FINITE GROUPS OF AUTOMORPHISMS OF ENRIQUES SURFACES AND THE MATHIEU GROUP M_{12}.

HISANORI OHASHI

Abstract. These notes were used for the talk given by the author at Fields Institute on 19th August, 2011.

1. Introduction

Let me thank the organizers for this opportunity of giving a talk. This is a joint work with S. Mukai at RIMS, Kyoto University. In my talk every variety is over $\mathbb{C}$.

First let me recall the following theorem of Mukai, which classifies the finite groups of symplectic automorphisms of $K3$ surfaces.

Theorem 1.1 (Mukai 1988.)**.** For a finite group G, the following properties are equivalent.

(1) G has an effective and symplectic action on some $K3$ surface.
(2) G is a subgroup of eleven maximal groups $G_1, \cdots, G_{11}$ which are explicitly determined (we omit them).
(3) G has an embedding into the Mathieu group M_{23} in such a way that the number of orbits of G on $\Omega := \{1, \cdots, 24\}$ is at least 5.

Today we consider a possible extention of this theorem; our goal is to show a version of this theorem for Enriques surfaces.

We recall that Mathieu groups are the oldest finite simple sporadic groups. As introduced in Kondo's lecture, M_{24} is a special subgroup of the symmetric group $\mathfrak{S}_{24}$ which acts quituply transitively on Ω. M_{23}, M_{22} are therefore defined as successive point stabilizer subgroups of M_{24}.

There is an another, although closely related, series of Mathieu groups. This is the small Mathieu group M_{12} in $\mathfrak{S}_{12}$, which acts on $\Omega^+ := \{1, \cdots, 12\}$ quintuply transitively. M_{11} is similarly the one point stabilizer subgroup of M_{12}.

There is a psychological evidence for our goal:

(1) Let G be acting on a $K3$ surface X. Then $H^*(X, \mathbb{Q})$ is a 24-dimensional representation of G. This representation is closely related to the natural representation of M_{24}, as theorem above implies.
(2) Let G be acting on an Enriques surface S. Then $H^*(S, \mathbb{Q})$ is a 12-dimensional representation of G. The first item in mind, shouldn't this representation be related to the natural representation of M_{12} ?

Date: 19th August, 2011.

Graduate School of Mathematics, Nagoya University, Nagoya 464-8602, JAPAN.

2. Characters

First we review the $K3$ case. Let φ be a symplectic automorphism of finite order n, acting on a $K3$ surface X. Then the following assertions are true.

(1) $n = \mathrm{ord}(\varphi) \leqq 8$.
(2) $\#\,\mathrm{Fix}(\varphi) < \infty$ if $\varphi \neq 1$.
(3) (Nikulin) The number of fixed points, or equivalently the Lefschetz number of φ,
$$L(\varphi) := \chi_{\mathrm{top}}(\mathrm{Fix}\,\varphi) = \mathrm{tr}(\varphi^* \mid_{H^*(X,\mathbb{Q})}),$$
depends only on n and is as follows.

$n = \mathrm{ord}\,\varphi$	1	2	3	4	5	6	7	8
$L(\varphi)$	24	8	6	4	4	2	3	2

(4) (Mukai) Any $g \in M_{23}$ with $\mathrm{ord}(g) = n$ has the same number of fixed points in Ω as $L(\varphi)$. Namely, $H^*(X,\mathbb{Q})$ and $\mathbb{Q}^{\Omega}$ (the natural representation as subgroup of M_{24}) has the same characters.

Let us proceed to automorphisms of Enriques surfaces. In what follows we use the notation
$$S = X/\varepsilon$$
to denote an Enriques surface S whose covering $K3$ surface is X, with the covering transformation ε. Recall that S satisfies and is characterized by the properties $q(S) = 0, p_g(S) = 0, P_2(S) = h^0(S, \mathcal{O}_S(2K_S)) = 1$.

Definition 2.1. $\sigma \in \mathrm{Aut}(S)$ is said to be semi-symplectic if it acts on the space $H^0(S, \mathcal{O}_S(2K_S))$ trivially.

An easy fact is that, any automorphism σ of order 2 is semi-symplectic. A harder fact is that, any automorphism σ of order 3 or 5 is semi-symplectic. Hence also for order 6. On the other hand, there exists non-semi-symplectic automorphisms of order 4.

We find that the connection to M_{12} is not so perfect in this case. Let σ be a semi-symplectic automorphism of order $n < \infty$.

(1) $n = \mathrm{ord}(\sigma) \leqq 6$.
(2) The fixed point set $\mathrm{Fix}(\sigma)$ is discrete for $n \geqq 3$, but possibly has curves for $n = 2$.
(3) The Lefschetz number $L(\sigma)$ *varies* even when we fix the order n.
(4) As a consequence of (3), we see that semi-symplectic automorphisms does not necessarily have the same character as the same-ordered element in M_{11}.

More precise description of fixed point sets can be given. Let σ as above. Let $P \in S$ be a fixed point of σ. From the definition of semi-symplectic property, the local action $(\mathrm{d}\sigma)_P \in \mathrm{GL}(T_P S)$ has two possiblities of determinants, namely ± 1. We call P with determinant 1 the symplectic fixed point:
$$\mathrm{Fix}(\sigma) = \mathrm{Fix}^+(\sigma) \amalg \mathrm{Fix}^-(\sigma).$$
$\mathrm{Fix}^-(\sigma)$ is the origin from where the diversity of Enriques automorphisms arises.

(1) order $n = 2$. $\mathrm{Fix}^+(\sigma)$ is exactly four points, while $\mathrm{Fix}^-(\sigma)$ is a disjoint union of smooth curves. The Lefschetz number $L(\sigma)$ takes every even value between -4 and 12.

(2) order $n = 3, 5$. The fixed point set is discrete and $\mathrm{Fix}^-(\sigma) = \emptyset$. They have the same characters as the same-ordered element of M_{11}.
(3) order $n = 4, 6$. The fixed point set is discrete but possibly $\mathrm{Fix}^-(\sigma) \neq \emptyset$.

Definition 2.2. Let G be a finite group of semi-symplectic automorphisms of an Enriques surface S. We say this action is Mathieu if every $\sigma \in G$ has the same $L(\sigma)$ as the character of the same-ordered element of M_{11}.

Concretely, this definition requires what follows.

$n = \mathrm{ord}\,\sigma$	1	2	3	4	5	6
$L(\sigma)$	12	4	3	4	2	1

The lower row presents the number of fixed points of an element $g \in M_{11}$ of same order.

3. Examples

One of the obstacles in the study of Enriques surfaces is the difficulty in giving projective models of them. Here we give three examples.

Example I. At the beginning Enriques himself described his surfaces as sextic hypersurfaces with prescribed (non-normal) singularities. Let $T = \{x_1x_2x_3x_4 = 0\} \subset \mathbb{P}^3$ be the coordinate tetrahedron. Let $\cup l_{ij} = \cup\{x_i = x_j = 0\}$ be the edges of T.

Theorem 3.1. (Enriques) Let S' be a sextic surface with ordinary double lines along $\cup l_{ij}$ and no other singularities. Then the desingularization $S \to S'$ gives an Enriques surface.

The first example comes with this model. Let us consider a general element S' of the linear pencil $\{a_1f_1 + a_2f_2 \mid (a_1 : a_2) \in \mathbb{P}^1\}$, where

$$f_1 = (xy + yz + zx + xt + yt + zt)xyzt = s_2s_4,$$
$$f_2 = (x^2 + y^2 + z^2 + t^2)xyzt + x^2y^2z^2t^2(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} + \frac{1}{z^2}) = s_1^2s_4 + s_3^2 - 4s_2s_4.$$

Here s_i are fundamental symmetric polynomials of degree i in four variables. On the surface S' and its desingularization S, we have the action of the symmetric group $\mathfrak{S}_4$ permutating coordinates and the standard Cremona transformation

$$\sigma\colon (x, y, z, t) \mapsto (\frac{1}{x}, \frac{1}{y}, \frac{1}{z}, \frac{1}{t}).$$

They constitute the biregular action of the finite group $\mathbb{Z}/2 \times \mathfrak{S}_4$ on S. This action is semi-sympletic and the subgroup $G = \mathbb{Z}/2 \times \mathfrak{A}_4$ acts Mathieu-semi-symplectically.

Example II. Next example comes out of the theory of elliptic surfaces. Recall that every Enriques surface has an elliptic fibration $S \to \mathbb{P}^1$ with exactly two double fibers. Conversely any elliptic surface $S \to \mathbb{P}^1$ with exactly two double fibers and $\chi_{\mathrm{top}}(S) = 12$ is an Enriques surface. They never have a section, hence Weierstrass models, but nevertheless we can describe Enriques surfaces using this structure.

A rational elliptic surface arises as the Jacobian of an elliptic Enriques surface S. Conversely in general S can be reconstructed from R via the logarithmic transformations, which are although highly transcendental. However, in some important cases we can describe these constructions explicitly in an algebro-geometric way. These are given in papers of Kondo and Hulek-Schütt. We use them.

Let $R\colon y^2 = x(x-1)(x-s^2)$ be the rational elliptic surface with singular fibers $I_2 + I_2 + I_4 + I_4$ and the Mordell-Weil group $\mathbb{Z}/2 \times \mathbb{Z}/4$. Let G be the group of automorphisms of this elliptic fibration generated by σ, ι, t_α where

$$\sigma\colon (x,y,s) \mapsto (\frac{x}{s^2}, \frac{y}{s^3}, \frac{1}{s})$$
$$\iota\colon (x,y,s) \mapsto (x,-y,s)$$
$$t_\alpha\colon \text{translations by } \alpha \in MW(R/\mathbb{P}^1).$$

The base change construction using the 2-torsion $(x,y) = (0,0)$ and fibers $R_p, R_{\sigma(p)}$ for general p gives the action of G on elliptic Enriques surface $S/\mathbb{P}^1$ and its subgroup $\mathbb{Z}/2\times\mathbb{Z}/4$ acts on S Mathieu-semi-symplectically.

Example III. Nikulin and Kondo classified Enriques surfaces S with $\#\operatorname{Aut}(S) < \infty$. The type VII surface in Kondo's paper (Fano's surface) has $\operatorname{Aut}(S) \simeq \mathfrak{S}_5$ and it can be shown that this is a Mathieu-semi-symplectic action.

4. Main theorem

First we give a remark on the relationship between M_{12} and M_{24}. The operator domain Ω has a special $12 + 12$ partition $\Omega^+ \amalg \Omega^-$. Then M_{12} is exactly the setwise stabilizer subgroup of Ω^+.

Our main theorem is as follows.

Theorem 4.1. For a finite group G, the following properties are equivalent.

(1) G has an effective Mathieu-semi-symplectic action on some Enriques surface S.
(2) G is a subgroup of the five maximal groups, $\mathfrak{A}_6, \mathfrak{S}_5, 3^2D_8, \mathbb{Z}/2\times\mathfrak{A}_4$ and $\mathbb{Z}/2\times\mathbb{Z}/4$.
(3) G can be embedded in the subgroup $\mathfrak{S}_6$ of M_{12} in such a way that, (A) for all $g \in G$, g has a fixed point in Ω^+, or (B) for all $g \in G$, g has a fixed point in Ω^-, or (C) $G \simeq \mathbb{Z}/2 \times \mathbb{Z}/4$.

G	$\#G$	moduli	description
$\mathbb{Z}/2 \times \mathbb{Z}/4$	8	1-dim.	
$\mathbb{Z}/2 \times \mathfrak{A}_4$	24	1-dim.	
3^2D_8	72	point(s)	3-Sylow normalizer in $\mathfrak{S}_6$
$\mathfrak{S}_5$	120	point(s)	symmetric
$\mathfrak{A}_6$	360	point(s)	alternating

The latter three groups are constructed via lattice-theoretic constructions. An account of this will be given in the Mukai's talk in this conference.

Mnemonic: The maximal groups are exactly subgroups of $\mathfrak{S}_6$ whose orders are not divisible by 16.

Sketch of Proof: The classification of Mathieu-semi-symplectic groups proceeds as follows.

- *Step 1.* Classification of possible $\mathrm{Fix}(\sigma)$.
- *Step 2.* Elimination of G by characters.
- *Step 3.* Elimination of G by geometry.

As an example, let us show that there exist no semi-symplectic action of $\mathbb{Z}/8$ on Enriques surface S. Suppose that σ is an semi-symplectic automorphism of order 8. Let $\tilde{\sigma}$ be the lift to the covering $K3$ surface X which is a symplectic automorphism of order 8. It has exactly two fixed points $P, Q \in X$. For the covering transformation ε to act on X freely, we must have $\varepsilon(P) = Q$. But by the holomorphic Lefschetz fixed point formula, the local actions of $\tilde{\sigma}$ on the tangent spaces at P and Q do not coincide. This is a contradiction.