# Direct Electrical Impedance Tomography for Nonsmooth Conductivities

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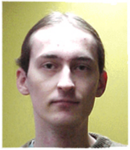

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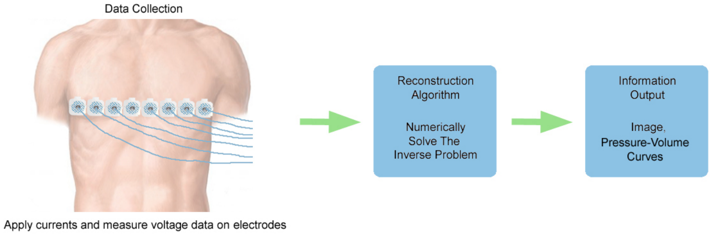
# The EIT Problem




Medical Applications in 2-D:

- Monitoring ventilation and perfusion in ARDS patients
- Detection of pneumothorax
- Diagnosis of pulmonary edema and pulmonary embolus

# The 2-D EIT Problem

Given a bounded domain $\Omega \in \mathbb{R}^2$, determine the conductivity $\sigma(z)$ where

$$\begin{aligned} \nabla \cdot (\sigma(z)\nabla u) &= 0 \text{ in } \Omega, \\ u &= f \text{ on } \partial\Omega \end{aligned}$$

from knowledge of the Dirichlet-to-Neumann map

$$\Lambda_\sigma f = \sigma \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega}$$

The approach presented here is based on the constructive proof of Astala and Päivärinta [Ann. of Math. **163** (2006)].

# CGO Solutions

The method is based on the existence of exponentially growing solutions to the conductivity and resistivity equations

$$\nabla \cdot (\sigma(z)\nabla u_1(z,k)) = 0, \quad u_1 \sim e^{ikz} \text{ when } |z| \to \infty$$

$$\nabla \cdot \left(\frac{1}{\sigma(z)}\nabla u_2(z,k)\right) = 0, \quad u_2 \sim ie^{ikz} \text{ when } |z| \to \infty$$

The exponential behaviour of the CGO solutions is used for nonlinear Fourier analysis for the inverse problem.

$k$ can be thought of as a frequency-domain variable.

# CGO Solutions

Defining

$$f_\mu(z,k) = u_1(z,k) + iu_2(z,k)$$

and

$$\mu(z) = \frac{1-\sigma(z)}{1+\sigma(z)}$$

one can show $f_\mu(z,k)$ satisfies the Beltrami equation

$$\bar\partial_z f_\mu = \mu \overline{\partial_z f_\mu}$$

and the solutions can be written as

$$f_\mu(z,k) = e^{ikz}(1+\omega(z,k)), \quad \text{with} \quad \omega(z,k) = \mathcal{O}\left(\frac{1}{z}\right) \text{ as } |z| \to \infty.$$

# Computing CGO Solutions (FP)

A computation shows that $\omega$ satisfies the equation

$$\bar{\partial}_z \omega - \nu \overline{\partial_z \omega} - \alpha \overline{\omega} - \alpha = 0. \tag{1}$$

Here $e_k(z) := \exp(i(kz + \overline{k}\overline{z}))$ and

$$\begin{aligned} \nu(z,k) &\equiv e_{-k}(z)\mu(z), & (2) \\ \alpha(z,k) &\equiv -i\overline{k}e_{-k}(z)\mu(z), & (3) \end{aligned}$$

We will make a substitution, defining $u \in L^p(\Omega)$ such that

$$\overline{u} = -\bar{\partial}_z \omega.$$

# Computing CGO Solutions (FP)

Then $\omega = -P\overline{u}$ and $\partial\omega = -S\overline{u}$, where

$$Pf(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{f(\lambda)}{\lambda - z} dm(\lambda), \quad Sg(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{g(\lambda)}{(\lambda - z)^2} dm(\lambda).$$

Then (1) becomes

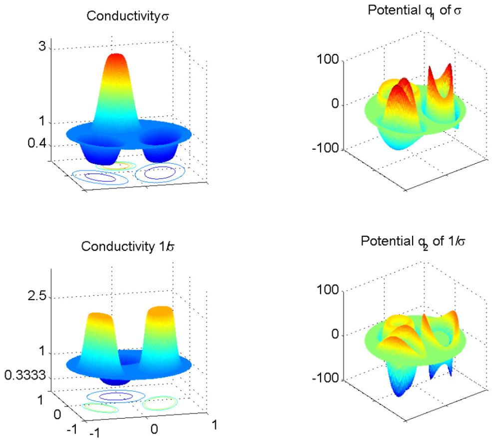
$$u + (-\overline{\nu}S - \overline{\alpha}P)\overline{u} = -\overline{\alpha}$$

or

$$(I + A\rho)u = -\overline{\alpha}, \quad \text{with} \quad \rho f = \overline{f} \quad \text{and} \quad A = -\overline{\nu}S - \overline{\alpha}P$$

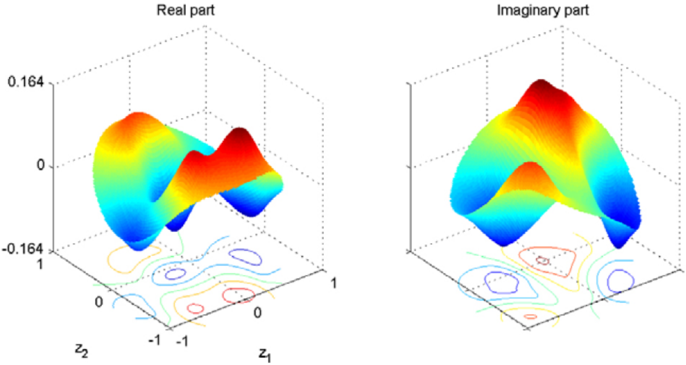
This is equation is then discretized and solved with GRMES using a preconditioner.

# An Example for the CGO Solutions



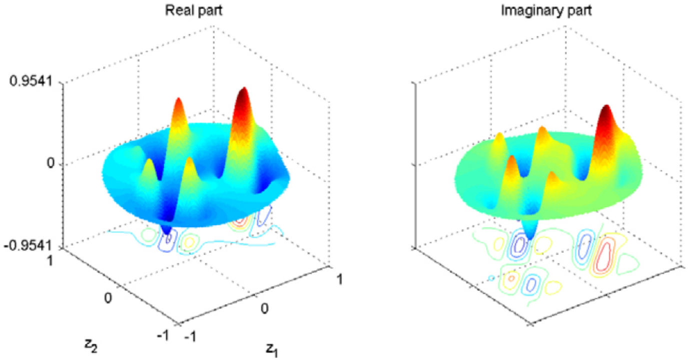

An example conductivity and corresponding resistivity

# Computing CGO Solutions




Real and imaginary parts of $\omega(z, 1)$

# Computing CGO Solutions




Real and imaginary parts of $\omega(z, -4.9497 - 4.9497i)$

# Overview of the reconstruction algorithm

The reconstruction procedure consists of these three steps:

(i) **Recover traces of CGO solutions at the boundary $\partial\Omega$ from the DN map by solving a boundary integral equation given by Astala and Päivärinta.**

(ii) **Compute approximate values of CGO solutions inside the unit disc using the low-pass transport matrix.**

(iii) **Reconstruct the conductivity.** The approximate conductivity is computed from the recovered values of the CGO solutions inside $\Omega$ using differentiation and simple algebra.

# A Boundary Integral Formula for CGO Solutions

Defining

$$M_\mu(z,k) = 1 + \omega(z,k),$$

the following boundary integral equation holds:

$$M_\mu(\,\cdot\,,k)|_{\partial\Omega} + 1 = (\mathcal{P}_\mu^k + \mathcal{P}_0)M_\mu(\,\cdot\,,k)|_{\partial\Omega}, \qquad (4)$$

where $\mathcal{P}_\mu^k$ and $\mathcal{P}_0$ are projection operators to be discussed.

Numerical solution of (4) is done by

- writing real and imaginary parts separately
- replacing all the operators by their $(4N+2) \times (4N+2)$ matrix approximations
- solving the resulting finite linear system for $|k| \leq R$ where $R > 0$ depends on the noise level.

# The Hilbert Transform

Define the $\mu - \mathit{Hilbert}$ transform $\mathcal{H}_\mu : H^{1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$ by

$$\mathcal{H}_\mu : u_1|_{\partial\Omega} \longrightarrow u_2|_{\partial\Omega}$$

To extend this to complex-valued functions in $H^{1/2}(\partial\Omega)$, define

$$\mathcal{H}_\mu(iu) = i\mathcal{H}_{-\mu}(u)$$

Theorem [AP]: The Dirichlet-to-Neumann map $\Lambda_\sigma$ uniquely determines $\mathcal{H}_\mu$, $\mathcal{H}_{-\mu}$, and $\Lambda_{\sigma^{-1}}$.

# The Hilbert Transform

From the proof, in the weak sense for real-valued $g \in H^{1/2}(\partial\Omega)$

$$\partial_T \mathcal{H}_\mu g = \Lambda_\sigma g$$

where $\partial_T$ is the tangential derivative map along the boundary. It can be approximated in the trig basis by the matrix $D_T$:

$$D_T = \begin{bmatrix} 0 & 1 & & & & & \\ -1 & 0 & & & & & \\ & & 0 & 2 & & & \\ & & -2 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & N \\ & & & & & -N & 0 \end{bmatrix}. \quad (5)$$

# The Projection Maps

Define an averaging operator

$$\mathcal{L}\phi := |\partial\Omega|^{-1} \int_{\partial\Omega} \phi \, ds.$$

The operator $\mathcal{P}_\mu : H^{1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$ is defined by

$$\mathcal{P}_\mu g = \frac{1}{2}(I + i\mathcal{H}_\mu)g + \frac{1}{2}\mathcal{L}g,$$

where $g$ may be complex-valued. Further, denote

$$\mathcal{P}_\mu^k g := e^{-ikz}\mathcal{P}_\mu(e^{ikz}g)$$

# Boundary Data

For $n = 1, \ldots, 2N$, define a set of trigonometric basis functions:

$$\phi_n(\theta) = \begin{cases} \pi^{-1/2} \cos\left((n+1)\theta/2\right), & \text{for odd } n, \\ \pi^{-1/2} \sin\left(n\theta/2\right), & \text{for even } n. \end{cases}$$

Any function $g \in L^2(\partial\Omega)$ representing current density on the boundary can then be approximated by

$$g(\theta) \approx \sum_{n=1}^{2N} \langle g, \phi_n \rangle \, \phi_n(\theta),$$

where the inner product is defined for real-valued functions $f, g \in L^2(\partial\Omega)$ by

$$\langle f, g \rangle := \int_0^{2\pi} f(\theta) g(\theta) \, d\theta.$$

# Boundary Data

Now define the $2N \times 2N$ matrix approximation $[R_{mn}]$ to the ND map by

$$R_{mn} = \langle u_n|_{\partial\Omega}, \phi_m \rangle$$

where $u_n|_{\partial\Omega}$ is the solution to the Neumann problem with $g = \phi_n$. Define

$$\widetilde{L}_\sigma := [R_{mn}]^{-1};$$

Now we can approximate $\mathcal{H}_\mu$ acting on real-valued, zero-mean functions expanded in the trig basis by

$$\widetilde{H}_\mu := D_T^{-1} L_\sigma.$$

# Step 2: The Low-Pass Transport Matrix

Useful Facts:

- The function $f_\mu$ is harmonic outside the unit disc since $\mu$ is supported inside $\Omega$.
- We know the trace of $f_\mu$ on $\partial\Omega$.
- Thus, the Fourier coefficients of $f_\mu$ can be used to expand $f_\mu$ as a power series outside $\Omega$.

The transport matrix is the matrix in a $2 \times 2$ linear system that connects the CGO solutions inside $\Omega$ to their values outside $\Omega$.

# Step 2: The Low-Pass Transport Matrix

For any $z_0 \in \mathbb{R}^2 \setminus \bar{\Omega}$, set $\nu_{z_0}^{(R)}(k) = 0$ if $|k| \geq R$, and if $|k| < R$

$$\nu_{z_0}^{(R)}(k) := \frac{-f_\mu(z_0,k) + \overline{f_{-\mu}}(z_0,k)}{f_\mu(z_0,k) + f_{-\mu}(z_0,k)}$$

We solve the truncated Beltrami equations

$$\begin{aligned} \bar{\partial}_k \alpha^{(R)} &= \nu_{z_0}^{(R)}(k)\,\overline{\partial_k \alpha^{(R)}}, \quad \alpha^{(R)}(z,z_0,k) = e^{ik(z-z_0)+k\phi(k)}, \\ \bar{\partial}_k \beta^{(R)} &= \nu_{z_0}^{(R)}(k)\,\overline{\partial_k \beta^{(R)}}, \quad \beta^{(R)}(z,z_0,k) = ie^{ik(z-z_0)+k\widetilde{\phi}(k)}, \end{aligned}$$

where $\phi(k) \to 0$ as $k \to \infty$. We also have the conditions

$$\alpha^{(R)}(z,z_0,0) = 1 \text{ and } \beta^{(R)}(z,z_0,0) = i.$$

# Step 2: The Low-Pass Transport Matrix

For any $z_0 \in \mathbb{R}^2 \setminus \bar{\Omega}$, set $\nu_{z_0}^{(R)}(k) = 0$ if $|k| \geq R$, and if $|k| < R$

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where $\phi(k) \to 0$ as $k \to \infty$. We also have the conditions

$$\alpha^{(R)}(z,z_0,0) = 1 \;\text{ and }\; \beta^{(R)}(z,z_0,0) = i.$$

# Step 2: The Low-Pass Transport Matrix

Fix any nonzero $k_0 \in \mathbb{C}$ and choose any point $z$ inside the unit disc. We can now use the approximate transport matrix

$$T^{(R)} = T^{(R)}_{z,z_0,k_0} := \begin{pmatrix} a_1^{(R)} & a_2^{(R)} \\ b_1^{(R)} & b_2^{(R)} \end{pmatrix} \qquad (6)$$

to compute

$$\begin{aligned} u_1^{(R)}(z,k_0) &= a_1^{(R)} u_1(z_0,k_0) + a_2^{(R)} u_2(z_0,k_0), \\ u_2^{(R)}(z,k_0) &= b_1^{(R)} u_1(z_0,k_0) + b_2^{(R)} u_2(z_0,k_0), \end{aligned} \qquad (7)$$

where $\alpha^{(R)} = a_1^{(R)} + i a_2^{(R)}$ and $\beta^{(R)} = b_1^{(R)} + i b_2^{(R)}$.

# Step 3: Reconstructing the Conductivity

From [AP] for any fixed $k_0 \in \mathbb{C}$, $\mu(z)$ is related to $f_\mu$ by

$$\mu(z) = \frac{\bar{\partial} f_\mu(z, k_0)}{\overline{\partial f_\mu}(z, k_0)}.$$

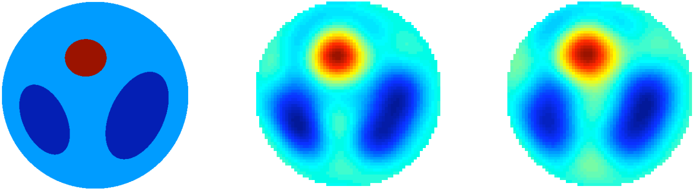
Thus, the conductivity $\sigma(z)$ can be recovered by

$$\sigma(z) = -i \frac{\bar{\partial}(\Im f)}{\bar{\partial}(\Re f)}$$

This can be computed independently for each $z$ in the ROI.

# Example: Heart-and-Lungs Phantom

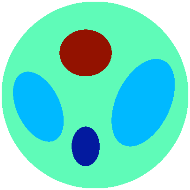
| Ideal Conductivity | | From ideal data | From 0.01% noise |
|---|---|---|---|
| Background : | 1.0 | $\sim 1.2$ | $\sim 1.2$ |
| Lungs : | 0.7 | 0.637 | 0.637 |
| Heart : | 2.0 | 1.997 | 1.870 |

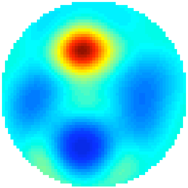


| | | |
|---|---|---|
| Relative error: | 11.6% | 12.7% |
| Dynamic range: | 105% | 95% |

# Example: Heart-and-Lungs + Spine

| Ideal Conductivity | | From ideal data |
|---|---|---|
| Background : | 1.0 | $\sim$ 1.1 |
| Spine : | 0.2 | 0.373 (*min*) |
| Lungs : | 0.7 | $\sim$ 0.7 |
| Heart : | 2.0 | 2.273 (*max*) |

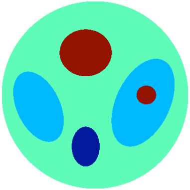


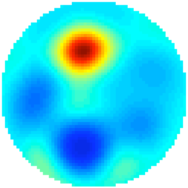


Relative error: 16.3%
Dynamic range: 106%

# Example: Heart-and-Lungs + Spine + Tumor

| Ideal Conductivity | From ideal data |
|---|---|
| Background : 1.0 | $\sim$ 1.1 |
| Spine : 0.2 | 0.378 (*min*) |
| Lungs : 0.7 | $\sim$ 0.7 |
| Heart : 2.0 | 2.332 (*max*) |

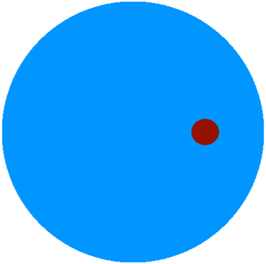




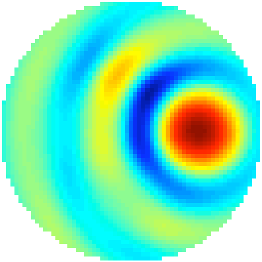
| | |
|---|---|
| Relative error: | 16.7% |
| Dynamic range: | 109% |

# Subtract the previous images:

Ideal Difference

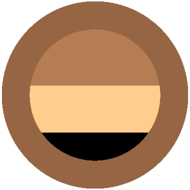
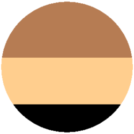
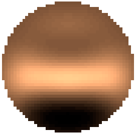


Reconstruction



# Example 3: Layered Medium

Top Layer: 1.2 Middle Layer: 2.0 Bottom Layer: 0.3

| Ideal | Reconstruction | Relative error |
|---|---|---|
|  |  | 21.7% |
|  |  | 24.7% |

Dynamic range: 134%