MOTIVIC ZETA FUNCTIONS FOR DEGENERATIONS OF CALABI-YAU VARIETIES

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1. Introduction

Let $f \in \mathbb{Z}[x_1, \ldots, x_n]$ be a non-constant polynomial and let $p \in \mathbb{Z}_{>0}$ be a prime. We define

$$S_m = \{a \in (\mathbb{Z}/p^{m+1}\mathbb{Z}) : f(a) \equiv 0 \ (mod\ p^{m+1})\}$$

and put $N_m = |S_m|$.

The generating series for the integers N_m,

$$P_f^p(T) = \sum_{m \geq 0} N_m T^m \in \mathbb{Z}[[T]],$$

is called the *Poincaré series* associated to f and p. A closely related object is the p-adic zeta function $Z_f^p(s)$, which is entirely determined by $P_{f,p}(p^{-s})$. It is a meromorphic function on the complex plane. Igusa's p-adic monodromy conjecture predicts in a precise way how the singularities of the complex hypersurface $V(f) \subset \mathbb{A}^n_{\mathbb{C}}$ influence the poles of $Z_f^p(s)$ and thus the asymptotic behaviour of the integers N_m as m tends to infinity. The conjecture states that, when p is sufficiently large, poles of $Z_f^p(s)$ should correspond to local monodromy eigenvalues of the polynomial map $\mathbb{C}^n \to \mathbb{C}$ defined by f.

In the mid-nineties, J. Denef and F. Loeser developed the theory of motivic integration, and constructed a motivic object $Z_f^{mot}(s)$ that interpolates the p-adic zeta functions $Z_f^p(s)$ for $p \gg 0$ and captures their geometric essence. Denef and Loeser also formulated a motivic upgrade of the monodromy conjecture.

It is the purpose of this talk to present joint work with J. Nicaise (cf. [4] [5] [6] [7]), where we introduce and study a global version of Denef and Loeser's motivic zeta functions. More precisely, I will explain how we associate a motivic generating series to any Calabi-Yau variety X defind over a complete discretely valued field K. This series encodes information concerning degenerations of X and the behaviour of certain invariants of X under tamely ramified extensions K'/K, and has properties analogous to Denef and Loeser's zeta function.

The link between Denef and Loeser's motivic zeta function $Z_f^{mot}(s)$ and our global variant is an alternative interpretation of $Z_f^{mot}(s)$ in terms of non-archimedean geometry, due to J. Sebag and J. Nicaise [9]. This interpretation is based on the theory of motivic integration on rigid varieties developed by F. Loeser and J. Sebag [8].

This is joint work with J. Nicaise.

2. Calabi-Yau varieties and weak Néron models

2.1. **Notation.** We let R denote a complete discrete valuation ring and fix a uniformizer $\pi \in R$. We put $K = Frac(R)$ and $k = \overline{k} = R/(\pi)$, and denote by p the characteristic exponent of k.

We fix a separable closure K^s of K. For any $d \in \mathbb{N}$ such that p does not divide d, we put $K(d) = K(\pi^{1/d})$. The union of the fields $K(d)$ is a subfield of K^s, called the *tame closure* K^t of K. It is a procyclic group, and we call every topological generator of $G(K^t/K)$ a *tame monodromy operator.*

Throughout, X/K will denote a smooth, proper and geometrically connected variety of dimension g. We assume that $X(K) \neq \emptyset$. Moreover, we assume that X is *Calabi-Yau*, by which we mean that $\Omega^g_X \cong \mathcal{O}_X$. A *gauge form* for X is a nowhere vanishing form $\omega \in \Omega^g_X(X)$.

Finally, we put $X(d) = X \otimes_K K(d)$ and we let $\omega(d)$ denote the pullback of ω to $X(d)$ for every d not divisible by p.

2.2. We will associate to X a motivic generating series $Z_X(T) \in \mathcal{M}_k[[T]]$. Here

$$\mathcal{M}_k = K_0(Var_k)[\mathbb{L}^{-1}],$$

where $K_0(Var_k)$ is the Grothendieck ring of k-varieties and $\mathbb{L} = [\mathbb{A}^1_k] \in K_0(Var_k)$.

Remark 2.2.1. Taking the class $[V] \in K_0(Var_k)$ should be considered the most general way to measure the size of a k-variety V. In particular, it generalizes the procedure of counting rational points that we encountered in Section 1.

The construction of $Z_X(T)$ is based on the motivic integration in the sense of Loeser and Sebag (cf. [8]). They developed a theory of motivic integration in the context of rigid varieties and formal schemes. We will next explain their main result, which doesn't require going into details concerning rigid/formal geometry.

Definition 2.2.2. *A smooth R-scheme $\mathcal{Y}$ of finite type is a Weak Néron Model of X if the following properties hold:*

1. $\mathcal{Y} \times_R K = X$, *and*
2. *The natural restriction map $\mathcal{Y}(R) \to X(K)$ is a bijection.*

Remark 2.2.3. In the situation considered in this talk, there always exists a Weak Néron Model (WNM). However, such a model is usually neither proper nor unique.

Theorem 2.2.4 (Loeser-Sebag [8]). *Let X/K be a Calabi-Yau variety and let ω be a gauge form. Let $\mathcal{Y}$ be a WNM of X. Then*

$$\int_X |\omega| = \mathbb{L}^{-g} \sum_{C \in \pi_0(\mathcal{Y}_s)} [C]\mathbb{L}^{-ord_C(\omega)} \in \mathcal{M}_k.$$

To interpret this theorem, one should think of $X^{an}(K)$ (X^{an} being the rigid analytification of X) as a family of open balls parametrized by $\mathcal{Y}_s$. The volume of each ball is renormalized by ω so that the total volume is independent of the choice $\mathcal{Y}$.

2.3. **Definition of the motivic zeta function.**

Proposition 2.3.1 ([7]). *Let X be a Calabi-Yau variety over K, and let ω be a gauge form on X. Then for every weak Néron model $\mathcal{Y}$ of X, the value*

$$\mathrm{ord}(X,\omega) \quad := \quad \min\{ord_C(\omega) \mid C \in \pi_0(\mathcal{Y}_s)\} \quad \in \mathbb{Z}$$

only depends on the pair (X, ω), and not on $\mathcal{Y}$.

Definition 2.3.2. *Let X be a Calabi-Yau variety over K. A distinguished gauge form on X is a gauge form ω such that* $\mathrm{ord}(X, \omega) = 0$.

Thus, a distinguished gauge form on X extends to a relative differential form on every weak Néron model, in a "minimal" way. One can show that X admits a distinguished gauge form since X has a K-rational point. Moreover, a distinguished gauge form is unique up to multiplication with a unit in R.

Definition 2.3.3. *Let X/K be a Calabi-Yau variety, and let ω be a distinguished gauge form on X. We define the motivic zeta function $Z_X(T)$ of X to be the generating series*

$$Z_X(T) = \mathbb{L}^g \cdot \sum_d \Big(\int_{X(d)} |\omega(d)|\Big) T^d \in \mathcal{M}_k[[T]].$$

It is easily seen that our definition of $Z_X(T)$ is independent of the choice of distinguished gauge form, hence it is an invariant of X.

3. Properties of $Z_X(T)$

In this section we assume that $char(k) = 0$. By embedded resolution of singularities, we can find an *sncd*-model $\mathcal{X}$ for X, i.e., a regular proper R-model such that $\mathcal{X}_s = \sum_{i \in I} N_i E_i$ is a divisor with strict normal crossings. For every $i \in I$, we define the order $\mu_i = \mathrm{ord}_{E_i}\omega$ of ω along E_i as in [9, 6.8]. These values do not depend on the choice of distinguished gauge form ω.

One can deduce from [9, 7.7] that the motivic zeta function $Z_X(T)$ can be expressed in the form

$$(3.1) \qquad Z_X(T) = \sum_{\emptyset \neq J \subset I} (\mathbb{L}-1)^{|J|-1} [\widetilde{E}_J^o] \prod_{j \in J} \frac{\mathbb{L}^{-\mu_j} T^{N_j}}{1 - \mathbb{L}^{-\mu_j} T^{N_j}} \quad \in \mathcal{M}_k[[T]]$$

where $\widetilde{E}_J^o$ is a certain finite étale cover of $E_J := \cap_{j \in J} E_j$ (see [7] for further details).

In particular, one sees from (3.1) that $Z_X(T)$ is a rational function and that every pole of $Z_X(\mathbb{L}^{-s})$ is of the form $s = -\mu_i/N_i$ for some $i \in I$. Every irreducible component E_i of the special fiber yields in this way a "candidate pole" $-\mu_i/N_i$ of the zeta function. Since the expression in (3.1) is independent of the chosen normal crossings model $\mathcal{X}$, one expects in general that not all of these candidate poles are actual poles of $Z_X(T)$. But even candidate poles that appear in *every* model will not always be actual poles. To explain this phenomenon, we propose in Section 4 a version of Denef and Loeser's Monodromy Conjecture for Calabi-Yau varieties.

Example 3.0.4. Let X/K be an elliptic curve, with reduction type IV^* over R. Then one computes that $s = 2/3$ is the only pole of $Z_X(\mathbb{L}^{-s})$. Hence, there is a unique component in the special fiber of the minimal regular model of X that corresponds to a pole of the zeta function.

3.1. **Log canonical threshold.** Choose a regular proper R-model $\mathcal{X}$ of X such that $\mathcal{X}_s$ is a strict normal crossings divisor $\mathcal{X}_s = \sum_{i \in I} N_i E_i$ and define the values μ_i, $i \in I$ as above. We put

$$\begin{aligned} lct(X) &= \min\{\mu_i/N_i \,|\, i \in I\}, \\ \delta(X) &= \max\{\, |J| \,|\, \emptyset \neq J \subset I,\ E_J \neq \emptyset,\ \mu_j/N_j = lct(X) \text{ for all } j \in J\} - 1. \end{aligned}$$

Definition 3.1.1. *We call $lct(X)$ the log canonical threshold of X, and $\delta(X)$ the degeneracy index of X.*

The following theorem shows that these values do not depend on the chosen model $\mathcal{X}$.

Theorem 3.1.2. *Let X be a Calabi-Yau variety with $X(K) \neq \emptyset$.*

1. *The value $s = -lct(X)$ is the largest pole of the motivic zeta function $Z_X(\mathbb{L}^{-s})$, and its order equals $\delta(X) + 1$. In particular, $lct(X)$ and $\delta(X)$ are independent of the model $\mathcal{X}$.*
2. *Assume moreover that $K = \mathbb{C}((t))$ and that X admits a projective model over the ring $\mathbb{C}\{t\}$ of germs of analytic functions at the origin of the complex plane. If we put $\alpha = lct(X)$, then, for every embedding of $\mathbb{Q}_\ell$ in $\mathbb{C}$, $\exp(-2\pi i\alpha)$ is an eigenvalue of every monodromy operator $\sigma \in Gal(K^s/K)$ on $H^g(X \times_K K^s, \mathbb{Q}_\ell)$.*

For an explanation of the relation of $lct(X)$ to the log canonical threshold in birational geometry, and an interpretation of $\delta(X)$, we refer to [7].

4. The motivic monodromy conjecture

It is natural to wonder if there is a relation between poles of $Z_X(T)$ and monodromy eigenvalues for Calabi-Yau varieties X, similar to the one predicted by Denef and Loeser's motivic monodromy conjecture for hypersurface singularities (cf. [3]).

Definition 4.0.3. *Let X be a Calabi-Yau variety with $X(K) \neq \emptyset$. For any element $a/b \in \mathbb{Q}$, let $\tau(a/b)$ denote the order of a/b in $\mathbb{Q}/\mathbb{Z}$, and let $\sigma \in Gal(K^t/K)$ be a tame monodromy operator. We say that X satisfies the Global Monodromy Property (GMP) if there exists a finite subset $\mathcal{S}$ of $\mathbb{Z} \times \mathbb{Z}_{>0}$ such that*

$$Z_X(T) \in \mathcal{M}_k \left[T, \frac{1}{1 - \mathbb{L}^a T^b}\right]_{(a,b)\in\mathcal{S}}$$

and such that for each $(a, b) \in \mathcal{S}$, the cyclotomic polynomial $\Phi_{\tau(a/b)}(t)$ divides the characteristic polynomial of σ on $H^i(X \times_K K^t, \mathbb{Q}_\ell)$ for some $i \in \mathbb{N}$.

We have, in particular, proved that abelian varieties satisfy the Global Monodromy Property. Recall that for any abelian variety A/K, there exists a finite separable extension K'/K such that the Néron model $\mathcal{A}'/R'$ of $A \times_K K'$ (R' being the integral closure of R in K') is a semi-abelian scheme. We say that A is tamely ramified if the field extension K'/K is tamely ramified.

The dimension of the maximal subtorus of $\mathcal{A}'_s$ is called the potential toric rank of A, and denoted $t_{pot}(A)$. Moreover, we denote by $c(A) \in \mathbb{Q}$ Chai's base change conductor (cf. [1]). This value is zero if and only if A has semi-abelian reduction over R.

Theorem 4.0.4 (Monodromy conjecture for abelian varieties [4], [5])**.** *Let A be a tamely ramified abelian variety of dimension g, and let σ be a tame monodromy operator in $Gal(K^t/K)$.*

1. *The motivic zeta function $Z_A(T)$ belongs to the subring*

$$\mathcal{M}_k \left[T, \frac{1}{1 - \mathbb{L}^a T^b}\right]_{(a,b)\in\mathbb{N}\times\mathbb{Z}_{>0}, a/b=c(A)}$$

of $\mathcal{M}_k[[T]]$. *The zeta function* $Z_A(\mathbb{L}^{-s})$ *has a unique pole at* $s = c(A)$, *of order* $t_{pot}(A) + 1$.

(2) *The cyclotomic polynomial* $\Phi_{\tau(c(A))}(t)$ *divides the characteristic polynomial of the tame monodromy operator* σ *on* $H^g(A \times_K K^t, \mathbb{Q}_\ell)$. *Thus for every embedding* $\mathbb{Q}_\ell \hookrightarrow \mathbb{C}$, *the value* $\exp(2\pi c(A)i)$ *is an eigenvalue of* σ *on* $H^g(A \times_K K^t, \mathbb{Q}_\ell)$.

We have recently generalized Theorem 4.0.4 to tamely ramified semi-abelian varieties of any dimension. Moreover, we have also investigated the case of $K3$-surfaces X (in the case $k = \mathbb{C}$) allowing a triple-point-free degeneration. We managed to show that if the degeneration is of flower pot type or of cyclic type (see [2] for this terminology) then X has the Global Monodromy Property. This will appear soon.

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