A stability result for electric impedance tomography by elastic perturbation

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Outline

- EIT by elastic perturbation
- Uniqueness and a reconstruction formula
- Stability

1. EIT by elastic perturbation

$\Omega \subset \mathbf{R}^n$ smooth bounded domain, $n = 2, 3$

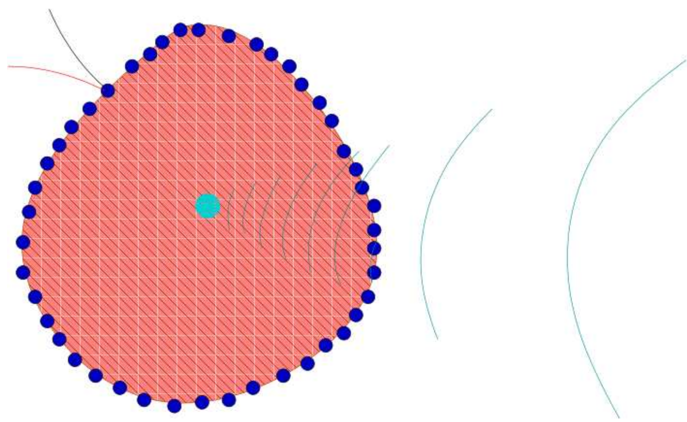
$\gamma(x)$ unknown conductivity. The test potentials satisfy

$$\begin{cases} \operatorname{div}(\gamma(x)\nabla u_i(x)) & = & 0 \quad \text{in } \Omega \\ u_i & = & f_i \quad \text{on } \partial\Omega \\ \gamma\partial_n u_i & = & g_i \quad \text{on } \partial\Omega \end{cases}$$

- In general, the recovery of γ from boundary measurements is severely ill-posed, unless additionnal information is available

- For example : If we knew that the medium contained small-volume inclusions, then we could extract (some) information from the data in a stable manner

We are interested in multi-wave imaging where EIT measurements are made while the medium is perturbed by ultrasounds



- One or several currents are imposed at the boundary and the voltage potential is measured there (get data $u_{/\partial\Omega}, \partial_n u_{/\partial\Omega}$)

- At the same time, a spherical region is mechanically excited by focused accoustic waves (the focal spot is a few mm's in diameter)

- The electric measurements are repeated as the ultrasound spot scans the whole domain

- Asymptotically, the measurements contain information on the pointwise energy density

$$E(x) = \gamma(x)\nabla u(x) \cdot \nabla u(x)$$

Reconstruction with the '0-Laplacian'

Assume that the conductivity γ is known in the vicinity of $\partial\Omega$

The asymptotic formula gives an approximation of the local values of the energy density $\gamma(x)|\nabla u(x)|^2 \sim E(x)$ thus we may obtain u by solving

$$\begin{cases} \operatorname{div}\left(\dfrac{E(x)\nabla u(x)}{|\nabla u(x)|^2}\right) = 0 & \text{in } \Omega \\ \dfrac{E(x)\nabla u(x)}{|\nabla u(x)|^2}\cdot n = \psi & \text{on } \partial\Omega \end{cases}$$

provided that the current g is chosen in such a way that $\nabla u \neq 0$

Numerical reconstruction :

We use 2 measurements E_1 and E_2, corresponding to 2 Dirichlet data g_1 and g_2 on $\partial\Omega$.

1. Select an initial conductivity γ^0 and compute the solution to

$$\begin{cases} \mathrm{div}(\gamma^0 \nabla u^0) = 0 & \text{in } \Omega \\ u^0 = g_1 & \text{on } \partial\Omega \end{cases}$$

2. Compute the error $e^0 = \dfrac{E_1(x)}{|\nabla u^0|^2} - \gamma^0$

3. Introduce a corrector w solution to

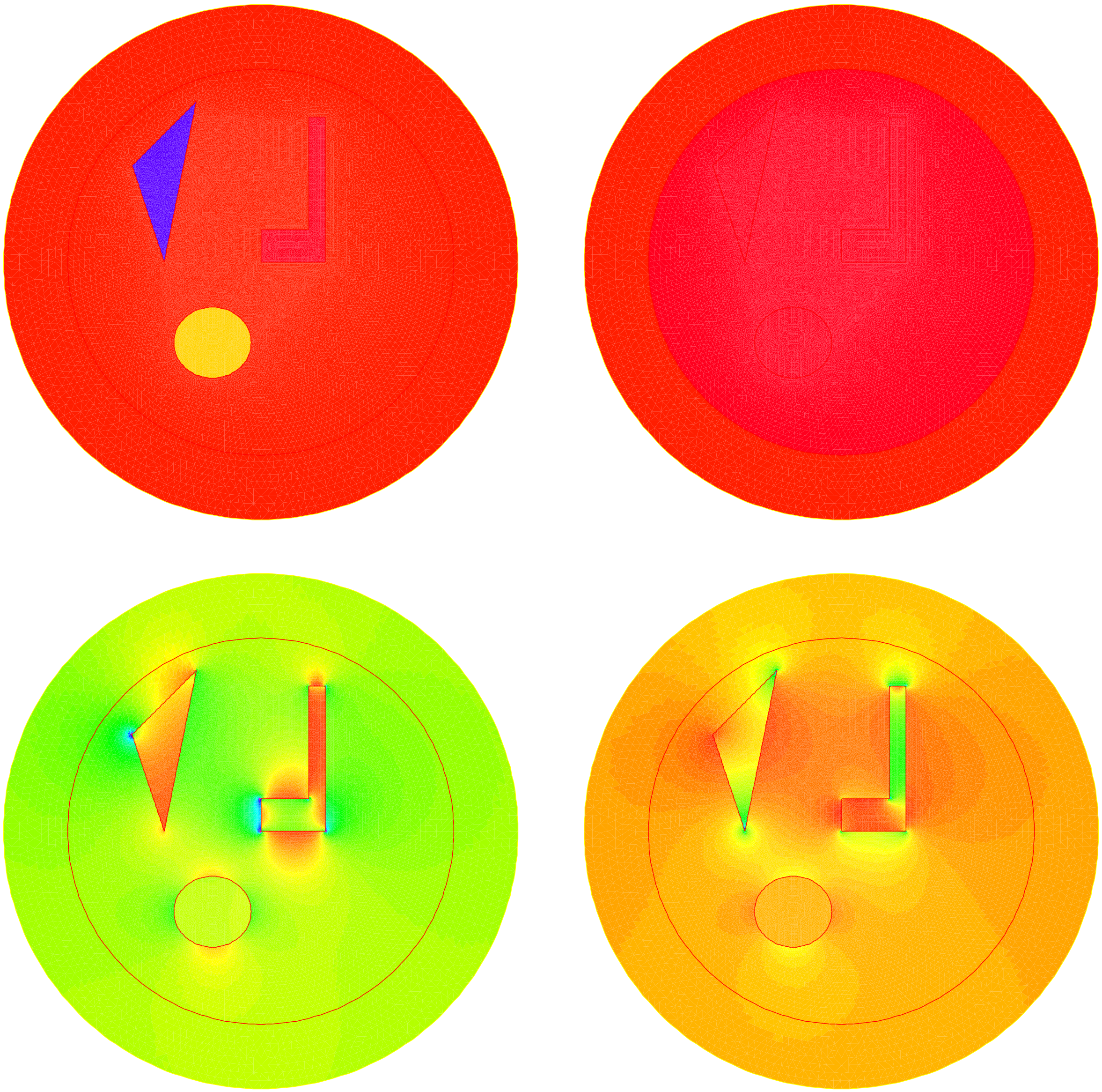
$$\begin{cases} \mathrm{div}(\gamma^0 \nabla w) = -\mathrm{div}(e^0 \nabla u^0) & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases}$$

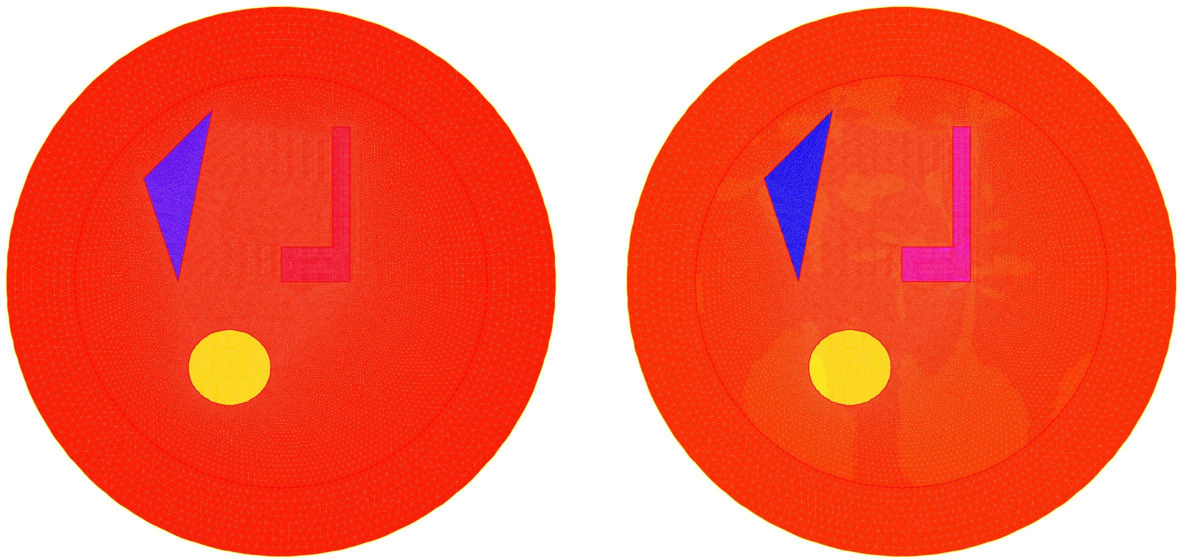
4. Update the conductivity

$$\gamma^1 = \frac{E_1(x) - 2\gamma^0(x)\nabla w \cdot \nabla u^0}{|\nabla u^0|^2}$$

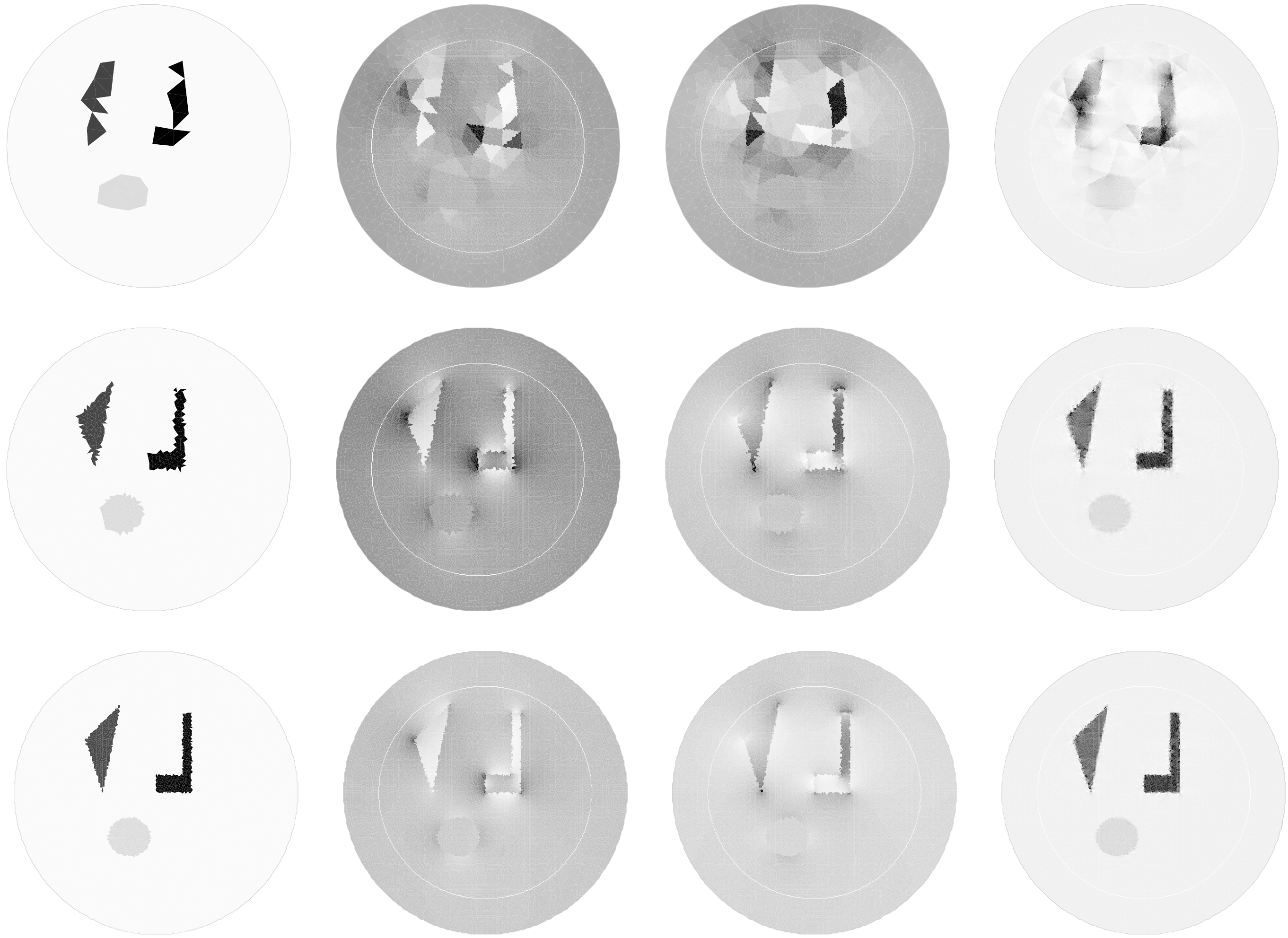
5. Iterate, alternating the currents g_1, g_2

An example of reconstruction (with 2 sets of measurements)





Sensitivity to the mesh



2. Uniqueness in 2D [Capdeboscq, De Gournay, Fehrenbach, Kavian]

Assume that $\Omega \subset \mathbf{R}^2$ is convex with smooth boundary, and that the conductivity γ is smooth ($\mathcal{C}^{1,\alpha}$)

Let $(g_1, g_2) \in \mathcal{C}^{2,\alpha}(\partial\Omega)$, such that $||g_i||_{2,\alpha,\partial\Omega} = 1$

Let $u_i, i = 1, 2$ be the solutions to
$$\begin{cases} \mathrm{div}(\gamma\nabla u_i) = 0 & \text{in } \Omega \\ u_i = g_i & \text{on } \partial\Omega. \end{cases}$$

The internal data is $E_{ij}(x) = \gamma(x)\nabla u_i(x) \cdot \nabla u_j(x) \quad a.e.\ x \in \omega$

Alessandrini and Nesi have shown that if $(g_1, g_2) : \partial\Omega \to \partial\Omega$ is a homeomorphism, then $|\det(\nabla u_1, \nabla u_2)| > 0, \text{ a.e. in } \Omega$

Prop : If (g_1, g_2) is a homeomorphism $\partial\Omega \longrightarrow \partial\Omega$, then the internal data E_{ij} in $\omega \subset \Omega$ determines γ up to a constant in ω

The proof is based on the following formula for computing $\ln(\gamma)$:

Let $s_i = \sqrt{\gamma}\,\nabla u_i, \quad i = 1, 2$

Assume that $\det(\nabla u_1, \nabla u_2) > 0$ and decompose $s_2 = \alpha s_1 + \beta J s_1$ where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \alpha = \frac{E_{12}}{E_{11}} \qquad \beta = \frac{\sqrt{E_{11}E_{22} - E_{12}^2}}{E_{11}}$$

Then we have in ω

$$\begin{cases} \operatorname{div}(\sqrt{\gamma} s_1) = \operatorname{div}(\gamma \nabla u_1) = 0 \\ \operatorname{div}(\frac{1}{\sqrt{\gamma}} J s_1) = \operatorname{div}(J \nabla u_1) = 0 \\ \operatorname{div}(\sqrt{\gamma} s_2) = \operatorname{div}(\sqrt{\gamma}[\alpha s_1 + \beta J s_1]) = 0 \\ \operatorname{div}(\frac{1}{\sqrt{\gamma}} J s_2) = \operatorname{div}(\frac{1}{\sqrt{\gamma}} J[\alpha s_1 + \beta J s_1]) = 0 \end{cases}$$

From these relations one obtains

$$\begin{cases} -2\mathrm{div}(s_1) = \nabla \ln(\gamma) \cdot s_1 = \mathcal{U} \cdot J s_1 \\ -2\mathrm{div}(J s_1) = -\nabla \ln(\gamma) \cdot J s_1 = \mathcal{U} \cdot s_1 \end{cases} \qquad (1)$$

where $\mathcal{U} = \dfrac{\nabla \alpha}{\beta} - J\nabla \ln(\beta)$ is a known function

Note that $|s_1|^2 = \gamma \nabla u_1 \cdot \nabla u_1$, so that we can write $s_1 = \sqrt{E_{11}}(\cos(\theta), \sin(\theta))$

Then we have in ω

$$\begin{cases} \mathrm{div}(s_1) = \left(\nabla \ln(\sqrt{E_{11}}) + J\nabla\theta\right) \cdot s_1 \\ \mathrm{div}(J s_1) = \left(\nabla \ln(\sqrt{E_{11}}) - J\nabla\theta\right) \cdot J s_1 \end{cases} \qquad (2)$$

Using (1) and (2) we obtain

$$\nabla\theta = \mathcal{U} - J\nabla \ln(\sqrt{E_{11}})$$

from which one can deduce s_1 and finally

$$\nabla \ln(\gamma) \quad = \quad J\mathcal{U} - \frac{2J\mathcal{U} \cdot s_1}{E_{11}} s_1$$

so that γ is determined up to a constant from the energy densities E_{ij}

In general, determining the constant requires additional information

3. Stability

3.1. The 2D case

Let $\gamma \in \mathcal{C}^{1,\alpha}(\overline{\Omega})$ such that $\quad 0 < \lambda \leq \gamma(x) \leq \Lambda < \infty.$

Let $g \in \mathcal{C}^{2,\alpha}(\partial\Omega)$. By elliptic regularity, the solution to

$$\begin{cases} \operatorname{div}(\gamma \nabla u) & = & 0 & \text{in } \Omega \\ u & = & g & \text{on } \partial\Omega \end{cases}$$

satisfies $\quad ||u||_{\mathcal{C}^{2,\alpha}(\overline{\Omega})} \; \leq \; C \, ||g||_{\mathcal{C}^{2,\alpha}(\partial\Omega)}$

where $\quad C = C(\Omega, \lambda, \Lambda, ||\gamma||_{1,\alpha,\overline{\Omega}})$

In particular, the maps $\gamma \in \mathcal{C}^{1,\alpha}(\Omega) \longrightarrow E_{ij} \in \mathcal{C}^{1,\alpha}(\Omega)$ are Frechet differentiable

Consider (g_1, g_2) diffeomorphic Dirichlet data (i.e. $\det(\nabla u_1, \nabla u_2) > 0$)

Assume that for all $x \in \omega \subset \Omega$,

$$\begin{cases} m & < \quad \min\{E_{11}, \sqrt{E_{11}E_{22} - E_{12}^2}\} \\ M & > \quad \max\{||E_{ij}||_{\mathcal{C}^1(\overline{\omega})}, \quad i = 1, 2\} \end{cases}$$

Then there exists $\rho > 0$ such that

$$\forall \; ||\tilde{\gamma} - \gamma||_{\mathcal{C}^{1,\alpha}} < \rho, \qquad \begin{cases} \frac{m}{2} & < \quad \min\{\tilde{E}_{11}, \sqrt{\tilde{E}_{11}\tilde{E}_{22} - \tilde{E}_{12}^2}\} \\ 2M & > \quad \max\{||\tilde{E}_{ij}||_{\mathcal{C}^1(\overline{\omega})}, \quad i = 1, 2\} \end{cases} \tag{3}$$

We observe that the maps

$$E_{ij} \longrightarrow \begin{cases} \mathcal{U} & = \dfrac{E_{11}\nabla(E_{12}/E_{11})}{\sqrt{E_{11}E_{22}-E_{12}^2}} - J\nabla\ln\big(\dfrac{\sqrt{E_{11}E_{22}-E_{12}^2}}{E_{11}}\big) \\ \nabla\theta & = \mathcal{U} - J\nabla\ln(\sqrt{E_{11}}) \\ s_1 & = \sqrt{E_{11}}(\cos(\theta),\sin(\theta)) \\ \nabla\gamma & = J\mathcal{U} - \dfrac{2(J\mathcal{U}\cdot s_1)}{E_{11}}s1 \end{cases}$$

are Lipschitz functions of the E_{ij}'s and of their gradients, given the bounds (3)

Prop :

Assume ω is a smooth connected subset of Ω with $z_0 \in \partial\omega \cap \partial\Omega$.

There exists $\rho = \rho(\Omega, \omega, g_1, g_2)$ such that

if $||\gamma - \tilde{\gamma}||_{\mathcal{C}^{1,\alpha}(\overline{\omega})} < \rho$ and if $\tilde{\gamma}(z_0) = \gamma(z_0), \partial_\nu u_1(z_0) = \partial_\nu \tilde{u}_1(z_0)$

then

$$||\gamma - \tilde{\gamma}||_{\mathcal{C}^1(\overline{\omega})} \quad \leq \quad C \sum_{i,j=1}^{2} ||E_{ij} - \tilde{E}_{ij}||_{\mathcal{C}^1(\overline{\omega})}$$

with

$$\begin{cases} C & = & \dfrac{C(\omega)}{m} \max(1, \dfrac{M^3}{m^3}) \\ 2m & = & \min\{E_{11}(\gamma), \sqrt{E_{11}(\gamma)E_{22}(\gamma) - E_{12}(\gamma)^2}\} \\ M/2 & = & \max\{||E_{ij}(\gamma)||_{\mathcal{C}^1(\overline{\omega})}, \quad i = 1, 2\} \end{cases}$$

3.2. The 3D case

Assume that in a subset O of Ω, we measure electrostatic energy densities $E_{ij}(x) = \gamma \nabla u_i(x) \cdot \nabla u_j(x)$ corresponding to 3 imposed boundary values $u_i = g_i$, such that

$$\det(E)(x) \geq c_0 > 0 \quad \text{in } O \qquad \text{with } E = (E_{ij})_{1 \leq i,j \leq 3}$$

We set $S_i(x) = \sqrt{\gamma} \nabla u_i(x), F = \nabla \log(\sqrt{\gamma})$

As $E_{ij}(x) = S_i(x) \cdot S_j(x)$, the data E determines $S_1(x), S_2(x), S_3(x)$ up to an orthogonal matrix $R(x) = [R_1(x), R_2(x), R_3(x)]$, which satisfies

$$\begin{aligned} \nabla \cdot R_i &= V_{ik} \cdot R_k - F \cdot R_i, \quad 1 \leq i \leq 3 \\ \nabla \times R_i &= V_{ik} \times R_k + F \times R_i \quad 1 \leq i \leq 3 \end{aligned}$$

where the V_{ij}'s are computed from the data H.

These conditions lead to a system of ODE's of the form

$$\partial_k R \quad = \quad \sum_{|\alpha|=3} Q_\alpha^k R^\alpha, \qquad R^\alpha = \Pi_{i=1}^9 R_i^{\alpha_i}$$

with a polynomial RHS.

This system can be solved by direct integration along pathes $\gamma_{x_0,x}$ in O, provided $R(x_0)$ is known, and from R one can determine $\nabla \log(\gamma)$.

Moreover, one can obtain a local stability result of the form

$$||R - \tilde{R}||(x) \quad \leq \quad C_0 ||R - \tilde{R}||(x_0) \; + \; C_1 ||H - \tilde{H}||_{W^{1,\infty}(O)}$$

About the hypothesis on $\det(H)$ (see also [Triki, Bal-Uhlmann])

Assume γ is smooth, and can be smoothly extended by 1 outside a large ball, and set $q(x) = -(\Delta\sqrt{\gamma})/\gamma$.

Then $v = \sqrt{\gamma}u$ solves $\Delta v + qv = 0$ in $\mathbf{R}^3$

Let $\rho \in \mathbf{C}^n$, $\rho = r(\mathbf{k} + i\mathbf{k}^\perp)$ with $\mathbf{k}, \mathbf{k}^\perp \in \mathbf{S}^2$, $\mathbf{k}\cdot\mathbf{k}^\perp = 0$, $r = |\rho|/\sqrt{2}$

$w(x) = e^{\rho\cdot x}$ is a harmonic complex plane wave (CGO solution)

Thm : [Sylvester, Uhlmann]

$$v_\rho = \sqrt{\gamma}u_\rho \quad = \quad e^{\rho\cdot x}(1+\psi_\rho) \quad \text{with} \quad r\psi_\rho = O(1) \text{ in } \mathcal{C}^1(\overline{\Omega})$$

In particular,

$$\sqrt{\gamma}\nabla u_\rho \quad = \quad e^{\rho\cdot x}\left(\rho + \rho\psi_\rho + \nabla\psi_\rho - (1+\psi_\rho)\nabla\sqrt{\gamma}\right)$$

Setting $\rho_1 = r(e_2 + ie_1), \quad \rho_2 = r(e_3 + ie_1)$ and

$$(S_1, S_2, S_3, S_4) \quad = \quad \sqrt{\gamma}(\nabla Re(u_{\rho_1}), \nabla Im(u_{\rho_1}), \nabla Re(u_{\rho_2}), \nabla Im(u_{\rho_2}))$$

yields

$$\det(S_1, S_2, S_3) \quad = \quad r^3 e^{r(2x_2+x_3)}(-\cos(rx_1) + f_1(x))$$
$$\det(S_1, S_2, S_4) \quad = \quad r^3 e^{r(2x_2+x_3)}(-\sin(rx_1) + f_2(x))$$

where the size of the supports of f_1, f_2 tends to 0 as $r \to \infty$

Thus, choosing r large enough, we see that we can find Dirichlet data $u = Re(u_{\rho_i}),\ u = Im(u_{\rho_i})$ on $\partial\Omega$ and a partition of Ω into sets O_k, so that

$$\det(S_1, S_2, S_3) \ > \ c_0 \qquad \text{or} \qquad \det(S_1, S_2, S_4) \ > \ c_0 \quad \text{in } O_k$$

Conclusion

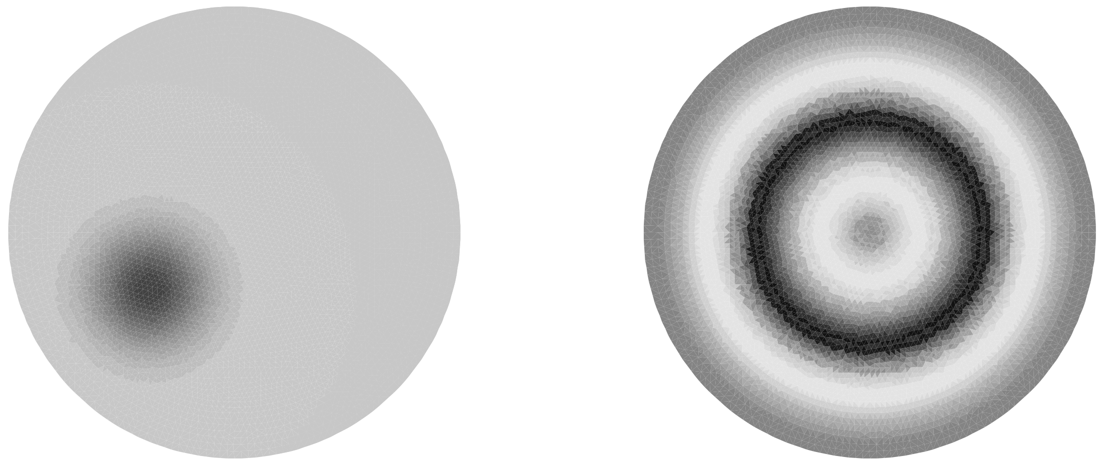
- Multiwave imaging provides internal data, which stabilizes the reconstruction (in numerical experiments)

- For US+EIT, the internal data is the elastic energy density. The feasibility of focusing ultrasound waves on regions of the order of mm's is not so clear however (but one can use plane waves, see the work of P. Kuchment and L. Kunyansky, G. Bal and J. Schotland).

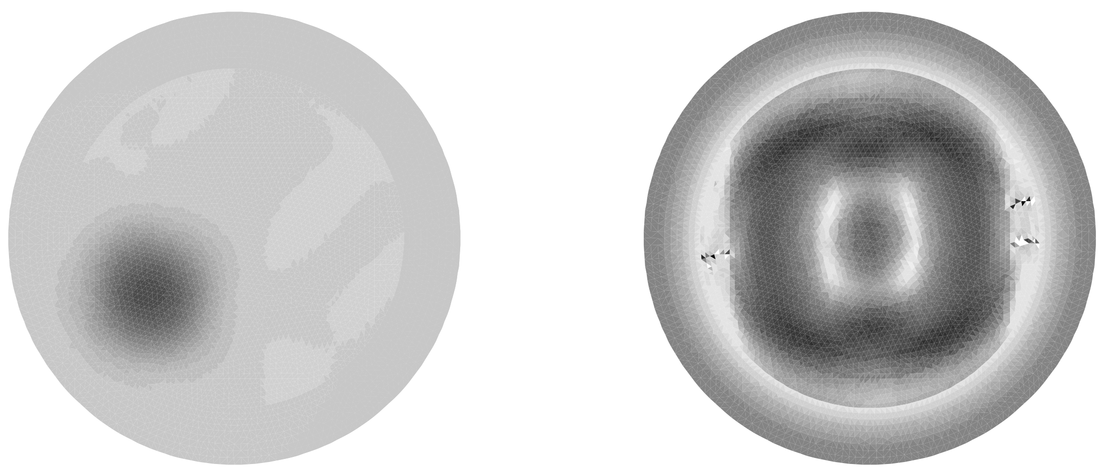
- For US+EIT in 2D, stability follows from an explicit reconstruction formula. The 3D situation is more complex.

- Extension to anisotropic medium ?

2 anisotropic conductivities with same Dirichlet to Neumann map



A_{11}, target conductivities



A_{11}, reconstructed