New approaches to the numerical solution of Maxwell's Equations

MITACS conference on Medical Imaging
Toronto, June 20–24, 2011

Charles L. Epstein (with some help from Michael O'Neil)

Departments of Mathematics
and Radiology
University of Pennsylvania

June 22, 2011

Acknowledgment

Theoretical work joint with **Leslie Greengard** of the Courant Institute.
Computational work joint with **Zydrunas Giambutas, Leslie Greengard, Andreas Klöckner, and Michael O'Neil,** all of the Courant Institute.

Thanks!

Thanks to Adrian for inviting me to participate in this very interesting meeting!

The Life History of a PDE, I

(0) Physicists describe a physical system in the language of PDEs. They write down the "general solution," make unsubstantiated assertions about the properties of all solutions, declare the subject closed and move on to something entirely different.

(1) If the equation describes something interesting, or if mathematicians have developed techniques to study it, they will prove existence, uniqueness, and regularity results for the equation, or some vastly simplified "model equation." Usually this takes about 50 years. The techniques used are frequently quite abstract, but at the very least non-constructive.
The Mathematicians declare the problem finished.

The Life History of a PDE, II

(2) If solving the equation may have some practical applications, some effort is made to find constructive (at least in principle) methods to find solutions.

(3) If money is involved, these "in principle methods" are turned into actual computer programs, which might produce something resembling solutions to the original equations.

(4) If real money is involved, the engineers solve the problem using the FEM.

Today I want to speak about some level (2) and preliminary level (3) work we've recently done with regard to solving the Time Harmonic Maxwell Equations (THME).

Representations of solutions $\Delta u + k^2 u = 0$

Much of what we discuss today has to do with how the representation of solutions to Maxwell's equations impacts numerical algorithms. To motivate this discussion we first consider the simpler case of Helmholtz' equation in an exterior domain $\Omega \subset \mathbb{R}^3$:

$$\Delta u + k^2 u = 0 \text{ in } \Omega, \text{ with } u \restriction_{b\Omega} = f. \tag{1}$$

The "outgoing" fundamental solution for this equation is:

$$g_k(x, y) = \frac{e^{ik\|x-y\|}}{4\pi\|x - y\|}, \tag{2}$$

where $\operatorname{Im} k \geq 0$.

Layer Potentials
$\Delta u + k^2 u = 0$

Functions defined in $b\Omega^c$ by the formulæ, called respectively single and double layer potentials, for $x \notin b\Omega$:

$$Sw(x) = \int\limits_{b\Omega} g_k(x, y)w(y)dA(y),$$

$$Dw(x) = \int\limits_{b\Omega} \partial_{\nu_y} g_k(x, y)w(y)dA(y), \quad (3)$$

are outgoing solutions to Helmholtz' equation so long as $x \in b\Omega^c$.

To impose a boundary condition we need to examine what happens to these functions as $x \to b\Omega$. The function $Sw(x)$ is continuous across $b\Omega$, whereas $Dw(x)$ has a jump.

For $x \in b\Omega$:

$$Sw(x) = \int_{b\Omega} g_k(x,y)w(y)dA(y),$$

$$D^{\pm}w(x) = \frac{\pm 1}{2}w(x) + \int_{b\Omega} n_k(x,y)w(y)dA(y). \quad (4)$$

Here $+$ indicates that the limit is taken from the unbounded component of $b\Omega$, and $-$ from the bounded component. The kernels $g_k(x,y), n_k(x,y)$ define pseudodifferential operators of order -1 on $b\Omega$, which are compact operators on either Sobolev or Hölder spaces.

Layer Potentials and the Dirichlet problem $\Delta u + k^2 u = 0$

To solve the Dirichlet problem stated above, with these representations, we would need to solve either

$$\begin{aligned} Sw(x) &= f(x) \text{ or} \\ D^{+}w(x) &= f(x). \end{aligned} \tag{5}$$

The first equation is extremely unstable numerically as S is compact. The second equation is Fredholm of second kind, and is solvable, in principle for all k not in a discrete subset $E \subset \mathbb{R}$. A number $k \in E$ if and only if k^2 is an eigenvalue of the *interior* Neumann problem. These are called interior, or spurious resonances, as they result from the choice of representation and have nothing to do with the exterior problem per se.

The double layer potential provides a good representation for small enough frequencies, but becomes unstable for larger frequencies. To avoid this, in practice, one uses a combination:

$$D^{+}w + i\eta S w = f, \tag{6}$$

for an appropriately chosen constant η. This representation reduces the solution of the Dirichlet problem to solving a Fredholm equation of second kind, with no interior resonances, and good behavior as $k \to 0$, i.e. no low-frequency breakdown. We are looking for a representation of solutions to Maxwell's equations with all these desirable properties.

Time Harmonic Equations $\nabla\times\boldsymbol{E}=ik\boldsymbol{H}\quad\nabla\times\boldsymbol{H}=-ik\boldsymbol{E}$

In this talk we will consider solutions to the Maxwell system with harmonic dependence in time:

$$\boldsymbol{\Xi}(x,t)=\boldsymbol{E}(x)e^{-i\omega t}\quad\boldsymbol{N}(x,t)=\boldsymbol{H}(x)e^{-i\omega t};\tag{7}$$

In this case Maxwell's equations take the form of a first order elliptic system:

$$\begin{aligned}\nabla\times\boldsymbol{E}=ik\boldsymbol{H}\quad\nabla\cdot\boldsymbol{E}=0\\ \nabla\times\boldsymbol{H}=-ik\boldsymbol{E}\quad\nabla\cdot\boldsymbol{H}=0.\end{aligned}\tag{THME}$$

If ϵ is the electrical permittivity, and μ the magnetic permeability of space, the the wave number, $k^2=\mu\epsilon\omega^2$. For the purposes of this talk we assume that σ, the conductivity, is zero, though we can also handle the dielectric problem. We denote this equation by THME[k].

Boundary conditions, at ∞

$$\nabla \times \boldsymbol{E} = ik\boldsymbol{H} \quad \nabla \times \boldsymbol{H} = -ik\boldsymbol{E}$$

We consider these equations in the complement, Ω of a region D.
We assume that $bD = b\Omega$.
For simplicity we assume that D is a perfect conductor, though other, more physically realistic situations can be considered as well.
For many choices of boundary condition, this problem has a unique solution, provided we impose the *outgoing radiation condition:*

$$i_{\hat{\boldsymbol{x}}} \nabla \times \boldsymbol{E} - ik\boldsymbol{E} = o\left(\frac{1}{|x|}\right), \tag{8}$$

along with the assumption that $\operatorname{Im} k \geq 0$.

For example:
We could specify the "tangential" components of $\boldsymbol{E}$: that is $\mathcal{T}\boldsymbol{E} \restriction_{Tb\Omega}$, where

$$\mathcal{T}\boldsymbol{w} = -\boldsymbol{n} \times \boldsymbol{n} \times \boldsymbol{w}.$$

We could specify the "tangential" components of $\boldsymbol{H}$.

We could specify relations between $\boldsymbol{E} \restriction_{Tb\Omega}$ and $\boldsymbol{H} \restriction_{Tb\Omega}$.

We could specify jumps in the tangential and normal components and obtain a solution in all of $\mathbb{R}^3 \setminus b\Omega$. (the dielectric problem).

Numerical Solution of Maxwell's Equation $\nabla \times \boldsymbol{E} = ik\boldsymbol{H} \quad \nabla \times \boldsymbol{H} = -ik\boldsymbol{E}$

From the pure math perspective (level (1)) quite a lot is known about BVPs for Maxwell's equation, in exterior domains.
For example: If $\operatorname{Im} k \geq 0$, then there is a unique outgoing solution to (THME[k]) with specified tangential components of $\boldsymbol{E}$ on $b\Omega$.

We would like a representation for solutions to the THME[k] so that the numerical method (level (2)!), for solving, e.g. the prefect conductor problem, has the following three properties:

1. We are reduced to solving Fredholm equations of second kind on the boundary of Ω.
2. These equations have no interior resonances.
3. Neither these equations nor the representation suffer from low frequency breakdown.

The Debye-Mie Solution $\nabla\times\boldsymbol{E}=ik\boldsymbol{H}\quad\nabla\times\boldsymbol{H}=-ik\boldsymbol{E}$

- For the case of the exterior of a round sphere, Debye and Mie found a representation that does not suffer from low frequency breakdown. This motivated a lot of our work.

$$\begin{aligned}\boldsymbol{E}&=\nabla\times\nabla\times(\boldsymbol{x}\psi)+i\omega\mu\nabla\times(\boldsymbol{x}\phi),\\ \boldsymbol{H}&=\nabla\times\nabla\times(\boldsymbol{x}\phi)-i\omega\epsilon\nabla\times(\boldsymbol{x}\psi)\end{aligned}\tag{9}$$

- Here ψ and ϕ are outgoing, scalar solutions to the Helmholtz equation $\Delta u+k^2u=0$.
- They can be determined from $\boldsymbol{n}\cdot\boldsymbol{E}$ and $\boldsymbol{n}\cdot\boldsymbol{H}$, but not by equations of second kind.
- The success of this method is *pure luck!* It relies on the diagonalization of the boundary equations, and makes extensive usage of properties of spherical harmonics.

Uniqueness Theorems

The starting point for most numerical methods that use Fredholm integral equations is a uniqueness theorem. This is the marvelous feature of this type of representation; as in finite dimensions:

$$\text{Existence} \equiv \text{Uniqueness} \tag{10}$$

Theorem

Let $(\boldsymbol{E}, \boldsymbol{H})$ be an outgoing solution to (THME[k]) in Ω with $\operatorname{Im} k \geq 0$. If either $\mathcal{T}\boldsymbol{E} = 0$, or $\mathcal{T}\boldsymbol{H} = 0$, then $(\boldsymbol{E}, \boldsymbol{H}) \equiv (0, 0)$ in Ω.

As we will see, for our representation it is more natural to look at the normal components

$$\boldsymbol{n} \cdot \boldsymbol{E} \restriction_{b\Omega} \text{ and } \boldsymbol{n} \cdot \boldsymbol{H} \restriction_{b\Omega}, \tag{11}$$

together. While this does not correspond to a physical boundary value problem, it is what is used in the Debye-Mie formula.

Representing solutions to the THME[k], I

$\nabla \times \boldsymbol{E} = ik\boldsymbol{H} \quad \nabla \times \boldsymbol{H} = -ik\boldsymbol{E}$

Without regard to any particular BVP, we can ask the following question:
Is there a parametrization for the *space* of outgoing solution to the THME[k] that behaves nicely as $k \to 0$?
First observe that the usual choice, the physical current on the boundary, cannot work.

If $k \neq 0$, then *any* tangential current $\boldsymbol{j}$ defined on $b\Omega$ defines a unique solution of the outgoing THME[k].

When $k = 0$, the current must satisfy the equation: $\nabla_{b\Omega} \cdot \boldsymbol{j} = 0$.

Representing solutions to the THME[k], II

$\nabla\times\boldsymbol{E}=ik\boldsymbol{H}\quad\nabla\times\boldsymbol{H}=-ik\boldsymbol{E}$

At zero frequency the space of allowable currents is of infinite codimension in the space of all tangential fields, for any $k\neq 0$. The same is true of the tangential components of $\boldsymbol{E}$ and $\boldsymbol{H}$, which must also satisfy a PDE on $b\Omega$ when $k=0$:
$\nabla_{b\Omega}\boldsymbol{n}\times\boldsymbol{E}=\nabla_{b\Omega}\boldsymbol{n}\times\boldsymbol{H}=0.$

Boundary currents are vector fields. We can describe them in terms of the Hodge decomposition:

$$\boldsymbol{j}=\nabla_{b\Omega}\phi+\boldsymbol{n}\times\nabla_{b\Omega}\chi+\boldsymbol{j}_H, \tag{12}$$

where $\boldsymbol{j}_H$ is harmonic: $\nabla_{b\Omega}\boldsymbol{j}_H=\nabla_{b\Omega}\boldsymbol{n}\times\boldsymbol{j}_H=0.$

If $k\neq 0$, then all components of this representation are arbitrary, while at $k=0$ we must have $\phi=0$.

Representing solutions to the THME[k], III

$$\nabla \times \boldsymbol{E} = ik\boldsymbol{H} \quad \nabla \times \boldsymbol{H} = -ik\boldsymbol{E}$$

This shows that neither the current, nor the tangential components of the field can provide a parametrization for the space solutions to THME[k] that behaves nicely as $k \to 0$.

Is there something else that behaves better?

Normal Components

$$\nabla \times \boldsymbol{E} = ik\boldsymbol{H} \quad \nabla \times \boldsymbol{H} = -ik\boldsymbol{E}$$

- In the physics literature it has been asserted for many years that the normal components of $\boldsymbol{E}$ and $\boldsymbol{H}$ determine an outgoing solution:

Theorem (Physics "Theorem")

Let $(\boldsymbol{E}, \boldsymbol{H})$ be an outgoing solution to (THME[k]) with $\operatorname{Im} k \geq 0$. If $\boldsymbol{n} \cdot \boldsymbol{E}$ and $\boldsymbol{n} \cdot \boldsymbol{H}$ are both zero then $(\boldsymbol{E}, \boldsymbol{H}) \equiv (0, 0)$ in Ω.

- If $b\Omega$ is simply connected, then this follows from a simple integrations by parts argument and Stokes theorem.
- In the non-simply connected case it is false!

What happens in the Non-Simply Connected Case $\nabla\times\boldsymbol{E}=ik\boldsymbol{H}\quad\nabla\times\boldsymbol{H}=-ik\boldsymbol{E}$

- Let $\alpha=\mathcal{T}\boldsymbol{E}|_{b\Omega}$ and $\beta=\mathcal{T}\boldsymbol{H}|_{b\Omega}$. If $(\boldsymbol{E},\boldsymbol{H})$ have vanishing normal components, then these are vector fields on $b\Omega$ that satisfy $\nabla_{b\Omega}\boldsymbol{n}\times\boldsymbol{v}=0$. A solution of THME[k] with vanishing normal components vanishes precisely when the integral

$$W(\alpha,\bar{\beta})=\operatorname{Re}\left[\int\limits_{b\Omega}\boldsymbol{n}\cdot(\alpha\times\overline{\beta})dA\right]\tag{13}$$

is zero. The total radiated energy is a constant multiple of this integral.

The Non-simply Connected Case, Continued $\nabla \times \boldsymbol{E} = ik\boldsymbol{H} \quad \nabla \times \boldsymbol{H} = -ik\boldsymbol{E}$

If the total genus of the boundary of $b\Omega$ is p, then using the classical uniqueness theorem and our integral representation we can prove:

Theorem (Correct theorem)

For each k, with $\operatorname{Im} k \geq 0$, there is a $2p$-dimensional space of outgoing solutions to the k-harmonic Maxwell equations in Ω with vanishing normal components on $b\Omega$.

Some history $\nabla \times \boldsymbol{E} = ik\boldsymbol{H} \quad \nabla \times \boldsymbol{H} = -ik\boldsymbol{E}$

After we proved this, we learned that the existence of these solutions was proved in the 1980s by Rainer Kress, but seems to have been forgotten. Recently it was hinted at by difficulties with the numerical solution of the THME[k] in the complement of a solid torus.

We call the solutions with vanishing normal components *k-Neumann fields*. When $k = 0$, these solutions are the classical "Neumann Fields," which are Hodge representatives of $H^1_{\text{dR}}(\Omega, b\Omega)$.

If $b\Omega$ has genus p, then there are $2p$ additional parameters that need to be specified, beyond the normal data to define a solution. A lot of the difficulty in avoiding low frequency breakdown in this case is connected to how this additional data is specified.

While the normal components of the field do not arise together in a physical BVP, this theorem suggests that we look for a representation in terms of data that is related to the normal components $(\boldsymbol{n} \cdot \boldsymbol{E}, \boldsymbol{n} \cdot \boldsymbol{H})$ by Fredholm equations of second kind. When $b\Omega$ is not simply connected, some additional constraints, related to the topology are also needed.

The indifference of this data to the frequency gives reasons to hope that such data will lead to numerical methods that do not suffer from low frequency breakdown. If we're lucky these equations will not have interior resonances.

We now consider different methods for representing solutions to Maxwell's equations in terms of integral kernels.

When solving Maxwell's equations numerically, especially in an exterior domain, it is useful to represent the solution in terms of layer potentials, which are integrals over $b\Omega$. All these representations start with

$$g(y) = \frac{e^{ik|y|}}{4\pi|y|},$$

which is the *outgoing* fundamental solution for the Helmholtz equation. Using g insures that the solutions satisfy the radiation condition "at infinity."

Representation formulæ
$\nabla \times \boldsymbol{E} = ik\boldsymbol{H} \quad \nabla \times \boldsymbol{H} = -ik\boldsymbol{E}$

A standard representation is based on using a vector and a scalar potential:

$$\boldsymbol{E} = ik\boldsymbol{A} - \nabla\phi, \boldsymbol{H} = \nabla \times \boldsymbol{A}.$$

Where:

$$\boldsymbol{A}(x) = \int_{b\Omega} g(x-y)\boldsymbol{j}(y)dS(y),$$

$$\phi = \frac{1}{ik}\int_{b\Omega} g(x-y)\nabla \cdot_{b\Omega} \boldsymbol{j}(y)dS(y) \quad (14)$$

There are a variety of representations of this general type.

MFIE

$\nabla \times \boldsymbol{E} = ik\boldsymbol{H} \quad \nabla \times \boldsymbol{H} = -ik\boldsymbol{E}$

This representation suffers from low frequency breakdown as ω tends to zero, there are $O(\omega)$-, $O(1)$-, and $O(\omega^{-1})$-terms.

If we try to specify the tangential component of $\boldsymbol{H}$ we get an integral equation of the form

$$\boldsymbol{n} \times \boldsymbol{H} = \left(\frac{1}{2} - K(k)\right)\boldsymbol{j} \quad \text{MFIE.}$$

Here $\boldsymbol{j}$ is the physical current, and $K(k)$ is a compact operator.

EFIE and CFIE

$\nabla \times \boldsymbol{E} = ik\boldsymbol{H} \quad \nabla \times \boldsymbol{H} = -ik\boldsymbol{E}$

If we try to specify the tangential component of $\boldsymbol{E}$ we get an equation of the form

$$\boldsymbol{n} \times \boldsymbol{E} = T(k)\boldsymbol{j} \qquad \text{EFIE.}$$

Here $T(k)$ is an elliptic operator of order 1.

Both the MFIE and EFIE have spurious resonances, which can be avoided by considering a linear combination of these conditions, $\boldsymbol{n} \times \boldsymbol{H} + i\eta\boldsymbol{n} \times \boldsymbol{E}$, known as the CFIE. It is hypersingular, and has low frequency breakdown.

Potentials
$\nabla \times \boldsymbol{E} = ik\boldsymbol{H} \quad \nabla \times \boldsymbol{H} = -ik\boldsymbol{E}$

Our approach starts with a different representation of $\boldsymbol{E}$ and $\boldsymbol{H}$ in terms of a 2 scalar functions ϕ, and Ψ, and two vector potentials $\boldsymbol{A}$, and $\boldsymbol{T}$, setting:

$$\boldsymbol{E} = (ik\boldsymbol{A} - \nabla\phi - \nabla \times \boldsymbol{T}) \quad \boldsymbol{H} = (ik\boldsymbol{T} - \nabla\Psi + \nabla \times \boldsymbol{A}). \tag{15}$$

All the potentials satisfy the Helmholtz equation:

$$\Delta\beta + k^2\beta = 0. \tag{16}$$

For $(\boldsymbol{E}, \boldsymbol{H})$ to solve the (THME[k]) we need to require:

$$\nabla \cdot \boldsymbol{A} = -ik\phi \quad \nabla \cdot \boldsymbol{T} = -ik\Psi. \tag{17}$$

Potentials, cont.
$\nabla \times \boldsymbol{E} = ik\boldsymbol{H} \quad \nabla \times \boldsymbol{H} = -ik\boldsymbol{E}$

The potentials are represented as integrals over the $b\Omega$

$$\boldsymbol{A} = \int_{b\Omega} g(x,y)\boldsymbol{j}(y) \cdot d\boldsymbol{x} dS(y)$$

$$\boldsymbol{T} = \int_{b\Omega} g(x,y)\boldsymbol{m}(y) \cdot d\boldsymbol{x} dS(y) \quad (18)$$

$$\phi = \int_{b\Omega} g(x,y)r(y)dS(y)$$

$$\Psi = \int_{b\Omega} g(x,y)q(y)dS(y) \quad (19)$$

Here $\boldsymbol{j}$ and $\boldsymbol{m}$ are tangential vector fields and r, q are scalar functions on $b\Omega$.

Potentials, cont.
$\nabla \times \boldsymbol{E} = ik\boldsymbol{H} \quad \nabla \times \boldsymbol{H} = -ik\boldsymbol{E}$

To get a solution to the THME[k] we need to require

$$\frac{1}{ik}\nabla_{b\Omega}\boldsymbol{n}\times\boldsymbol{j} = r \quad \frac{-1}{ik}\nabla_{b\Omega}\boldsymbol{n}\times\boldsymbol{m} = qdA. \tag{20}$$

There are, apparently too many unknowns. For the perfect conductor we let: $\boldsymbol{m} = \boldsymbol{n}\times\boldsymbol{j}$ and then these equations have a unique solution, provided $b\Omega$ is simply connected. In the non-simply connected case one needs to add a harmonic vector field, $\boldsymbol{j}_H$, to a particular solution $\boldsymbol{j}_R$.

The harmonic vector fields, $\boldsymbol{j}_H$, are the solutions to the system of equations

$$\nabla_{b\Omega}[\boldsymbol{n}\times\boldsymbol{j}_H] = 0 \quad \nabla_{b\Omega}\boldsymbol{j}_H = 0. \tag{21}$$

We denote this vector space by $\mathcal{H}^1(b\Omega)$.

r and q are scalar functions with mean zero on every component of $b\Omega$. These relations allow us to use the scalar functions r and q, along with $\boldsymbol{j}_H$, (in the non-simply connected case) as the basic unknowns. We call $(r, q, \boldsymbol{j}_H)$, with r and q of mean zero on every component of $b\Omega$,

Debye Source Data.

To find $\boldsymbol{j}$ from (r,q) we need to solve:

$$\Delta_{b\Omega,1}\boldsymbol{j} = -ik(\nabla_{b\Omega} r + \boldsymbol{n} \times \nabla_{b\Omega} q). \tag{22}$$

We can reduce this to solving the scalar Laplace equation, setting

$$\boldsymbol{j}_R = -ik(\nabla_{b\Omega} R_0 r + \boldsymbol{n} \times \nabla_{b\Omega} R_0 q). \tag{23}$$

R_0 is the partial inverse of $\Delta_{b\Omega,0}$. The solution $\boldsymbol{j}_R$ is orthogonal to the harmonic vector fields $\mathcal{H}^1(b\Omega)$, so the general solution is

$$\boldsymbol{j} = \boldsymbol{j}_R + \boldsymbol{j}_H.$$

The operator taking (r,q) to $\boldsymbol{j}_R$ has order -1.

Boundary Integral Equations

If we represent solutions to the THME[k] in Ω in terms of integrals over $b\Omega$, then letting the point of evaluation tend to $b\Omega$ gives integral equations that the boundary data must satisfy.

For example a harmonic function represented as a double layer potential

$$u(x) = \int_{b\Omega} w(y)\partial_{\nu_y} g_0(x,y)dA(y), \tag{24}$$

this limit would be:

$$u(y') = \frac{1}{2}w(y') \mp \int_{b\Omega} k(y',y)w(y)dA(y). \tag{25}$$

Here $k(y',y)$ is a kernel defining an operator of order -1.

The Normal Components
$\nabla \times \boldsymbol{E} = ik\boldsymbol{H} \quad \nabla \times \boldsymbol{H} = -ik\boldsymbol{E}$

Taking limits in our representation formula we easily obtain that the normal components satisfy a Fredholm equation of second kind on $b\Omega$:

$$\lim_{x \to b\Omega^{\pm}} \begin{pmatrix} \boldsymbol{n} \cdot \boldsymbol{E} \\ \boldsymbol{n} \cdot \boldsymbol{H} \end{pmatrix} = \begin{pmatrix} \frac{\pm \mathrm{Id}}{2} - K_1 & 0 \\ 0 & \frac{\mp \mathrm{Id}}{2} + K_1 \end{pmatrix} \begin{pmatrix} r \\ q \end{pmatrix} + \begin{pmatrix} ik\boldsymbol{n}_x \cdot G & -K_0 \\ K_0 & ik\boldsymbol{n}_x \cdot G \end{pmatrix} \begin{pmatrix} \boldsymbol{j}(r, q, \boldsymbol{j}_H) \\ \boldsymbol{m}(r, q, \boldsymbol{j}_H) \end{pmatrix}. \quad (26)$$

This exactly what we were hoping for.
We denote the rows of this matrix by $\mathcal{N}_{\boldsymbol{E}}^{\pm}(k), \mathcal{N}_{\boldsymbol{H}}^{\pm}(k)$.

The system of equations is analytic in k; recall that $\boldsymbol{m} = \boldsymbol{n} \times \boldsymbol{j}$, and

$$\Delta_{b\Omega,1}\boldsymbol{j} = -ik(\nabla_{b\Omega} r + \boldsymbol{n} \times \nabla_{b\Omega} q). \tag{27}$$

Hence

$$\left(\begin{pmatrix} \frac{\pm \mathrm{Id}}{2} - K_1(k) & 0 \\ 0 & \frac{\mp \mathrm{Id}}{2} + K_1(k) \end{pmatrix} + \widetilde{K}_1(k) \right) \begin{pmatrix} r \\ q \\ \boldsymbol{j}_H \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}. \tag{28}$$

Here $K_1(k), \widetilde{K}_1(k)$ are analytic families of operators of order -1. This equation is solvable in the complement of a discrete set E_+, which is disjoint from $\overline{\mathbb{C}_+}$ when $b\Omega$ is simply connected.

The Tangential Components $\nabla \times \boldsymbol{E} = ik\boldsymbol{H} \quad \nabla \times \boldsymbol{H} = -ik\boldsymbol{E}$

The tangential components satisfy a similar system of equations:

$$\lim_{x \to b\Omega^{\pm}} \begin{pmatrix} \boldsymbol{n} \times \boldsymbol{E} \\ \boldsymbol{n} \times \boldsymbol{H} \end{pmatrix}_t =$$

$$\frac{1}{2}\begin{pmatrix} \pm\boldsymbol{n} \times \boldsymbol{m} \\ \mp\boldsymbol{j} \end{pmatrix} + \begin{pmatrix} -M_0 & 0 & ikM_1 & -\widetilde{M}_0 \\ 0 & -\boldsymbol{n} \times M_0 & -\boldsymbol{n} \times \widetilde{M}_0 & -ik\boldsymbol{n} \times M_1 \end{pmatrix} \begin{pmatrix} r \\ q \\ \boldsymbol{j} \\ \boldsymbol{m} \end{pmatrix} . \quad (2$$

We let $\mathcal{T}_{\boldsymbol{E}}^{\pm}(k)$, $\mathcal{T}_{\boldsymbol{H}}^{\pm}(k)$ denote the rows of the tangential equation. Each row is a system of two equations for two unknown functions, but these are not of second kind in r and q.

The Jumps Across $b\Omega$

$$\nabla \times \boldsymbol{E} = ik\boldsymbol{H} \quad \nabla \times \boldsymbol{H} = -ik\boldsymbol{E}$$

The fields defined by the integral representation satisfy the THME in the complement of $b\Omega$ and have a jump across the boundary:

$$\begin{bmatrix} \boldsymbol{n} \cdot \boldsymbol{E}_{\pm} \\ \boldsymbol{n} \cdot \boldsymbol{H}_{\pm} \end{bmatrix} = \begin{pmatrix} r \\ -q \end{pmatrix} \quad \begin{bmatrix} \boldsymbol{n} \times \boldsymbol{E}_{\pm} \\ \boldsymbol{n} \times \boldsymbol{H}_{\pm} \end{bmatrix} = \begin{pmatrix} \boldsymbol{n} \times \boldsymbol{m} \\ -\boldsymbol{j} \end{pmatrix}. \tag{30}$$

Using these jump relations and a simple integration by parts arguments we can show that each row of the tangential operator, $\mathcal{T}_{\boldsymbol{E}}^{+}(k)$, $\mathcal{T}_{\boldsymbol{H}}^{+}(k)$ has a trivial nullspace, when $\operatorname{Im} k \geq 0$.

The Tangential Operator Has a Trivial Nullspace, I

$\nabla \times \boldsymbol{E} = ik\boldsymbol{H} \quad \nabla \times \boldsymbol{H} = -ik\boldsymbol{E}$

If $(\boldsymbol{E}_+, \boldsymbol{H}_+) = (0, 0)$, then the jump conditions for such potentials imply that

$$\mathcal{T}\boldsymbol{E}_- \restriction_{bD} = \boldsymbol{j} \quad \mathcal{T}\boldsymbol{H}_- \restriction_{bD} = \boldsymbol{j}. \tag{IntRes}$$

These boundary conditions define a **non**-self adjoint boundary value problem for the Maxwell equations in D.

This is why the spurious resonances are not forced to lie on the real axis. That they actually occur on the lower half plane seems like good luck

The Tangential Operator Has a Trivial Nullspace, II

$\nabla \times \boldsymbol{E} = ik\boldsymbol{H} \quad \nabla \times \boldsymbol{H} = -ik\boldsymbol{E}$

Integration by parts and using $\mathcal{T}\boldsymbol{E}_- = \boldsymbol{j} = \mathcal{T}\boldsymbol{H}_-$ gives:

$$-ik\int_{bD}(\boldsymbol{j},\boldsymbol{j})dA = k^2\int_D(\boldsymbol{E}_-,\boldsymbol{E}_-)dV - \int_D(\nabla\times\boldsymbol{E}_-,\nabla\times\boldsymbol{E}_-)dV. \tag{31}$$

Using the quadratic formula we see that

$$k_\pm = \frac{-ia \pm \sqrt{4bc-a^2}}{2b}. \tag{32}$$

where a, b, c are all positive. This shows that $\operatorname{Im} k < 0$.

No Physical Resonances $\nabla \times \boldsymbol{E} = ik\boldsymbol{H} \quad \nabla \times \boldsymbol{H} = -ik\boldsymbol{E}$

Theorem

If $b\Omega$ is as above, then the boundary value problem

$$\nabla \times \boldsymbol{E} = ik\boldsymbol{H}, \quad \nabla \times \boldsymbol{H} = -ik\boldsymbol{E}, \tag{33}$$

satisfying

$$\mathcal{T}\boldsymbol{E}_- \restriction_{bD} = \boldsymbol{j} = \mathcal{T}\boldsymbol{H}_- \restriction_{bD} . \tag{34}$$

has no solutions if $\operatorname{Im}\omega \geq 0$.

Using this result we show, in the simply connected case, that using our parameters (r, q) one can find a unique outgoing solution to the (THME[k]) with arbitrarily specified normal components by solving Fredholm equations of second kind.

We still need to find a system of second kind equations for solving the THME[k] for the perfect conductor.

Scattering Off of a Perfect Conductor $\nabla \times \boldsymbol{E} = ik\boldsymbol{H} \quad \nabla \times \boldsymbol{H} = -ik\boldsymbol{E}$

For this problem, the solution of the THME[k] is the sum of a specified incoming part, and an oiutgoing solution:

$$(\boldsymbol{E}^{\text{tot}}, \boldsymbol{H}^{\text{tot}}) = (\boldsymbol{E}, \boldsymbol{H}) - (\boldsymbol{E}^{\text{in}}, \boldsymbol{H}^{\text{in}}).$$

The incoming wave is data, which we take to be a solution of THME[k] in D^c. The outgoing solution $(\boldsymbol{E}, \boldsymbol{H})$ is called the scattered field.

Physical considerations show that the tangential components of the sum must vanish on $b\Omega$:

$$\boldsymbol{n} \times (\boldsymbol{E} - \boldsymbol{E}^{\text{in}}) = 0.$$

Hence we need to solve a BVP for the THME[k] with specified tangential components. We do this somewhat indirectly.

Specifying the Tangential Components $\nabla \times \boldsymbol{E} = ik\boldsymbol{H} \quad \nabla \times \boldsymbol{H} = -ik\boldsymbol{E}$

To solve the THME[k] with specified tangential components we use a system of equations built out of the row of the tangential equations for $\boldsymbol{E}$ and the row of the normal equations for $\boldsymbol{H}$. The operator is defined by

$$\mathcal{Q}^{\pm}(k)\begin{pmatrix} r \\ q \\ \boldsymbol{j}_H \end{pmatrix} = \begin{pmatrix} G_0\nabla_{b\Omega}\cdot\mathcal{T}^{\pm}_{\boldsymbol{E}}(k) \\ \mathcal{N}^{\pm}_{\boldsymbol{H}}(k) \end{pmatrix}\begin{pmatrix} r \\ q \\ \boldsymbol{j}_H \end{pmatrix}. \tag{35}$$

Here G_0 is the Newtonian potential (for $k = 0$) restricted to the boundary of Ω. It is a pseudodifferential operator of order -1. The operator $\mathcal{Q}^{+}(k)$ is a system of Fredholm equations of second kind. It is invertible for k in the complement of a discrete set F_{+}.

The Simply Connected Case $\nabla\times\boldsymbol{E}=ik\boldsymbol{H}\quad\nabla\times\boldsymbol{H}=-ik\boldsymbol{E}$

In the simply connected case, F_+ is disjoint from the closed upper half plane, and $\boldsymbol{j}_H$ is absent.

Theorem

If $b\Omega$ is simply connected, then the integral equation

$$\mathcal{Q}^+(k)\begin{pmatrix} r \\ q \end{pmatrix}=\begin{pmatrix} f \\ h \end{pmatrix} \qquad \text{(ModSys)}$$

provides a unique solution to the problem of scattering from a perfect conductor, for any k in the closed upper half plane. Here,

$$f=G_0(\nabla_{b\Omega}\cdot\boldsymbol{E}_t^{\text{in}}),\qquad i\omega\mu h=\boldsymbol{n}\cdot\nabla_{b\Omega}\boldsymbol{E}_t^{\text{in}}=i\omega\mu\boldsymbol{n}\cdot\boldsymbol{H}^{\text{in}}, \quad (36)$$

where $\boldsymbol{E}_t^{\text{in}}$ is the tangential component of an incoming electric field, and $\boldsymbol{n}\cdot\boldsymbol{H}^{\text{in}}\restriction_{Tb\Omega}$, the normal component of the incoming magnetic field.

The non-simply connected case

When the boundary of Ω is not simply connected, then our scalar potentials do not include the k-Neumann fields. We need to replace the current $\boldsymbol{j}_R$ with $\boldsymbol{j}_R + \boldsymbol{j}_H$, where $\boldsymbol{j}_H$ is an (unknown) harmonic vector field. To determine this additional data we augment our equations with $2p$ additional equations.

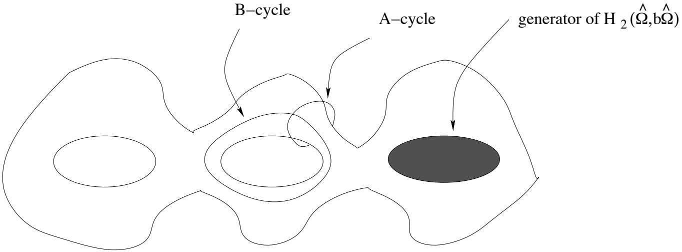
To describe these equations we need to say a word about the topology of a surface $\Gamma \subset \mathbb{R}^3$. The 1-dimensional homology splits into two subspaces

$$H_1(\Gamma) = H_1(D) \restriction_\Gamma + H_1(\Omega) \restriction_\Gamma, \tag{37}$$

where $\Gamma^c = D \cup \Omega$. We let $\{A_1, \ldots, A_p\}$ span $H_1(\Omega) \restriction_\Gamma$, and $\{B_1, \ldots, B_p\}$ span $H_1(D) \restriction_\Gamma$.

Homology of embedded surfaces

The B-cycles are homologically trivial in Ω, thus there are surfaces $S_j \subset \Omega$, such that $bS_j = B_j$. These surfaces are generators of the relative homology group $H_2(\Omega, b\Omega)$.



Auxiliary conditions when $p \neq 0$, I

The additional conditions are

$$\int_{A_j} \boldsymbol{E} \cdot d\boldsymbol{s} = \int_{A_j} \boldsymbol{E}^{\text{in}} \cdot d\boldsymbol{s}, \quad \frac{1}{ik} \int_{B_j} \boldsymbol{E} \cdot d\boldsymbol{s} = \frac{1}{ik} \int_{B_j} \boldsymbol{E}^{\text{in}} \cdot d\boldsymbol{s}. \tag{38}$$

We need to divide by k in the second set, because these integrals tend to zero as $k \to 0$.

Auxiliary conditions when $p \neq 0$, II

Integrating by parts in the second set of equations and using the fact that $\nabla \times \boldsymbol{E} = ik\boldsymbol{H}$, and $\nabla \times \boldsymbol{E}^{\text{in}} = ik\boldsymbol{H}^{\text{in}}$, in D^c, we see that the second set of equations can be replaced with

$$\int_{S_j} \boldsymbol{H} \cdot \boldsymbol{n} dA = \int_{S_j} \boldsymbol{H}^{\text{in}} \cdot \boldsymbol{n} dA. \tag{39}$$

These conditions behave well as $k \to 0$. Because

$$\int_{B_j} \boldsymbol{E}(0) \cdot d\boldsymbol{s} = 0, \tag{40}$$

for numerical purposes we also can use the hybrid form:

$$\frac{1}{ik} \int_{B_j} (\boldsymbol{E}(k) - \boldsymbol{E}(0)) \cdot d\boldsymbol{s} = \int_{S_j} \boldsymbol{H}^{\text{in}} \cdot \boldsymbol{n} dA, \tag{41}$$

The General Case $\nabla \times \boldsymbol{E} = ik\boldsymbol{H} \quad \nabla \times \boldsymbol{H} = -ik\boldsymbol{E}$

Theorem

If $b\Omega$ is a smooth surface in $\mathbb{R}^3$, then the integral equations

$$\mathcal{Q}^+(k)\begin{pmatrix} r \\ q \\ \boldsymbol{j}_H \end{pmatrix} = \begin{pmatrix} f \\ h \end{pmatrix} \qquad \text{(ModSys)}$$

augmented by

$$\int_{A_j} \boldsymbol{E}\cdot d\boldsymbol{s} = \int_{A_j} \boldsymbol{E}^{\text{in}}\cdot d\boldsymbol{s}, \quad \int_{S_j} \boldsymbol{H}\cdot \boldsymbol{n}dA = \int_{S_j} \boldsymbol{H}^{\text{in}}\cdot \boldsymbol{n}dA., \qquad (42)$$

has a unique solution for any k in the closed upper half plane. Here, $f = G_0(\nabla_{b\Omega}.\boldsymbol{E}_t^{\text{in}})$, and $i\omega\mu h = \boldsymbol{n}\cdot\nabla_{b\Omega}\boldsymbol{E}_t^{\text{in}} = i\omega\mu\boldsymbol{n}\cdot\boldsymbol{H}^{\text{in}}$.

A New Representation
$\nabla \times \boldsymbol{E} = ik\boldsymbol{H} \quad \nabla \times \boldsymbol{H} = -ik\boldsymbol{E}$

Our new representation, essentially, in terms of two scalar potential functions defined on $b\Omega$ and a harmonic vector field. It has some $+$s and $-$s:

$+$

- It does not suffer from low frequency breakdown.
- For $\operatorname{Im} k \geq 0$, The integral equations, are of second kind.
- They have no "spurious resonances."

$-$

- It requires the ability to solve the scalar Laplace equation on $b\Omega$: $\Delta_{b\Omega} u = f$.
- If the boundary is not simply connected, it requires determining the harmonic vector fields, but something of this sort would be needed for any method....

Numerical Experiments on surfaces of revolution (slides by Michael O'Neil)

The following slides describe numerical experiments, which are a proof of concept that our representations translate to numerical algorithms, with the expected properties. We first describe our method to solve the perfect conductor problem, with D a solid torus of revolution. We then describe how this approach is adapted to solve the dielectric problem.

These slides, and much of the computational work itself, are **Michael O'Neil's** work.

Axisymmetric representation

A genus 1 surface of revolution Γ is specified as

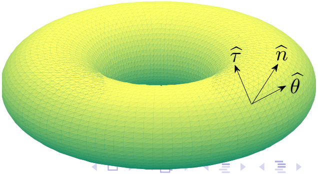
$$\begin{aligned} x(s,\theta) &= \rho(s)\cos\theta \\ y(s,\theta) &= \rho(s)\sin\theta \\ z(s,\theta) &= z(s) \end{aligned}$$

where $s \in [0, L]$ is counterclockwise arc length along a smooth generating curve γ in the ρ-z plane

$$\gamma(s) = (\rho(s), z(s)).$$

Local orthonormal coordinate system is

$$\begin{aligned} \hat{\boldsymbol{\tau}}(s,\theta) &= \rho'(s)\hat{\boldsymbol{\rho}}(\theta) + z'(s)\hat{\boldsymbol{z}} \\ \boldsymbol{T}(\theta) &= \boldsymbol{T}(\theta) \\ \boldsymbol{n}(s,\theta) &= z'(s)\hat{\boldsymbol{\rho}}(\theta) - \rho'(s)\hat{\boldsymbol{z}} \end{aligned}$$



An axisymmetric integral equation

A scalar integral equation with axisymmetric kernel

$$\sigma(s_x,\theta_x) + \int_\Gamma k(s_x,s_y,\theta_x-\theta_y)\sigma(s_y,\theta_y)da_y = f(s_x,\theta_x)$$

can be separated into a series of uncoupled equations

$$\sigma_n(s_x) + 2\pi\int_0^L k_n(s_x,s_y)\sigma_n(s_y)\rho(s_y)ds_y = f_n(s_x)$$

where

$$\sigma(s,\theta) = \sum_n \sigma_n(s)e^{in\theta} \qquad f(s,\theta) = \sum_n f_n(s)e^{in\theta}$$

$$k(s_x,s_y,\theta_x-\theta_y) = \sum_n k_n(s_x,s_y)e^{in(\theta_x-\theta_y)}$$

Axisymmetric representation

Analogously, start by representing r and q as

$$r(\boldsymbol{x}) = \sum_n r_n(s) e^{in\theta} \qquad q(\boldsymbol{x}) = \sum_n q_n(s) e^{in\theta}$$

Next, construct $\boldsymbol{j}$ and $\boldsymbol{m}$ according to

$$\boldsymbol{j} = ik(\nabla_\Gamma \Delta_\Gamma^{-1} r - \boldsymbol{n} \times \nabla_\Gamma \Delta_\Gamma^{-1} q) + \alpha_1 \boldsymbol{j}_{H_1} + \alpha_2 \boldsymbol{j}_{H_2}$$
$$\boldsymbol{m} = \boldsymbol{n} \times \boldsymbol{j}$$

This means that $\boldsymbol{j}$ and $\boldsymbol{m}$ are of the form

$$\boldsymbol{j} = \sum_n \boldsymbol{j}_n(r_n, q_n)\, e^{in\theta} + \alpha_1 \boldsymbol{j}_{H_1} + \alpha_2 \boldsymbol{j}_{H_2}$$

$$\boldsymbol{m} = \sum_n \boldsymbol{m}_n(r_n, q_n)\, e^{in\theta} + \alpha_1 \boldsymbol{m}_{H_1} + \alpha_2 \boldsymbol{m}_{H_2}$$

Axisymmetric representation

Decompose the data $\boldsymbol{E}^{\text{in}}, \boldsymbol{H}^{\text{in}}$ as

$$\boldsymbol{E}^{\text{in}} = \sum_n \boldsymbol{E}_n^{\text{in}} e^{in\theta} \qquad \boldsymbol{H}^{\text{in}} = \sum_n \boldsymbol{H}_n^{\text{in}} e^{in\theta}$$

Then solve a series of uncoupled equations on γ

$$\mathcal{S}_{n,0}\nabla_\Gamma \cdot \mathcal{T}\left(ik\mathcal{S}_{n,k}\boldsymbol{j}_n - \nabla\mathcal{S}_{n,k}r_n - \nabla\times\mathcal{S}_{n,k}\boldsymbol{m}_n\right) = -\mathcal{S}_{n,0}\nabla_\Gamma\cdot\boldsymbol{E}_n^{\text{in}}$$

$$ik\boldsymbol{n}\cdot\mathcal{S}_{n,k}\boldsymbol{m}_n - \frac{\partial}{\partial\boldsymbol{n}}\mathcal{S}_{n,k}q_n + \boldsymbol{n}\cdot\nabla\times\mathcal{S}_{n,k}\boldsymbol{m}_n = \boldsymbol{n}\cdot\boldsymbol{H}_n^{\text{in}}$$

$$\int_{A_i}\boldsymbol{E}_n\cdot d\boldsymbol{s} = -\int_{A_i}\boldsymbol{E}_n^{\text{in}}\cdot d\boldsymbol{s}$$

$$\int_{B_i}\frac{\boldsymbol{E}_n - \boldsymbol{E}_n\big|_{k=0}}{ik}\cdot d\boldsymbol{s} = -\int_{S_i}\boldsymbol{H}_n^{\text{in}}\cdot d\boldsymbol{a}$$

Low-frequency behavior

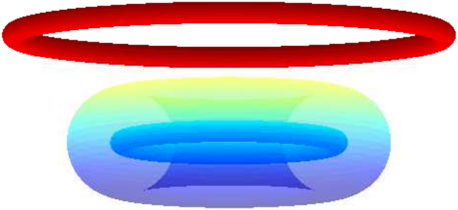
For small k, the discretized operator in matrix form for mode $n = 0$ scales as

$$\left(\begin{array}{c|c|c|c} \frac{1}{4} + \mathcal{K} & \mathcal{O}(k) & \mathcal{O}(1) & \\ \hline \mathcal{O}(k) & -\frac{1}{2} + \mathcal{K} & & \mathcal{O}(1) \\ \hline \mathcal{O}(k) & & \mathcal{O}(1) & \\ \hline & \mathcal{O}(1) & & \mathcal{O}(1) \end{array}\right) \left(\begin{array}{c} r \\ \hline q \\ \hline \alpha \\ \hline \beta \end{array}\right) = \left(\begin{array}{c} -\mathcal{S}_{0,0}\nabla_\Gamma \cdot \mathcal{T}\boldsymbol{E}_0^{\text{in}} \\ \hline -\boldsymbol{n} \cdot \boldsymbol{H}_0^{\text{in}} \\ \hline -\int_A \boldsymbol{E}_0^{\text{in}} \cdot d\boldsymbol{s} \\ \hline -\int_B \boldsymbol{E}_0^{\text{in}} \cdot d\boldsymbol{s} \end{array}\right)$$

where $\mathcal{K}$ is an $\mathcal{O}(1)$ compact operator. Some additional effort is required to make sure the solutions have mean zero.

Numerical examples

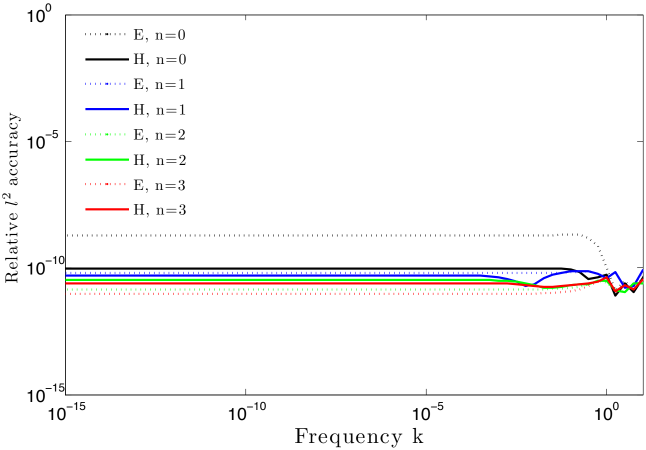
How do we test this code?



1. Generate outgoing solution due to r^{test},q^{test} on inner toroid
2. Grab $\boldsymbol{E}^{test}$, $\boldsymbol{H}^{test}$ on the surface of Γ and use as data
3. Solve integral equation for r, q
4. Compare $\boldsymbol{E}$, $\boldsymbol{H}$ to known $\boldsymbol{E}^{test}$, $\boldsymbol{H}^{test}$ on the outer toroid

Numerical examples

The relative accuracies for the first few modes as a function of k are



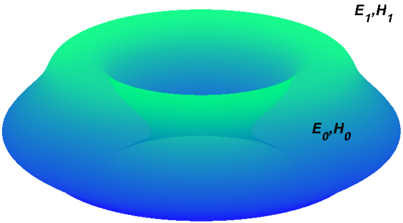
Example: The dielectric problem

The dielectric problem

The dielectric problem describes scattering in regions with different permittivities and permeabilities

$$\nabla \times \boldsymbol{E}_j = i\mu_j\omega\boldsymbol{H}_j \qquad \nabla \times \boldsymbol{H}_j = -i\epsilon_j\omega\boldsymbol{E}_j$$
$$\nabla \cdot \boldsymbol{E}_j = 0 \qquad \nabla \cdot \boldsymbol{H}_j = 0$$

Wavenumber in region j is $k_j = \sqrt{\epsilon_j\mu_j}\omega$.



Generalized Debye for the dielectric

The generalized Debye representation for the dielectric problem

$$\boldsymbol{E}_j = \sqrt{\mu_j}(ik_j\boldsymbol{A}_j - \nabla\varphi_j - \nabla\times\boldsymbol{Q}_j)$$
$$\boldsymbol{H}_j = \sqrt{\epsilon_j}(ik_j\boldsymbol{Q}_j - \nabla\psi_j + \nabla\times\boldsymbol{A}_j)$$

Vector and scalar potentials

$$\boldsymbol{A}_j = \mathcal{S}_k\boldsymbol{j}_j \qquad \boldsymbol{Q}_j = \mathcal{S}_k\boldsymbol{m}_j$$
$$\varphi_j = \mathcal{S}_k r_j \qquad \psi_j = \mathcal{S}_k q_j$$

- *Exterior* $\boldsymbol{E}_1, \boldsymbol{H}_1$ are generated from $\boldsymbol{j}_1$, $\boldsymbol{m}_1$, r_1, and q_1
- *Interior* $\boldsymbol{E}_0, \boldsymbol{H}_0$ are generated from $\boldsymbol{j}_0$, $\boldsymbol{m}_0$, r_0, and q_0

Generalized Debye for the dielectric

On genus p surface, the dielectric boundary conditions we impose are

$$\begin{aligned} \mathcal{S}_0 \nabla_\Gamma \cdot [\mathcal{T}\boldsymbol{E}^{tot}] &= 0 & \mathcal{S}_0 \nabla_\Gamma \cdot [\mathcal{T}\boldsymbol{H}^{tot}] &= 0 \\ [\boldsymbol{n} \cdot \epsilon \boldsymbol{E}^{tot}] &= 0 & [\boldsymbol{n} \cdot \mu \boldsymbol{H}^{tot}] &= 0 \\ \int_{A_i} [\boldsymbol{E}^{tot} \cdot d\boldsymbol{s}] &= 0 & \int_{A_i} [\boldsymbol{H}^{tot} \cdot d\boldsymbol{s}] &= 0 \\ \int_{B_i} [\boldsymbol{E}^{tot} \cdot d\boldsymbol{s}] &= 0 & \int_{B_i} [\boldsymbol{H}^{tot} \cdot d\boldsymbol{s}] &= 0 \end{aligned}$$

for $i = 1, \ldots, p$, and where

$$\begin{aligned} [\boldsymbol{E}] &= \boldsymbol{E}_1 - \boldsymbol{E}_0 + \boldsymbol{E}_1^{\text{in}} - \boldsymbol{E}_0^{\text{in}} \\ [\epsilon \boldsymbol{E}] &= \epsilon_1 \boldsymbol{E}_1 - \epsilon_0 \boldsymbol{E}_0 + \epsilon_1 \boldsymbol{E}_1^{\text{in}} - \epsilon_0 \boldsymbol{E}_0^{\text{in}} \end{aligned}$$

Generalized Debye for the dielectric

The same continuity conditions are satisfied as for the PEC

$$\nabla_\Gamma \cdot \boldsymbol{j}_j = ik_j r_j \qquad \nabla_\Gamma \cdot \boldsymbol{m}_j = ik_j q_j$$

except now $\boldsymbol{j}$ and $\boldsymbol{m}$ are constructed as

$$\boldsymbol{j}_1 = i\omega\left(\sqrt{\mu_1\epsilon_1}\nabla_\Gamma\Delta_\Gamma^{-1}r_1 - \sqrt{\epsilon_0\mu_0}\sqrt{\frac{\epsilon_0}{\epsilon_1}}\nabla_\Gamma\Delta_\Gamma^{-1}r_0\right) + \boldsymbol{j}_H$$

$$\boldsymbol{m}_1 = i\omega\left(\sqrt{\mu_1\epsilon_1}\nabla_\Gamma\Delta_\Gamma^{-1}q_1 - \sqrt{\epsilon_0\mu_0}\sqrt{\frac{\mu_0}{\mu_1}}\nabla_\Gamma\Delta_\Gamma^{-1}q_0\right) + \boldsymbol{m}_H$$

and

$$\boldsymbol{j}_0 = \sqrt{\frac{\epsilon_1}{\epsilon_0}}\mathcal{U}\boldsymbol{j}_1 \qquad \boldsymbol{m}_0 = \sqrt{\frac{\mu_1}{\mu_0}}\mathcal{U}\boldsymbol{m}_1$$

The operator $\mathcal{U}$ is called the *clutching map*.

The clutching map

The clutching map, $\mathcal{U}$, relates interior the $\boldsymbol{j}$, $\boldsymbol{m}$ to the exterior $\boldsymbol{j}$, $\boldsymbol{m}$. It has the following properties:

- If $\boldsymbol{v}$ is a harmonic vector field, then $\mathcal{U}\boldsymbol{v}$ is also a harmonic vector field
- If $\boldsymbol{w} \cdot \boldsymbol{j}_{H_i} = 0$, then $\mathcal{U}\boldsymbol{w} = \boldsymbol{n} \times \boldsymbol{w}$

The simplest choice for $\mathcal{U}\boldsymbol{w}$ would be $\boldsymbol{n} \times \boldsymbol{w}$ - but mild low-frequency breakdown still occurs...

To fix: Set $\mathcal{U} = \mathcal{I}$ on the harmonic vector field subspace.

The dielectric problem

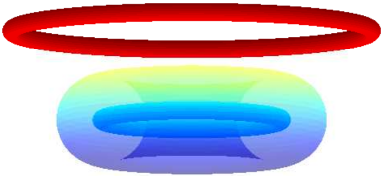
Theorem

With proper choice of the clutching map $\mathcal{U}$, the following integral equation system provides a unique solution to the dielectric problem for any set of k_i's with non-negative imaginary parts

$$\begin{aligned} \mathcal{S}_0 \nabla_\Gamma \cdot [\mathcal{T}\boldsymbol{E}^{tot}] &= 0 \\ [\boldsymbol{n} \cdot \epsilon \boldsymbol{E}^{tot}] &= 0 \\ \int_{A_i} [\boldsymbol{E}^{tot} \cdot d\boldsymbol{s}] &= 0 \\ \int_{B_i} [\boldsymbol{E}^{tot} \cdot d\boldsymbol{s}] &= 0 \end{aligned} \qquad \begin{aligned} \mathcal{S}_0 \nabla_\Gamma \cdot [\mathcal{T}\boldsymbol{H}^{tot}] &= 0 \\ [\boldsymbol{n} \cdot \mu \boldsymbol{H}^{tot}] &= 0 \\ \int_{A_i} [\boldsymbol{H}^{tot} \cdot d\boldsymbol{s}] &= 0 \\ \int_{B_i} [\boldsymbol{H}^{tot} \cdot d\boldsymbol{s}] &= 0 \end{aligned}$$

Numerical examples

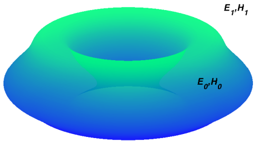
How do we test the dielectric code? Same idea...

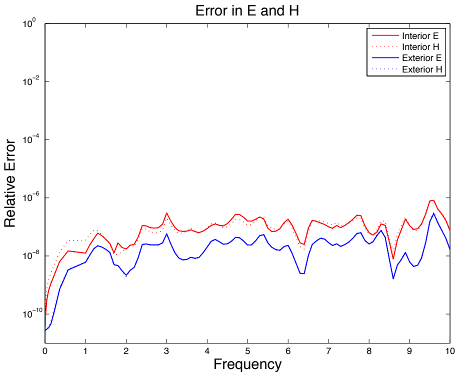


1. Generate Maxwell fields due to r_1^{test},q_1^{test} on outer toroid
2. Generate Maxwell fields due to r_0^{test},q_0^{test} on inner toroid
3. Grab $\boldsymbol{E}_1^{test}$, $\boldsymbol{H}_1^{test}$, $\boldsymbol{E}_0^{test}$, $\boldsymbol{H}_0^{test}$ on Γ and use as data
4. Solve integral equation for r_1, q_1, r_0, q_0
5. Compare $\boldsymbol{E}_1$, $\boldsymbol{H}_1$ to known $\boldsymbol{E}_0^{test}$, $\boldsymbol{H}_0^{test}$ on the outer toroid
6. Compare $\boldsymbol{E}_0$, $\boldsymbol{H}_0$ to known $\boldsymbol{E}_1^{test}$, $\boldsymbol{H}_1^{test}$ on the outer toroid

Numerical examples

The relative accuracy as a function of ω for scattering from



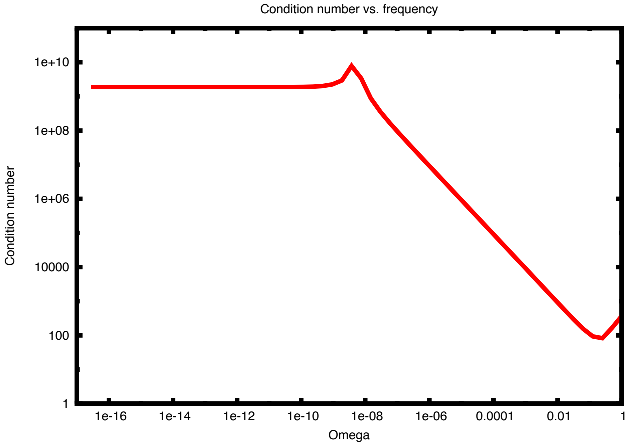


Material parameters were held constant at $\epsilon_0 = 0.90$, $\mu_0 = 1.10$, $\epsilon_1 = 1.30$, and $\mu_1 = 0.83$.

Numerical examples

Specify the clutching map on the harmonic vector fields to be

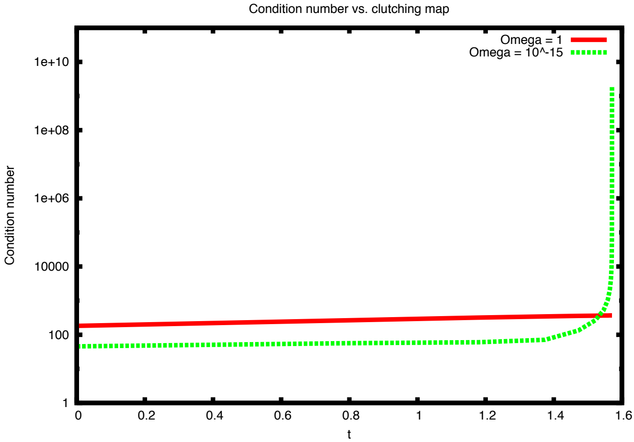
$$\mathcal{U}\boldsymbol{j}_{H_i} = \boldsymbol{n} \times \boldsymbol{j}_{H_i}$$



Numerical examples

Specify the clutching map on the harmonic vector fields to be

$$\mathcal{U}_t \boldsymbol{j}_{H_i} = \cos t\, \boldsymbol{j}_{H_i} + \sin t\, \boldsymbol{n} \times \boldsymbol{j}_{H_i}$$



Thanks!

Thanks very much for your attention!

Thanks to the NSF and AFOSR for their financial support.