpha)\,{}^t\overline{\beta}\right], \qquad T \in \mathfrak{S}_2.$$

Recall that Igusa's Siegel modular form Δ_5 is defined as the product of all even theta constants

$$\Delta_5(T) := \prod_{(\alpha,\beta)\,\text{even}} \theta_{\alpha,\beta}(T), \qquad T \in \mathfrak{S}_2.$$

For a genus 2 curve C with period $T \in \mathfrak{S}_2$, its Petersson norm

$$\|\Delta_5(C)\|^2 := (\det \Im T)^5 |\Delta_5(T)|^2$$

is independent of the choice of a symplectic basis of $H_1(C, \mathbb{Z})$. Hence $\|\Delta_5(C)\|$ is an invariant of C.

Corollary

Δ_5 is the average of Φ by the 10 switches:

$$\prod_{\binom{ijk}{lmn}} \left\| \Phi(K(C_\lambda))/\iota_{\binom{ijk}{lmn}} \right\| = \|\Delta_5(C_\lambda)\|^8.$$

Some problems

Fact (Gritsenko-Nikulin '97)

The Igusa cusp form $\Delta_5(T)$ admits an infinite product expansion near the cusp of $\mathfrak{S}_2$.

After our corollaries, the following statements are very likely to be the case.

Infinite product expansion of theta function in higher dimension ?

- For all partitions $\binom{ijk}{lmn}$, the Freitag theta function $\Theta_{\binom{ijk}{lmn}}(\Omega)$ admits an infinite product expansion near the cusp of $\mathbb{D}$.
- For even characteristic $(a, b) \in \mathbb{Z}[i]^2$, the product of the Riemann theta constants of degree 2

$$\theta_{\Re(\frac{a}{1+i}),\Re(\frac{b}{1+i})}(T)\,\theta_{\Im(\frac{a}{1+i}),\Im(\frac{b}{1+i})}(T)$$

  admits an infinite product expansion near the cusp of $\mathfrak{S}_2$.

Problem (The inverse of the period mapping for Enriques surfaces)

For $1 \leq i < j \leq 3$ *and* $4 \leq k < l \leq 6$, *find a system of automorphic forms*

$$\alpha_{ij}^{(1)}(z),\ \alpha_{kl}^{(2)}(z),\ \beta_{ij}^{(1)}(z),\ \beta_{kl}^{(2)}(z),\ \gamma_{ij}^{(1)}(z),\ \gamma_{kl}^{(2)}(z)$$

on $\mathbb{L} \otimes \mathbb{R} + i\mathcal{C}_{\mathbb{L}}^{+}$ *for (a finite index subgroup of)* Γ *such that*

$$Y_z := X_z/\iota, \qquad \iota(x) = (x_1, x_2, x_3, -x_4, -x_5, -x_6)$$

with

$$X_z = \left\{ [x] \in \mathbb{P}^5;\ \begin{array}{ll} \sum_{1\leq i<j\leq 3} \alpha_{ij}^{(1)}(z)x_ix_j + \sum_{4\leq k<l\leq 6} \alpha_{kl}^{(2)}(z)x_kx_l & = 0 \\ \sum_{1\leq i<j\leq 3} \beta_{ij}^{(1)}(z)x_ix_j + \sum_{4\leq k<l\leq 6} \beta_{kl}^{(2)}(z)x_kx_l & = 0 \\ \sum_{1\leq i<j\leq 3} \gamma_{ij}^{(1)}(z)x_ix_j + \sum_{4\leq k<l\leq 6} \gamma_{kl}^{(2)}(z)x_kx_l & = 0 \end{array} \right\}$$

is the Enriques surface whose period is the given $z \in \mathbb{L} \otimes \mathbb{R} + i\mathcal{C}_{\mathbb{L}}^{+}$.

For elliptic curves, this was solved by Jacobi using theta constants.

Problem

On a generic Jacobian Kummer surface, there exists 31 *conjugacy classes of free involutions (Mukai-Ohashi '09). They split into three families:*

- 10 *switches,*
- 15 *Hutchinson-Göpel involutions,*
- 6 *Hutchinson-Weber involutions.*

Recall that, as the average of the Borcherds Φ-function by 10 *switches, we get Igusa's Siegel modular form Δ_5:*

$$\prod_{\binom{ijk}{lmn}} \left\| \Phi(K(C_\lambda))/\iota_{\binom{ijk}{lmn}} \right\| = \|\Delta_5(C_\lambda)\|^8.$$

Determine the Siegel modular form constructed as the average of the Borcherds Φ-function by the 15 *Hutchinson-Göpel involutions (resp.* 6 *Hutchinson-Weber involutions).*