Optimization algorithm for the reconstruction of a conductivity inclusion

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Inclusion Ω in $\mathbb{R}^d$

For a given entire harmonic function H, consider

$$\begin{cases} \nabla \cdot (\chi(\mathbb{R}^d \setminus \overline{\Omega}) + k\chi(\Omega))\nabla u = 0 & \text{in } \mathbb{R}^d, \\ u(x) - H(x) = O(|x|^{1-d}) & \text{as } |x| \to \infty. \end{cases}$$

Multipolar expansions:

$$u(x) = H(x) + \sum_\alpha \sum_\beta \frac{(-1)^{|\beta|}}{\alpha!\beta!} \partial^\alpha H(0) M_{\alpha\beta}(k,\Omega) \partial^\beta \Gamma(x), \quad |x| \to \infty.$$

$\{M_{\alpha\beta}\}$: *Generalized* Polarization Tensors (GPT).

($\Gamma(x)$: the fundamental solution for the Laplacian.)

Small Inclusion

Let $D = \delta B + z$ be a single inclusion or multiple inclusions contained in Ω and $0 < k < +\infty$ be a constant. Consider

$$\begin{cases} \nabla \cdot (\chi(\Omega \setminus \overline{D}) + k\chi(D))\nabla u = 0, & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = g, & \text{on } \partial\Omega. \end{cases}$$

For $d = 2, 3$,

$$u(x) = U(x) + \sum_{\alpha} \sum_{\beta} \frac{(-1)^{|\beta|} \delta^{|\alpha|+|\beta|}}{\alpha! \beta!} \partial^{\alpha} U(z) M_{\alpha\beta}(k, B) \partial^{\beta} N(x, z) + O(\delta^{2d}).$$

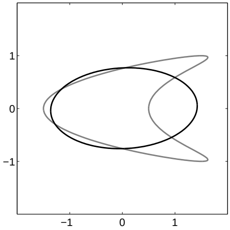
A complete asymptotic expansion is possible, but terms higher than $2d - 1$ involves not only GPTs but also the boundary-inclusion interaction. ($U(x)$: the back ground solution, $N(x)$: the Neumann function)

Equivalent ellipse

There is a canonical correspondence between the class of ellipses (ellipsoids) and the class of PTs:
If E is an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$, then

$$M(k, E) = (k-1)|E| \begin{bmatrix} \dfrac{a+b}{a+kb} & 0 \\ 0 & \dfrac{a+b}{b+ka} \end{bmatrix}.$$

Equivalent ellipse (ellipsoid)= ellipse with the same PT:



GPT and Imaging

by Ammari-Kang-L-Zribi

Aim: Make use of $\displaystyle\sum_{|\alpha|+|\beta|\leq K} a_\alpha b_\beta m_{\alpha\beta}$ for a fixed $K \geq 2$ to image finer details of the shape of the inclusion.

- If $K = 2$, it is imaging by PT (equivalent ellipse).

Optimization Problem: Let Ω be the target domain. Minimize over D

$$J[D] := \frac{1}{2} \sum_{|\alpha|+|\beta|\leq K} w_{|\alpha|+|\beta|} \left| \sum_{\alpha,\beta} a_\alpha b_\beta M_{\alpha\beta}(k, D) - \sum_{\alpha,\beta} a_\alpha b_\beta M_{\alpha\beta}(k, \Omega) \right|^2 .$$

- $w_{|\alpha|+|\beta|}$ are binary weights: $w_{|\alpha|+|\beta|} = 1$(on) or 0(off).
- A good choice for the initial guess: the equivalent ellipse.

An asymptotic expansion of GPT due to small boundary change:

- ϵ-perturbation of D:

$$\partial D_\epsilon := \{\tilde{x} = x + \epsilon h(x)\nu(x) \mid x \in \partial D\}.$$

- Asymptotic formula: Suppose that $H = \sum_\alpha a_\alpha x^\alpha$ and $F = \sum_\beta b_\beta x^\beta$ are harmonic polynomials. Then

$$\sum_{\alpha,\beta} a_\alpha b_\beta M_{\alpha\beta}(k, D_\epsilon) - \sum_{\alpha,\beta} a_\alpha b_\beta M_{\alpha\beta}(k, D)$$
$$= \epsilon(k-1)\int_{\partial D} h\left[\frac{\partial v}{\partial \nu}\Big|_- \frac{\partial u}{\partial \nu}\Big|_- + \frac{1}{k}\frac{\partial u}{\partial T}\Big|_- \frac{\partial v}{\partial T}\Big|_-\right] d\sigma$$
$$+O(\epsilon^2),$$

where

$$\begin{cases} \Delta u = 0, & \text{in } D \cup (\mathbb{R}^2 \setminus \overline{D}), \\ u|_+ - u|_- = 0, & \text{on } \partial D, \\ \dfrac{\partial u}{\partial \nu}\Big|_+ - k \dfrac{\partial u}{\partial \nu}\Big|_- = 0, & \text{on } \partial D, \\ (u - H)(x) = O(|x|^{-1}) & \text{as } |x| \to \infty, \end{cases}$$

and

$$\begin{cases} \Delta v = 0, & \text{in } D \cup (\mathbb{R}^2 \setminus \overline{D}), \\ kv|_+ - v|_- = 0, & \text{on } \partial D, \\ \dfrac{\partial v}{\partial \nu}\Big|_+ - \dfrac{\partial v}{\partial \nu}\Big|_- = 0, & \text{on } \partial D, \\ (v - F)(x) = O(|x|^{-1}) & \text{as } |x| \to \infty. \end{cases}$$

- Shape derivative: Let

$$\phi_D^{HF}(x) = (k-1)\left[\frac{\partial v}{\partial \nu}\Big|_- \frac{\partial u}{\partial \nu}\Big|_- + \frac{1}{k}\frac{\partial u}{\partial T}\Big|_- \frac{\partial v}{\partial T}\Big|_-\right],$$

and

$$\delta_D^{HF} = \sum_{\alpha,\beta} a_\alpha b_\beta M_{\alpha\beta}(k,D) - \sum_{\alpha,\beta} a_\alpha b_\beta M_{\alpha\beta}(k,B).$$

Then

$$\langle d_S J[D], h\rangle_{L^2(\partial D)} = \sum_{|\alpha|+|\beta|\le K} w_{|\alpha|+|\beta|}\delta_D^{HF}\langle \phi_D^{HF}, h\rangle_{L^2(\partial D)}.$$

- Determination of location: Let $i_l := e_l$ and $j_l := 2e_l$, $l = 1,\ldots,d$. Then $(\frac{m_{i_l j_l}}{m_{i_l i_l}})$ is the center of mass of B if B is a ball.

Gradient Descent Method

To get a minimum of $F : \mathbb{R}^m \to \mathbb{R}$, one starts with an initial guess x_0 and modify it as

$$x_{n+1} = x_n - \gamma_n \nabla F(x_n),$$

where γ_n is a positive real number.
Note that $\nabla F(x) = \sum_{j=1}^m \frac{d}{dt} F(x + t\mathbf{e}_j)\Big|_{t=0} \mathbf{e}_j$.

To approximate the inclusion, modify $D^{(0)}$ (initial guess) as

$$\partial D^{(n+1)} = \partial D^{(n)} - \gamma_n \left(\sum_j \langle d_S J[D^{(n)}], \psi_j \rangle \psi_j \right) \nu,$$

where ν is the outward normal direction to $\partial D^{(n)}$ and $d_S J$ is the shape derivative.

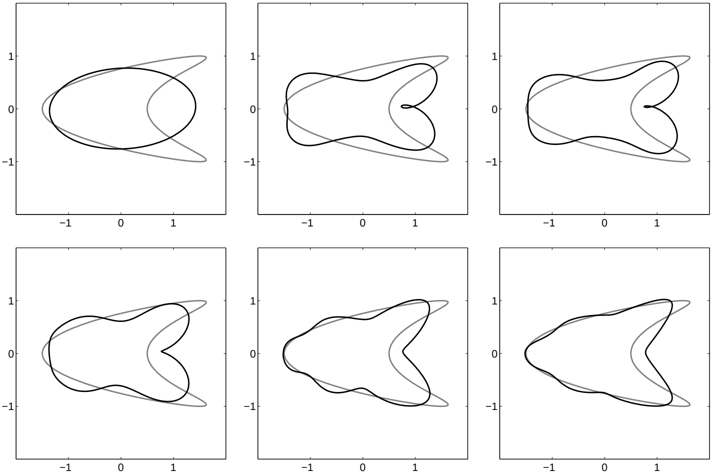


Figure: $K = 6$, 6 iterations.

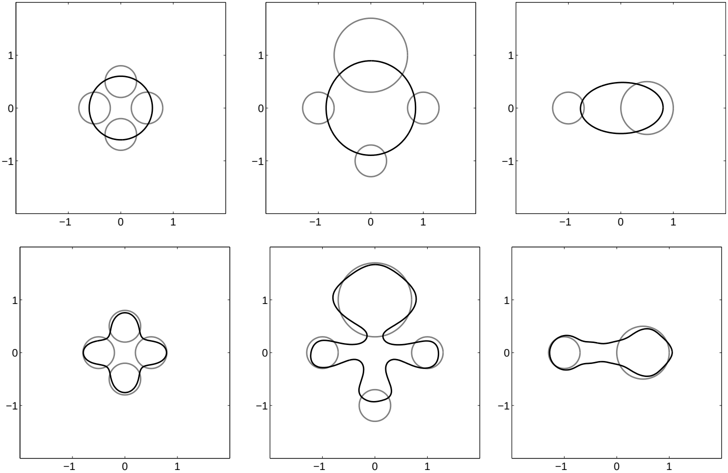


Figure: Reconstruction of clusters of inclusions. The upper images: the equivalent ellipses, and the lower ones: results after 6 iterations.

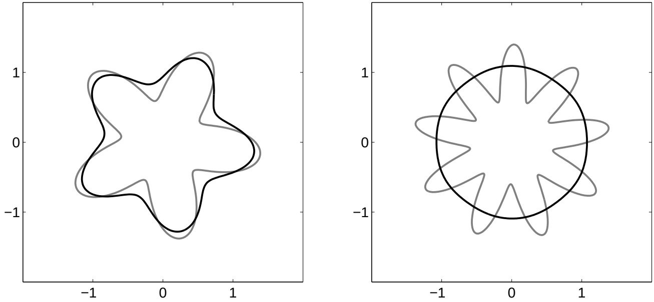
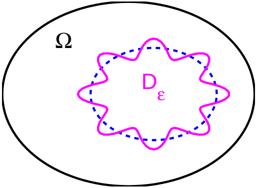


Figure: After 6 iterations

Related Works

Conductivity Equation, NtD map



- $\partial D_\epsilon = \{\tilde{x} = x + \epsilon h(x)\nu_x\}$
- $$\int_{\partial\Omega} f(u_\epsilon - u)\, d\sigma = \epsilon(1-k)\int_{\partial D} h\left[\frac{\partial v}{\partial T}\frac{\partial u}{\partial T} + \frac{1}{k}\frac{\partial v}{\partial \nu}\bigg|_+ \frac{\partial u}{\partial \nu}\bigg|_+\right] d\sigma + O(\epsilon^2),$$
  where v satisfies
  $$\begin{cases} \nabla\cdot((1+(k-1)\chi(D)\nabla v) = 0, & \text{in } \Omega, \\ \dfrac{\partial v}{\partial \nu} = f, & \text{on } \partial\Omega. \end{cases}$$

proof

Note that

- $\int_\Omega (1+(k-1)\chi_{D_\epsilon})\nabla u_\epsilon \cdot \nabla v = \int_{\partial\Omega} \frac{\partial u_\epsilon}{\partial \nu} v$
- $\int_\Omega (1+(k-1)\chi_{D})\nabla u_\epsilon \cdot \nabla v = \int_{\partial\Omega} u_\epsilon \frac{\partial v}{\partial \nu}$
- $\int_{\partial\Omega} \left(v\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu} \right) = 0$

We have

$$\begin{aligned}
&\int_{\partial\Omega} \left(v\frac{\partial (u_\epsilon - u)}{\partial \nu} - \frac{\partial v}{\partial \nu}(u_\epsilon - u) \right) \\
&= (k-1)\int_\Omega \left(\chi_{D_\epsilon} - \chi_D \right) \nabla u_\epsilon \cdot \nabla v \\
&= (k-1)\int_{D_\epsilon \setminus D} \nabla u_\epsilon^i \cdot \nabla v^e - (k-1)\int_{D\setminus D_\epsilon} \nabla u_\epsilon^e \cdot \nabla v^i \\
&\approx (k-1)\epsilon \int_{\partial D} h\mathcal{M}\nabla u^e : \nabla v^e,
\end{aligned}$$

where $\mathcal{M} = \tau \otimes \tau + \frac{1}{k}\nu \otimes \nu$.

Similarly, we can derive

- Modal Measurements: the Conductivity case

$$\int_{\partial\Omega} g(u_\epsilon - u) + (\omega_0^2 - \omega_\epsilon^2)\int_\Omega w_g u$$
$$\approx \epsilon(k-1)\int_{\partial D} h(x)\mathcal{M}\nabla u^e : \nabla w_g^e d\sigma_x,$$

where $\mathcal{M} = \tau\otimes\tau + \frac{1}{k}\nu\otimes\nu$

- Modal Measurements: the Elastic case

$$\int_{\partial\Omega} g\cdot\mathbb{C}_0(\widehat{\nabla}u_\epsilon - \widehat{\nabla}u)\nu + (\omega_0^2 - \omega_\epsilon^2)\int_\Omega w_g\cdot u$$
$$\approx -\epsilon\int_{\partial D} h(x)\mathcal{M}[\widehat{\nabla}u^e](x) : \widehat{\nabla}w_g^e(x)d\sigma(x)$$

where $\mathcal{M}$ is defined from the lame constants.

Basis functions: Optimal vectors

by Ammari-Beretta-Francini-Kang-L

Remind

$$\int_{\partial\Omega} g \cdot \mathbb{C}_0(\widehat{\nabla} u_\epsilon - \widehat{\nabla} u_0)\nu + (\omega_0^2 - \omega_\epsilon^2)\int_\Omega w_g \cdot u_0$$
$$= -\epsilon \int_{\partial D} h(x) \mathcal{M}[\widehat{\nabla} u_0^e](x) : \widehat{\nabla} w_g^e(x) d\sigma(x) + O(\epsilon^{1+\beta}).$$

Let $\mathcal{V} := \left\{ g \in L^2(\partial\Omega) : \int_{\partial\Omega} g \cdot (\mathbb{C}_D \widehat{\nabla} u_0)\nu = 0 \right\}$ and define $\Lambda : \mathcal{V} \to L^2(\partial D)$ by

$$\Lambda(g) := \mathcal{M}[\widehat{\nabla} u_0^e] : \widehat{\nabla} w_g^e \quad \text{on } \partial D.$$

If

$$h(x) = \sum_{l=1}^{L} \alpha_l v_{g_l},$$

where

$$v_{g_l} := \mathcal{M}[\widehat{\nabla} u_0^e] : \widehat{\nabla} w_{g_l}^e \quad \text{on } \partial D, \quad l = 1, \ldots, L,$$

and g_l are the significant singular vectors of Λ, then we have good reconstructed images.

Incomplete Measurements

Suppose that $\mathbb{C}_0(\widehat{\nabla} u_\epsilon - \widehat{\nabla} u_0)\nu$ is measured only in an open part Γ_1 of the boundary $\partial\Omega$.

For $g \in L^2(\partial\Omega)$ such that $g = 0$ on Γ_2 and $\int_{\Gamma_1} g \cdot (\mathbb{C}_D \widehat{\nabla} u_0)\nu = 0$, we can prove that the following asymptotic formula holds as $\epsilon \to 0$:

$$\int_{\Gamma_1} g \cdot \mathbb{C}_0(\widehat{\nabla} u_\epsilon - \widehat{\nabla} u_0)\nu + (\omega_0^2 - \omega_\epsilon^2) \int_\Omega w_g \cdot u_0$$

$$= -\epsilon \int_{\partial D} h(x) \mathcal{M}[\widehat{\nabla} u_0^e](x) : \widehat{\nabla} w_g^e(x) d\sigma(x) + O(\epsilon^{1+\beta})$$

for some $\beta > 0$.

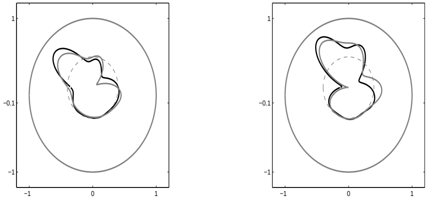


Figure: Incomplete measurements reconstruction: we use the data only measured on the part of $\partial\Omega$, that is $\{e^{i\theta} : \theta \in [0, \pi]\}$.

Thank you!