Resistor Networks and Optimal Grids for Electrical Impedance Tomography with Partial Boundary Measurements

Alexander Mamonov[1],
Liliana Borcea[2], Vladimir Druskin[3],
Fernando Guevara Vasquez[4]

[1]Institute for Computational Engineering and Sciences, University of Texas at Austin,
[2]Rice University, [3]Schlumberger-Doll Research, [4]University of Utah

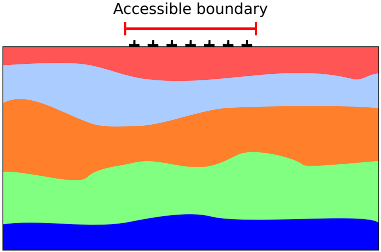
June 20, 2011

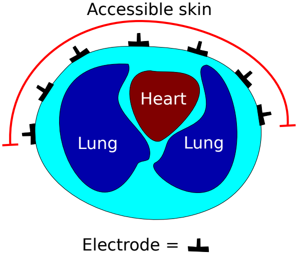
Support: Schlumberger R62860, NSF DMS-0604008, DMS-0934594.

Electrical Impedance Tomography

Electrical Impedance Tomography: Physical problem

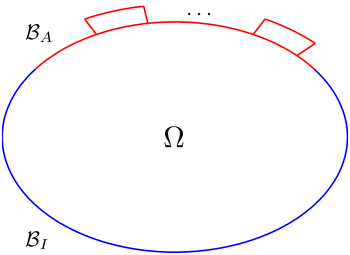
- Physical problem: determine the electrical conductivity inside an object from the simultaneous measurements of voltages and currents on (a part of) its boundary
- Applications:
  - Original: geophysical prospection
  - More recent: medical imaging
- Both cases in practice have measurements restricted to a part of object's boundary





Figures: Fernando Guevara Vasquez

Partial data EIT: mathematical formulation



- Two-dimensional problem $\Omega \subset \mathbb{R}^2$
- Equation for electric potential u

$$\nabla \cdot (\sigma \nabla u) = 0, \quad \text{in } \Omega$$

- Dirichlet data $u|_{\mathcal{B}} = \phi \in H^{1/2}(\mathcal{B})$ on $\mathcal{B} = \partial\Omega$
- Dirichlet-to-Neumann (DtN) map $\Lambda_\sigma : H^{1/2}(\mathcal{B}) \to H^{-1/2}(\mathcal{B})$

$$\Lambda_\sigma \phi = \sigma \frac{\partial u}{\partial \nu}\bigg|_{\mathcal{B}}$$

Partial data case:

- Split the boundary $\mathcal{B} = \mathcal{B}_A \cup \mathcal{B}_I$, accessible $\mathcal{B}_A$, inaccessible $\mathcal{B}_I$
- Dirichlet data: $\text{supp}\phi_A \subset \mathcal{B}_A$
- Measured current flux: $J_A = (\Lambda_\sigma \phi_A)|_{\mathcal{B}_A}$
- Partial data EIT: find σ given all pairs (ϕ_A, J_A)

Existence, uniqueness and stability

Existence and uniqueness:

- Full data: solved completely for any positive $\sigma \in L^\infty(\Omega)$ in 2D (Astala, Päivärinta, 2006)
- Partial data: for $\sigma \in C^{4+\alpha}(\overline{\Omega})$ and an arbitrary open $\mathcal{B}_A$ (Imanuvilov, Uhlmann, Yamamoto, 2010)

Stability (full data):

- For $\sigma \in L^\infty(\Omega)$ the problem is unstable (Alessandrini, 1988)
- Logarithmic stability estimates (Barcelo, Faraco, Ruiz, 2007) under certain regularity assumptions

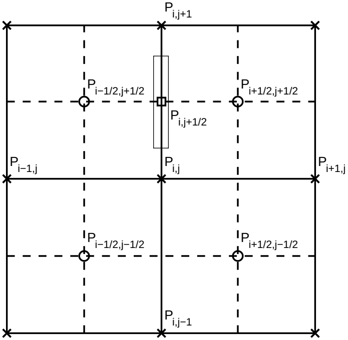
$$\|\sigma_1 - \sigma_2\|_\infty \leq C \left|\log \|\Lambda_{\sigma_1} - \Lambda_{\sigma_2}\|_{H^{1/2}(\mathcal{B}) \to H^{-1/2}(\mathcal{B})}\right|^{-a}$$

- The estimate is sharp (Mandache, 2001), additional regularity of σ does not help
- Exponential ill-conditioning of the discretized problem
- Resolution is severely limited by the noise, regularization is required

Numerical methods for EIT

1. Linearization: Calderon's method, one-step Newton, backprojection.
2. Optimization: typically output least squares with regularization.
3. Layer peeling: find σ close to $\mathcal{B}$, peel the layer, update Λ_σ, repeat.
4. D-bar method: non-trivial implementation.
5. **Resistor networks and optimal grids**
    - Uses the close connection between the continuum inverse problem and its discrete analogue for resistor networks
    - Fit the measured continuum data exactly with a resistor network
    - Interpret the resistances as averages over a special (optimal) grid
    - Compute the grid once for a known conductivity (constant)
    - Optimal grid depends weakly on the conductivity, grid for constant conductivity can be used for a wide range of conductivities
    - Obtain a pointwise reconstruction on an optimal grid
    - Use the network and the optimal grid as a non-linear preconditioner to improve the reconstruction using a single step of traditional (regularized) Gauss-Newton iteration

Finite volume discretization and resistor networks

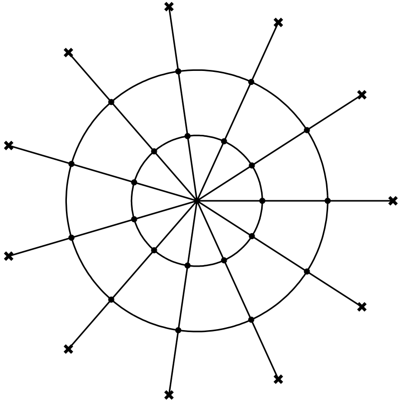


$$\gamma^{(1)}_{i,j+1/2} = \frac{L(P_{i+1/2,j+1/2}, P_{i-1/2,j+1/2})}{L(P_{i,j+1}, P_{i,j})}$$

$$\gamma_{i,j+1/2} = \sigma(P_{i,j+1/2})\gamma^{(1)}_{i,j+1/2}$$

- Finite volume discretization, staggered grid
- Kirchhoff matrix $K = A\,\text{diag}(\gamma)A^T \succeq 0$
- Interior I, boundary B, $|B| = n$
- Potential u is γ-harmonic $K_{I,:}u = 0$, $u_B = \phi$
- Discrete DtN map $\Lambda_\gamma \in \mathbb{R}^{n\times n}$
- Schur complement: $\Lambda_\gamma = K_{BB} - K_{BI}K_{II}^{-1}K_{IB}$
- **Discrete inverse problem:** knowing Λ_γ, A, find γ
- What network topologies are good?

Discrete inverse problem: circular planar graphs



C(5, 11)

- Planar graph Γ
- I embedded in the unit disk $\mathbb{D}$
- B in cyclic order on $\partial\mathbb{D}$

- Circular pair $(P; Q)$, $P \subset B$, $Q \subset B$
- $\pi(\Gamma)$ all $(P; Q)$ connected through Γ by disjoint paths
- **Critical** Γ: removal of any edge breaks some connection in $\pi(\Gamma)$
- Uniquely recoverable from Λ iff Γ is critical (Curtis, Ingerman, Morrow, 1998)
- Characterization of DtN maps of critical networks Λ_γ
  - Symmetry $\Lambda_\gamma = \Lambda_\gamma^T$
  - Conservation of current $\Lambda_\gamma \mathbf{1} = \mathbf{0}$
  - Total non-positivity $\det[-\Lambda_\gamma(P; Q)] \geq 0$

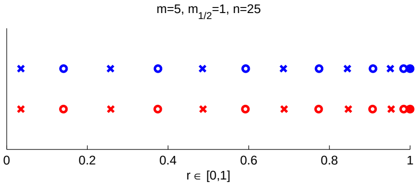
Discrete vs. continuum

- Measurement (electrode) functions χ_j, supp$\chi_j \subset \mathcal{B}_A$
- Measurement matrix $\mathcal{M}_n(\Lambda_\sigma) \in \mathbb{R}^{n \times n}$: $[\mathcal{M}_n(\Lambda_\sigma)]_{i,j} = \int\limits_{\mathcal{B}} \chi_i \Lambda_\sigma \chi_j dS$, $i \neq j$
- $\mathcal{M}_n(\Lambda_\sigma)$ has the properties of a DtN map of a resistor network (Morrow, Ingerman, 1998)
- How to interpret γ obtained from $\Lambda_\gamma = \mathcal{M}_n(\Lambda_\sigma)$?
- From finite volumes define the **reconstruction** mapping

  $\mathcal{Q}_n[\Lambda_\gamma]: \ \sigma^\star(P_{\alpha,\beta}) = \dfrac{\gamma_{\alpha,\beta}}{\gamma^{(1)}_{\alpha,\beta}}$, piecewise linear interpolation away from $P_{\alpha,\beta}$
- **Optimal grid** nodes $P_{\alpha,\beta}$ are obtained from $\gamma^{(1)}_{\alpha,\beta}$, a solution of the discrete problem for constant conductivity $\Lambda_{\gamma^{(1)}} = \mathcal{M}_n(\Lambda_1)$.
- The reconstruction is improved using a single step of preconditioned Gauss-Newton iteration with an initial guess $\sigma^\star$

$$\min_\sigma \| \mathcal{Q}_n[\mathcal{M}_n(\Lambda_\sigma)] - \sigma^\star \|$$

Optimal grids in the unit disk: full data



- Tensor product grids
  uniform in θ, adaptive in r
- Layered conductivity $\sigma = \sigma(r)$
- Admittance $\Lambda_\sigma e^{ik\theta} = R(k) e^{ik\theta}$
- For $\sigma \equiv 1$ $R(k) = |k|$,
  $\Lambda_1 = \sqrt{-\frac{\partial^2}{\partial \theta^2}}$
- Discrete analogue
  $\mathcal{M}_n(\Lambda_1) = \sqrt{\text{circ}(-1, 2, -1)}$

- Discrete admittance $R_n(\lambda) =$
  $$\cfrac{1}{\cfrac{1}{\gamma_1} + \cfrac{1}{\widehat{\gamma}_2 \lambda^2 + \ldots + \cfrac{1}{\widehat{\gamma}_{m+1} \lambda^2 + \gamma_{m+1}}}}$$
- Rational interpolation
  $$R(k) = \frac{k}{\omega_k^{(n)}} R_n(\omega_k^{(n)})$$
- Optimal grid $R_n^{(1)}(\omega_k^{(n)}) = \omega_k^{(n)}$
- Closed form solution available (Biesel, Ingerman, Morrow, Shore, 2008)
- Vandermonde-like system, exponential ill-conditioning

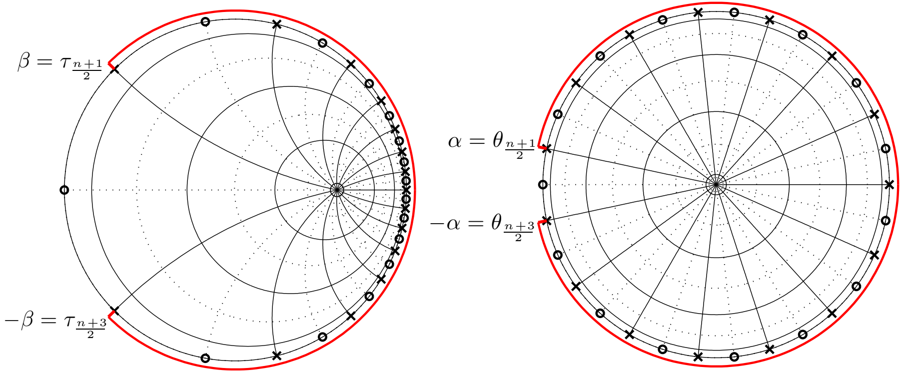
Transformation of the EIT under diffeomorphisms

- Optimal grids were used successfully to solve the full data EIT in $\mathbb{D}$
- Can we reduce the partial data problem to the full data case?
- Conductivity under diffeomorphisms G of Ω: **push forward** $\widetilde{\sigma} = G_*(\sigma)$, $\widetilde{u}(x) = u(G^{-1}(x))$,

$$\widetilde{\sigma}(x) = \left. \frac{G'(y)\sigma(y)(G'(y))^T}{|\det G'(y)|} \right|_{y=G^{-1}(x)}$$

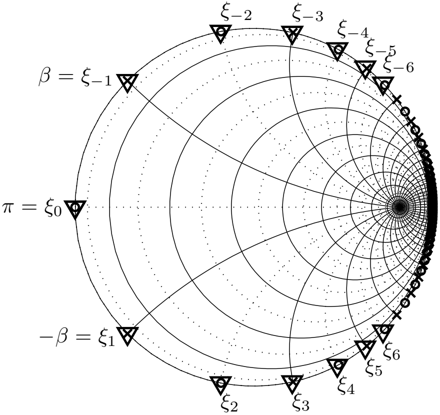
- Matrix valued $\widetilde{\sigma}(x)$, anisotropy!
- Anisotropic EIT is not uniquely solvable
- Push forward for the DtN: $(g_*\Lambda_\sigma)\phi = \Lambda_\sigma(\phi \circ g)$, where $g = G|_{\mathcal{B}}$
- Invariance of the DtN: $g_*\Lambda_\sigma = \Lambda_{G_*\sigma}$
- Push forward, solve the EIT for $g_*\Lambda_\sigma$, pull back
- Must preserve isotropy, $G'(y)(G'(y))^T = I \Rightarrow$ **conformal** G
- Conformal automorphisms of the unit disk are Möbius transforms

Conformal automorphisms of the unit disk



$F : \theta \to \tau$, $G : \tau \to \theta$. Primary $\times$, dual $\circ$, $n = 13$, $\beta = 3\pi/4$.
Positions of point-like electrodes prescribed by the mapping.

Conformal mapping grids: limiting behavior



Primary $\times$, dual $\circ$, limits $\triangledown$,
$n = 37$, $\beta = 3\pi/4$.

- No conformal limiting mapping
- Single pole moves towards $\partial\mathbb{D}$ as $n \to \infty$
- Accumulation around $\tau = 0$
- No asymptotic refinement in angle as $n \to \infty$
- Hopeless?
- Resolution bounded by the instability, $n \to \infty$ practically unachievable

Quasi-conformal mappings

- Conformal w, Cauchy-Riemann: $\frac{\partial w}{\partial \overline{z}} = 0$, how to relax?
- Quasi-conformal w, Beltrami: $\frac{\partial w}{\partial \overline{z}} = \mu(z) \frac{\partial w}{\partial z}$
- Push forward $w_*(\sigma)$ is no longer isotropic
- Anisotropy of $\widetilde{\sigma} \in \mathbb{R}^{2 \times 2}$ is $\kappa(\widetilde{\sigma}, z) = \frac{\sqrt{L(z)} - 1}{\sqrt{L(z)} + 1}$, $L(z) = \frac{\lambda_1(z)}{\lambda_2(z)}$

Lemma

Anisotropy of the push forward is given by $\kappa(w_*(\sigma), z) = |\mu(z)|$.

- Mappings with fixed values at $\mathcal{B}$ and min $\|\mu\|_\infty$ are **extremal**
- Extremal mappings are Teichmüller (Strebel, 1972)

$\mu(z) = \|\mu\|_\infty \frac{\overline{\phi(z)}}{|\phi(z)|}$, ϕ holomorphic in Ω

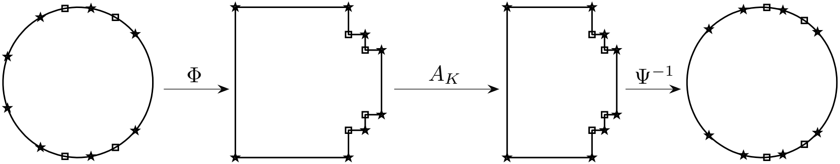
Computing the extremal quasi-conformal mappings

- Polygonal Teichmüller mappings
- Polygon is a unit disk with N marked points on the boundary circle
- Can be decomposed as

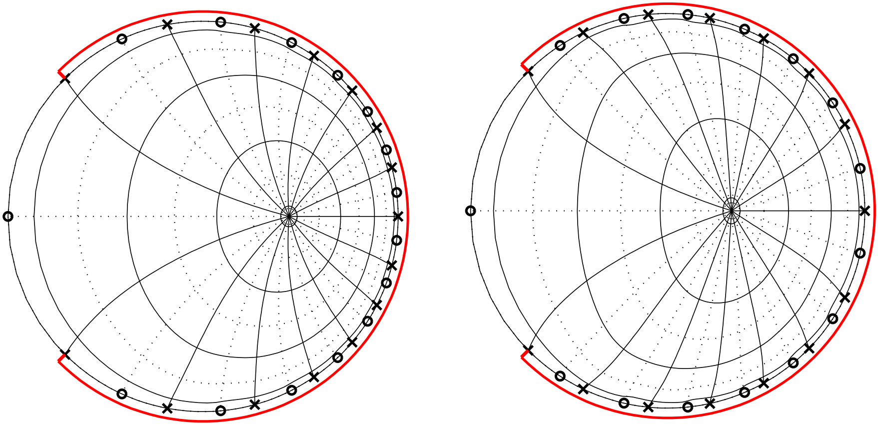
$$W = \Psi^{-1} \circ A_K \circ \Phi,$$

where $\Psi = \int \sqrt{\psi(z)}dz$, $\Phi = \int \sqrt{\phi(z)}dz$, A_K - constant affine stretching

- ϕ, ψ are rational with poles and zeros of order one on $\partial\mathbb{D}$
- Recall Schwarz-Christoffel $s(z) = a + b \int^z \prod_{k=1}^{N} \left(1 - \frac{\zeta}{z_k}\right)^{\alpha_k - 1} d\zeta$
- Ψ, Φ are Schwarz-Christoffel mappings to rectangular polygons

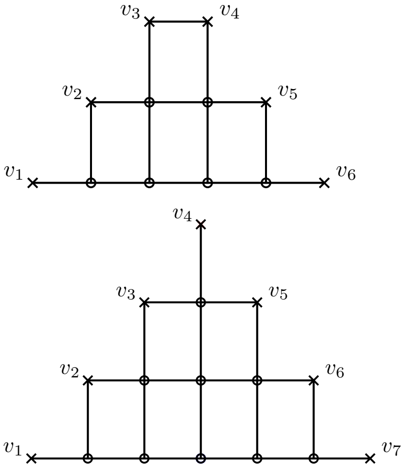


Polygonal Teichmüller mapping: the grids



The optimal grid with $n = 15$ under the Teichmüller mappings.
Left: $K = 0.8$; right: $K = 0.66$.

EIT with pyramidal networks: motivation



- Pyramidal (standard) graphs Σ_n
- Topology of a network accounts for the inaccessible boundary
- Criticality and reconstruction algorithm proved for pyramidal networks
- How to obtain the grids?
- Grids have to be purely 2D (no tensor product)
- Use the sensitivity analysis (discrete an continuum) to obtain the grids
- General approach works for any simply connected domain

Special solutions and recovery

Theorem

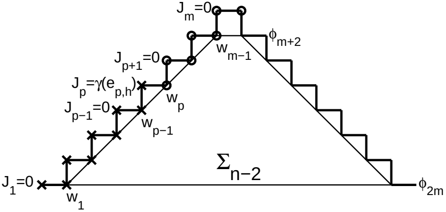
Pyramidal network (Σ_n, γ), $n = 2m$ is uniquely recoverable from its DtN map $\Lambda^{(n)}$ using the layer peeling algorithm. Conductances are computed with

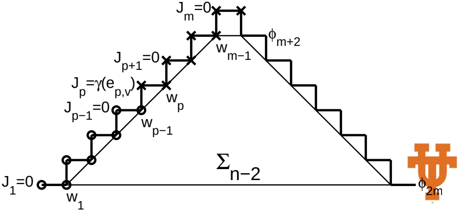
$$\gamma(e_{p,h}) = \left(\Lambda_{p,E(p,h)} + \Lambda_{p,C}\, \Lambda_{Z,C}^{-1}\, \Lambda_{Z,E(p,h)}\right) \mathbf{1}_{E(p,h)},$$
$$\gamma(e_{p,v}) = \left(\Lambda_{p,E(p,v)} + \Lambda_{p,C}\, \Lambda_{Z,C}^{-1}\, \Lambda_{Z,E(p,v)}\right) \mathbf{1}_{E(p,v)}.$$

The DtN map is updated using

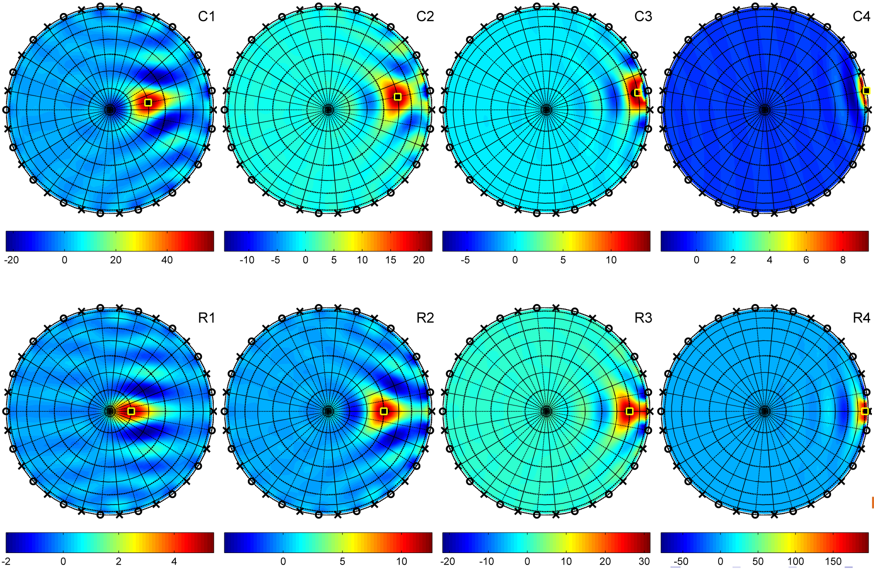
$$\Lambda^{(n-2)} = -K_S - K_{SB}\, P^T \left(P\,(\Lambda^{(n)} - K_{BB})\, P^T\right)^{-1} P\, K_{BS}.$$

The formulas are applied recursively to $\Sigma_n, \Sigma_{n-2}, \ldots, \Sigma_2$.

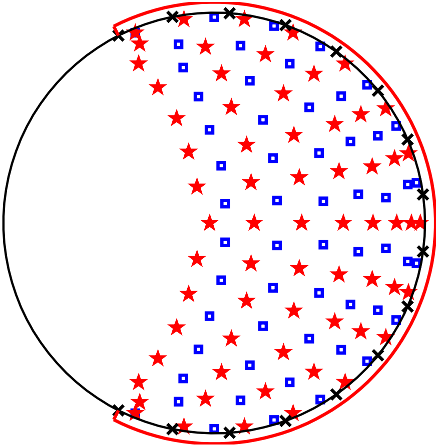




Sensitivity grids: motivation



Sensitivity grids



Sensitivity grid, $n = 16$.

- Proposed by F. Guevara Vasquez
- Sensitivity functions

$$\frac{\delta \gamma_{\alpha,\beta}}{\delta \sigma} = \left[\left(\frac{\partial \Lambda_\gamma}{\partial \gamma} \right)^{-1} \mathcal{M}_n \left(\frac{\delta \Lambda_\sigma}{\delta \sigma} \right) \right]_{\alpha,\beta}$$

where $\Lambda_\gamma = \mathcal{M}_n(\Lambda_\sigma)$

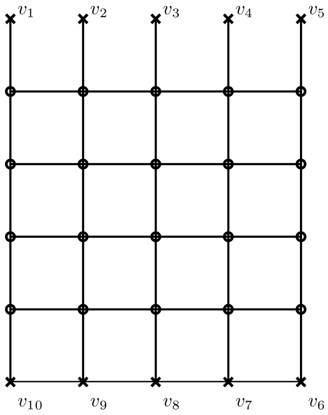
- The optimal grid nodes $P_{\alpha,\beta}$ are roughly

$$P_{\alpha,\beta} \approx \arg\max_{x \in \Omega} \frac{\delta \gamma_{\alpha,\beta}}{\delta \sigma}(x)$$

- Works for any domain and any network topology!

Two sided problem and networks

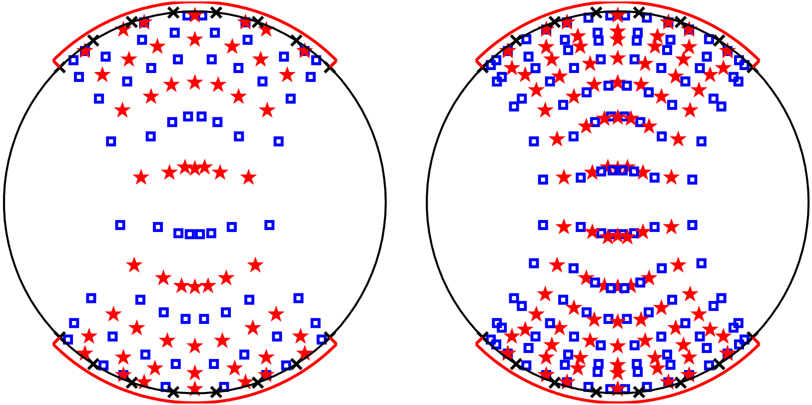
Two-sided problem: $\mathcal{B}_A$ consists of two disjoint segments of the boundary. Example: cross-well measurements.



- Two-sided optimal grid problem is known to be irreducible to 1D (Druskin, Moskow)
- Special choice of topology is needed
- Network with a *two-sided* graph T_n is proposed (left: $n = 10$)
- Network with graph T_n is critical and well-connected
- Can be recovered with layer peeling
- Grids are computed using the sensitivity analysis exactly like in the pyramidal case

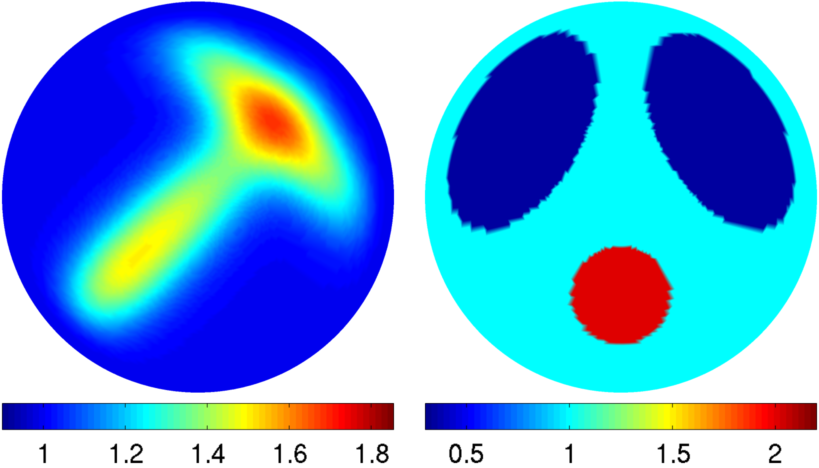
Sensitivity grids for the two-sided problem

Two-sided graph T_n lacks the top-down symmetry. Resolution can be doubled by also fitting the data with a network turned upside-down.



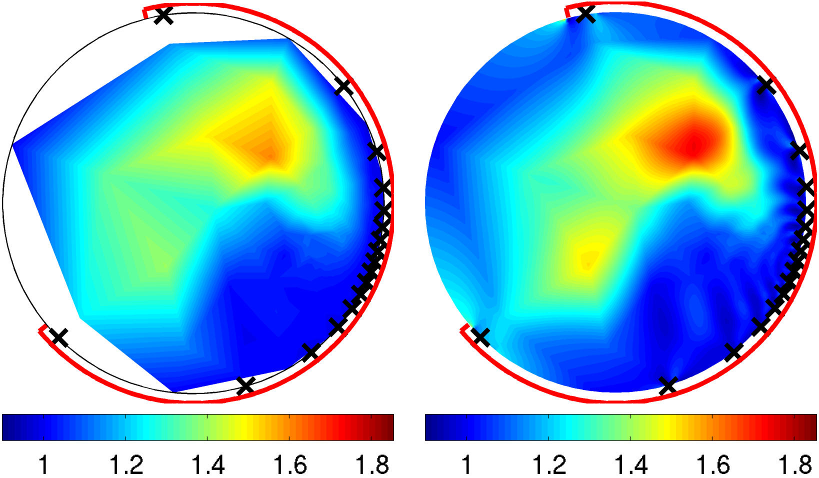
Left: single optimal grid; right: double resolution grid; $n = 16$.

Numerical results: test conductivities



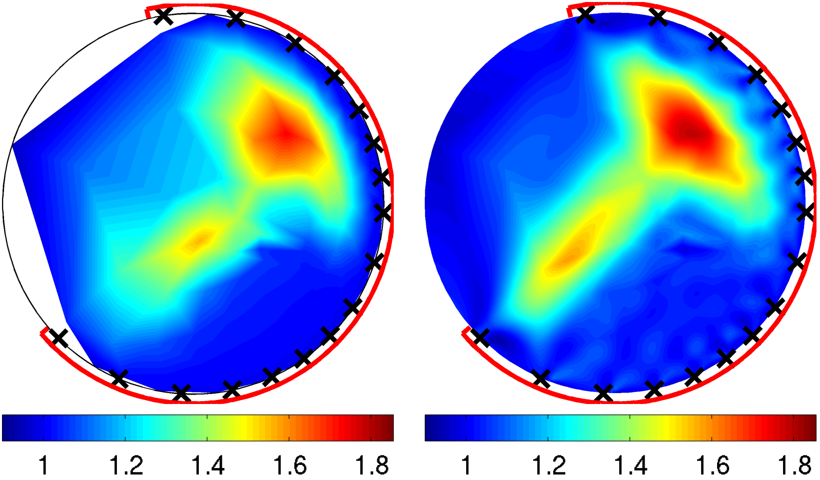
Left: smooth; right: piecewise constant chest phantom.

Numerical results: smooth σ + conformal



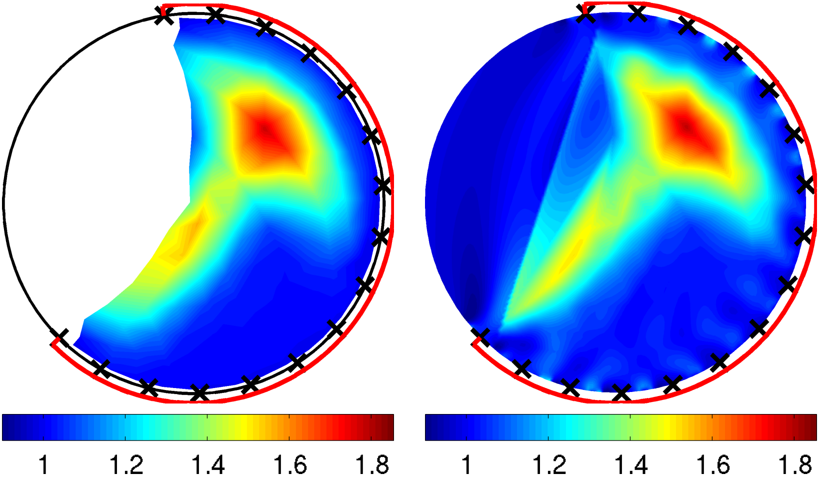
Left: piecewise linear; right: one step Gauss-Newton, $\beta = 0.65\pi$, $n = 17$, $\omega_0 = -\pi/10$.

Numerical results: smooth σ + quasiconformal



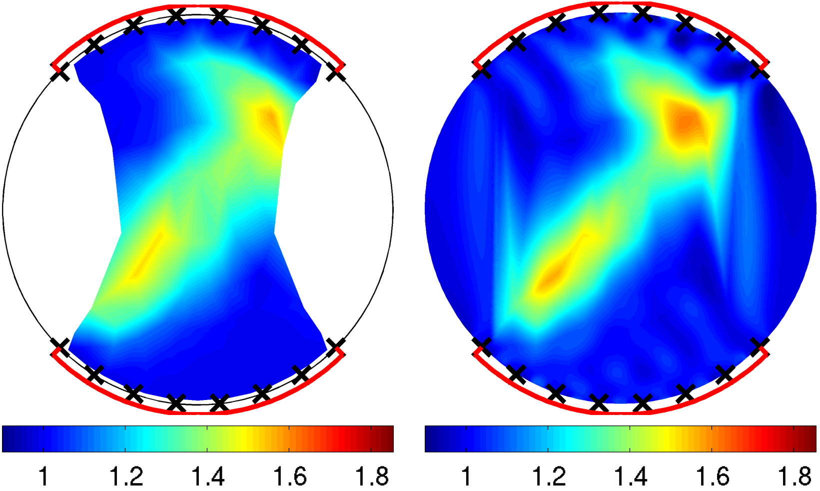
Left: piecewise linear; right: one step Gauss-Newton, $\beta = 0.65\pi$, $K = 0.65$, $n = 17$, $\omega_0 = -\pi/10$.

Numerical results: smooth σ + pyramidal



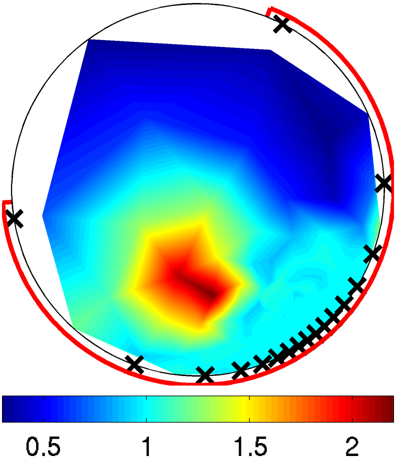
Left: piecewise linear; right: one step Gauss-Newton,
$\beta = 0.65\pi$, $n = 16$, $\omega_0 = -\pi/10$.

Numerical results: smooth σ + two-sided



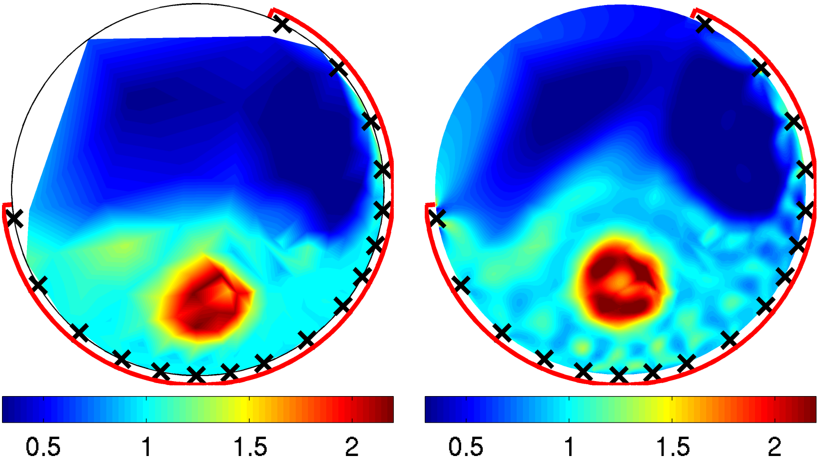
Left: piecewise linear; right: one step Gauss-Newton, $n = 16$, $\mathcal{B}_A$ is solid red.

Numerical results: piecewise constant σ + conformal



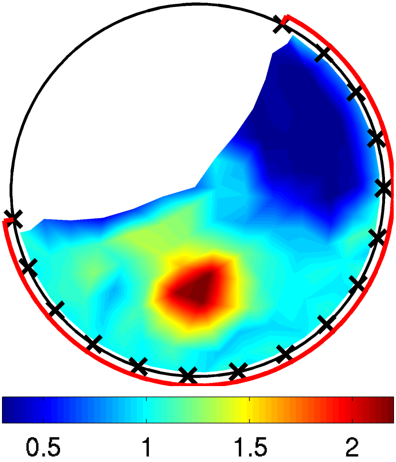
Left: piecewise linear; right: one step Gauss-Newton, $\beta = 0.65\pi$, $n = 17$, $\omega_0 = -3\pi/10$.

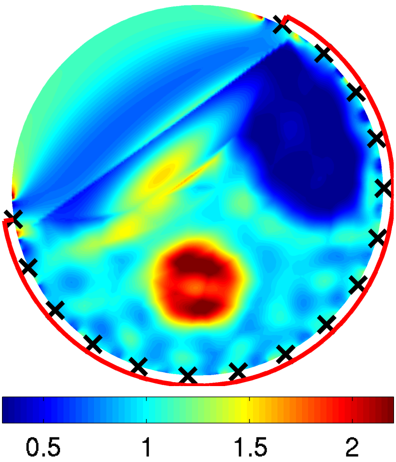
Numerical results: piecewise constant σ + quasiconf.



Left: piecewise linear; right: one step Gauss-Newton,
$\beta = 0.65\pi$, $K = 0.65$, $n = 17$, $\omega_0 = -3\pi/10$.

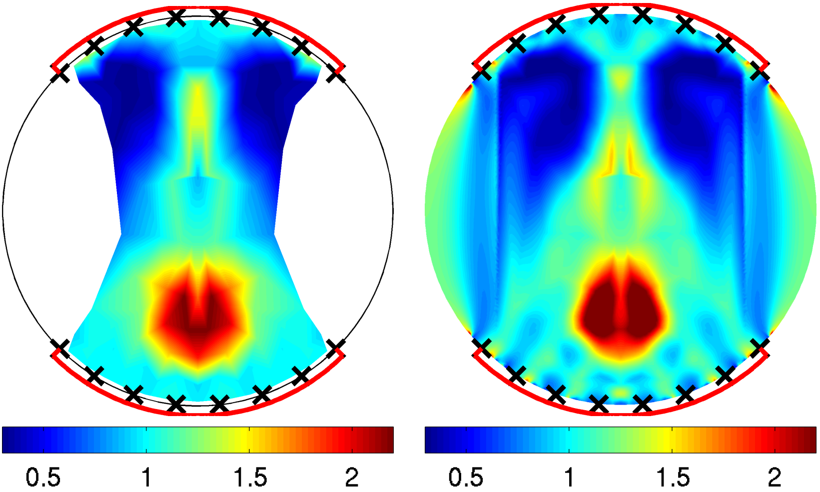
Numerical results: piecewise constant σ + pyramidal





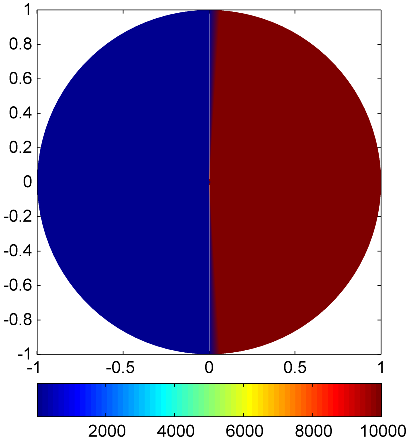
Left: piecewise linear; right: one step Gauss-Newton, $\beta = 0.65\pi$, $n = 16$, $\omega_0 = -3\pi/10$.

Numerical results: piecewise constant σ + two-sided



Left: piecewise linear; right: one step Gauss-Newton, $n = 16$, $\mathcal{B}_A$ is solid red.

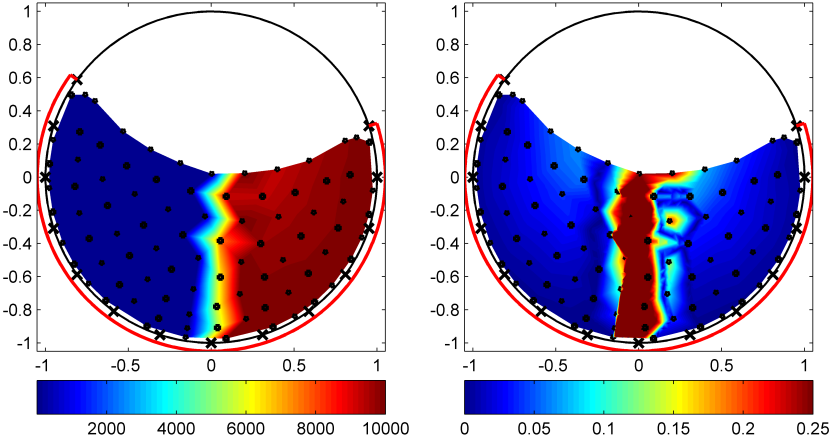
Numerical results: high contrast conductivity



Test conductivity, contrast 10^4.

- We solve the full non-linear problem
- No artificial regularization
- No linearization
- Big advantage: can capture really high contrast behavior
- Test case: piecewise constant conductivity, contrast 10^4
- Most existing methods fail
- Our method: relative error less than 5% away from the interface

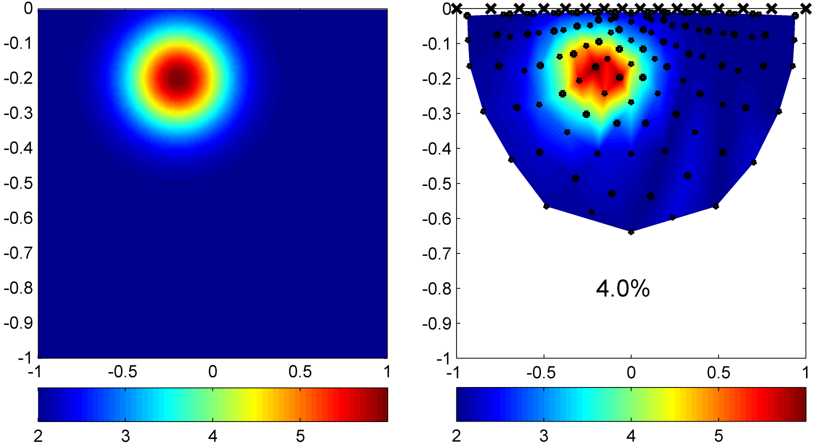
Numerical results: high contrast conductivity



High contrast reconstruction, $n = 14$, $\omega_0 = -11\pi/20$, contrast 10^4.
Left: reconstruction; right: pointwise relative error.

Numerical results: EIT in the half plane

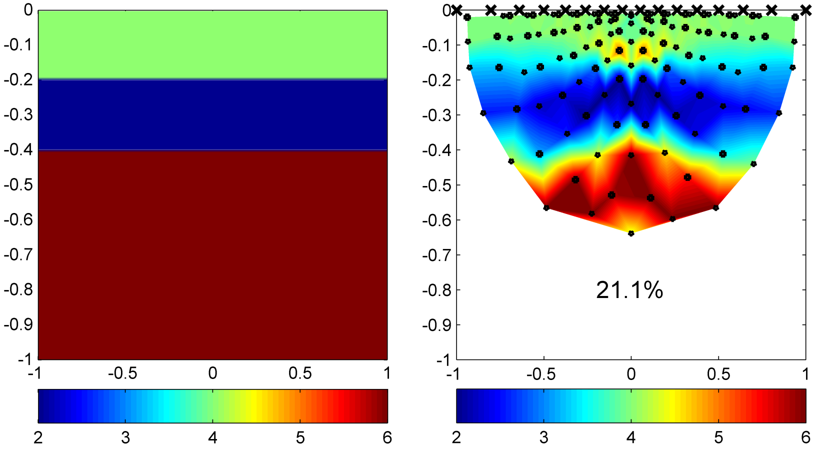
Can be used in different domains. Example: half plane, smooth σ.



Left: true; right: reconstruction, $n = 16$.

Numerical results: EIT in the half plane

Can be used in different domains. Example: half plane, layered σ.



Left: true; right: reconstruction, $n = 16$.

Conclusions

Two distinct computational approaches to the partial data EIT:

1. Circular networks and (quasi)conformal mappings
   - Uses existing theory of optimal grids in the unit disk
   - Tradeoff between the uniform resolution and anisotropy
   - Conformal: isotropic solution, rigid electrode positioning, grid clustering leads to poor resolution
   - Quasiconformal: artificial anisotropy, flexible electrode positioning, uniform resolution, some distortions
   - Geometrical distortions can be corrected by preconditioned Gauss-Newton
2. Sensitivity grids and special network topologies (pyramidal, two-sided)
   - No anisotropy or distortions due to (quasi)conformal mappings
   - Theory of discrete inverse problems developed
   - Sensitivity grids work well
   - Independent of the domain geometry

References

- *Uncertainty quantification for electrical impedance tomography with resistor networks*, L. Borcea, F. Guevara Vasquez and A.V. Mamonov. Preprint: arXiv:1105.1183 [math-ph]
- *Pyramidal resistor networks for electrical impedance tomography with partial boundary measurements*, L. Borcea, V. Druskin, A.V. Mamonov and F. Guevara Vasquez. Inverse Problems 26(10):105009, 2010.
- *Circular resistor networks for electrical impedance tomography with partial boundary measurements*, L. Borcea, V. Druskin and A.V. Mamonov. Inverse Problems 26(4):045010, 2010.
- *Electrical impedance tomography with resistor networks*, L. Borcea, V. Druskin and F. Guevara Vasquez. Inverse Problems 24(3):035013, 2008.