Inverse Problems with Internal Functionals

Guillaume Bal

Department of Applied Physics & Applied Mathematics

Columbia University

The Calderón problem

Consider the elliptic model:

$$-\nabla \cdot \gamma(x) \nabla u = 0 \quad \text{in} \quad X \qquad \text{and} \qquad u = g \quad \text{on} \quad \partial X.$$

The **Calderón problem** consists of reconstructing the unknown $\gamma(x)$ from knowledge of *all possible* Cauchy data $(u, \gamma\nu \cdot \nabla u)$ on ∂X with u solution of the above equation.

Calderón (1980) showed injectivity of the *linearized* Calderón problem using complex geometric optics (CGO) solutions. **Sylvester and Uhlmann** (1987) showed injectivity of the Calderón problem for C^2 functions γ and **Nachman** (1988) and then **Astala and Päivärinta** (2006) for L^∞ functions γ in dimension two.

Alessandrini (1988) showed that the modulus of continuity of the inverse problem was (essentially) logarithmic: severe *loss of resolution*.

High Contrast and High Resolution

Optical Tomography and Electrical Impedance Tomography, modeled by the Calderón problem, are **low resolution** but **High Contrast** modalities.

High resolution modalities include Ultrasound, M.R.I., X-ray CT. These modalities are sometimes **low contrast**.

Hybrid Inverse Problems are problems resulting from the physical coupling between a **High Contrast** modality and a **High Resolution** modality.

In this lecture, the *high resolution* modality is Ultrasound or M.R.I. *High contrast* comes from elastic, electrical, or optical properties of tissues.

Hybrid inverse problems and internal functionals

Hybrid inverse problems (HIP) typically involve a two-step process. In a first step, a **high resolution** inverse boundary problem is solved. This could be an *inverse wave problem* (reconstruction of an initial condition in a wave equation) or the *inversion of a Fourier transform* (similar to reconstructions in M.R.I.). We do not consider this step here.

The *outcome* of the first step is the availability of specific internal functionals of the parameters of interest. HIP theory aims to address:

- Which parameters can be uniquely determined
- With which stability (resolution)
- Under which illumination (probing) mechanism.

Quantitative Photo-Acoustic Tomography (QPAT)

In the diffusive regime, **optical radiation** is modeled by:

$$-\nabla\cdot\gamma(x)\nabla u+\sigma(x)u=0 \text{ in } X \qquad u=g \text{ on } \partial X \qquad \textbf{Illumination,}$$

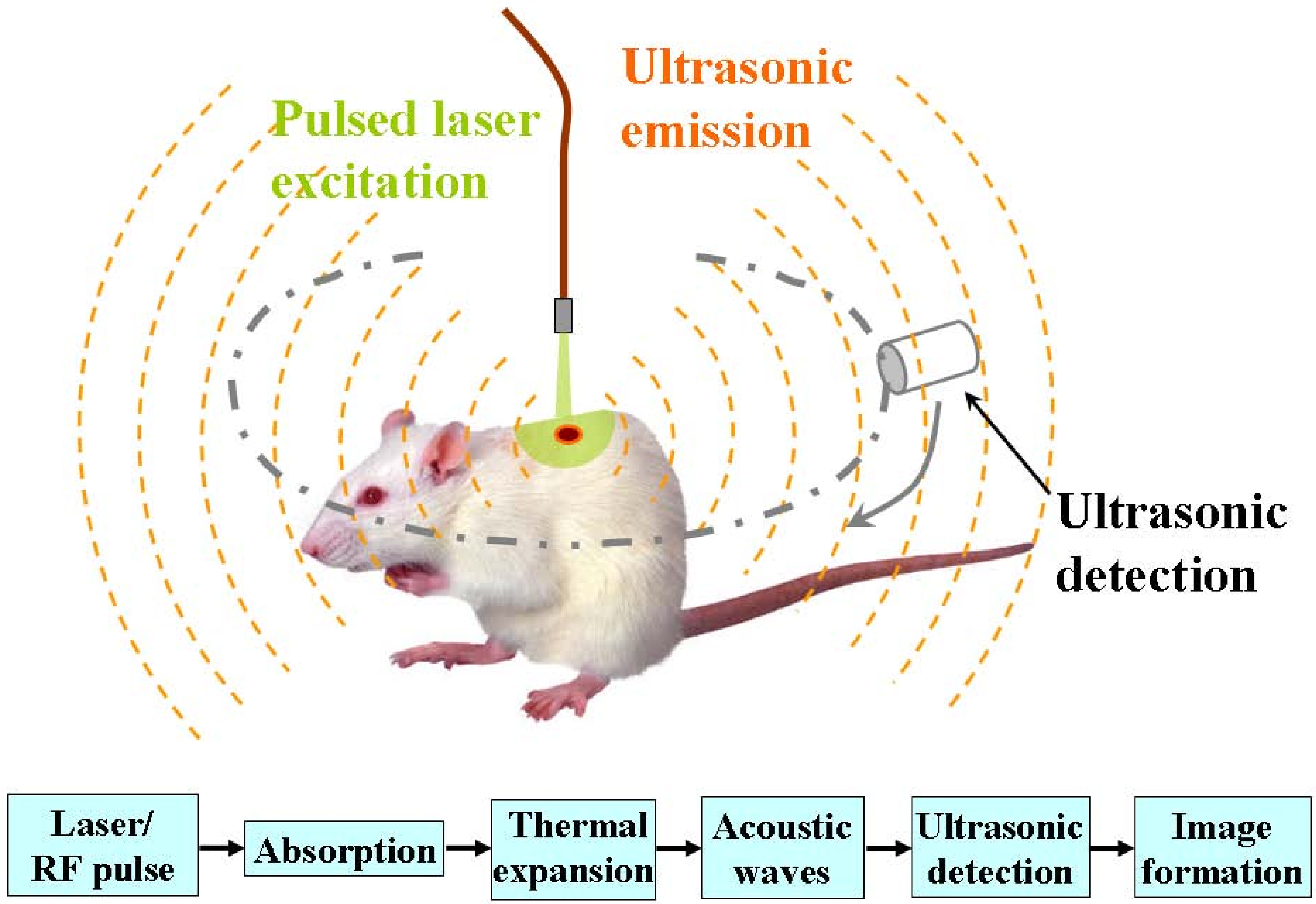
$$H(x)=\Gamma(x)\sigma(x)u(x) \text{ in } X \qquad \textbf{Internal Functional}.$$

The **objectives** of *quantitative PAT* are to understand:

- What we can reconstruct of $(\gamma(x),\sigma(x),\Gamma(x))$ from knowledge of $H_j(x)$, $1\le j\le J$ obtained for illuminations $g=g_j$, $1\le j\le J$.
- How stable the reconstructions are.
- How to choose J and the illuminations g_j.

Similar mathematical problem in Magnetic Resonance Elastography.

Experiments in Photo-acoustics



Some references on PAT and QPAT

S. K. Finch, D. Patch and Rakesh., SIAM J. Math. Anal., 35 (2004), pp. 1213–1240.

M. Haltmeier, O. Scherzer, P. Burgholzer, and G. Paltauf, Inverse Problems, 20 (2004).

M. Xu and L. V. Wang, Rev. Sci. Instr., 77 (2006).

P. Kuchment and L. Kunyansky, Euro. J. Appl. Math., 19 (2008).

Y. Hristova, P. Kuchment, and L. Nguyen, Inverse Problems, 24 (2008)

B. T. Cox, S. R. Arridge, and P. C. Beard, J. Opt. Soc. Am. A, 26 (2009).

Y. Xu, L. Wang, P. Kuchment, and G. Ambartsoumian, CRC Press, Ch. 6, (2009).

G. Bal, A. Jollivet, and V. Jugnon, Inverse Problems, 26 (2010).

H. Ammari, E. Bossy, V. Jugnon, and H. Kang, SIAM Review, (2010).

R. J. Zemp, Applied Optics, 49 (2010)

G. Bal and G. Uhlmann, Inverse Problems, 26(8) (2010).

G. Bal, K. Ren, G. Uhlmann, and T. Zhou, Inverse Problems, (2011).

P. Stefanov and G. Uhlmann, *Thermoacoustic tomography arising in brain imaging*.

G. Bal and K. Ren, *Multiple-source quantitative photoacoustic tomography*.

Quantitative Thermo-Acoustic Tomography (QTAT)

In **Thermo-Acoustic Tomography**, low-frequency radiation is used.

Using a (scalar) **Helmholtz model** for radiation, **quantitative TAT** is

$\Delta u + n(x)k^2 u + ik\sigma(x)u = 0$ in X, $\quad u = g$ on ∂X **Illumination,**

$H(x) = \sigma(x)|u|^2(x)$ in X **Internal Functional**.

QTAT consists of uniquely and stably reconstructing $\sigma(x)$ from knowledge of $H(x)$ for appropriate illuminations g.

Ultrasound Modulation

In **Ultrasound modulated** Optical Tomography (UMOT) or Electrical Impedance Tomography (UMEIT), ultrasonic waves are used to **modify** electrical or optical properties tissues.

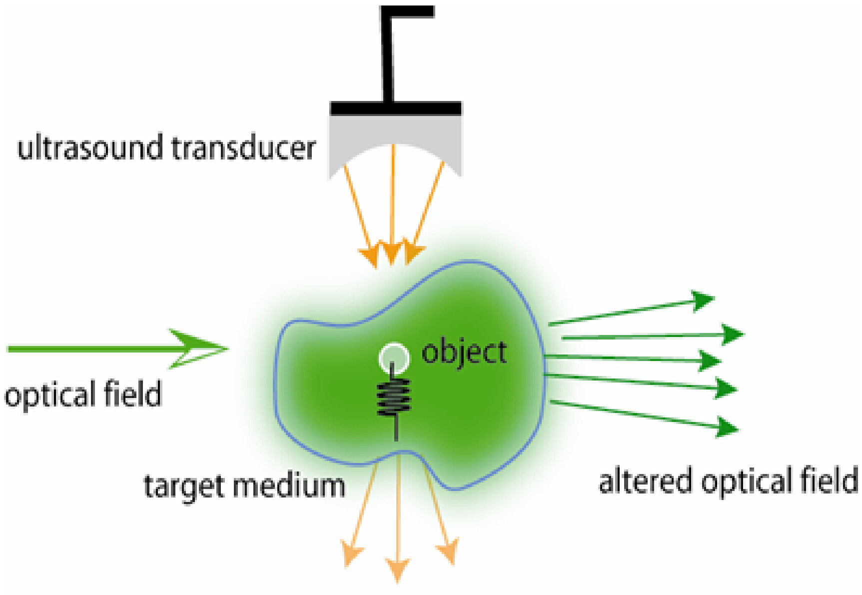
After modeling (à la MRI; see e.g., [B.-Schotland PRL'10]), the UMEIT and UMOT HIP take the form:

$$-\nabla \cdot \gamma(x)\nabla u + \sigma(x)u = 0 \text{ in } X \qquad u = g \text{ on } \partial X \qquad \textbf{Illumination,}$$

$$H(x) = \alpha_1 \gamma(x)|\nabla u|^2(x) + \alpha_2 \sigma(x)|u|^2(x) \quad \text{in } X \quad \textbf{Internal Functional}.$$

The objective is to *reconstruct* $\gamma(x)$ and $\sigma(x)$ from knowledge of **internal functionals** $H(x)$ for **one or several illuminations** $g(x)$ on ∂X.

Ultrasound Modulation



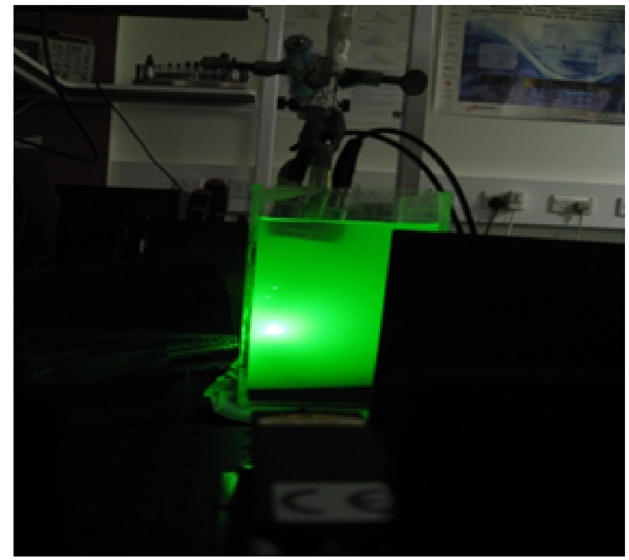


Figure 6: An illustrative diagram (left) and a photo(right) of our UOT system with a 532nm laser, a 5MHz ultrasound transducer, and a CCD camera.

Courtesy Dr. Tang, Imperial College, London.

Some references on Ultrasound Modulation

M. Kempe, M. Larionov, D. Zaslavsky, and A. Z. Genack, JOSA A, 14(5) (1997)

H. Zhang and L. Wang, *Acousto-electric tomography*, Proc. SPIE, 5320 (2004)

M. Atlan, B. C. Forget, F. Ramaz, A. C. Boccara, and M. Gross, Optics Letters, 30(11) (2005)

H. Ammari, E. Bonnetier, Y. Capdeboscq, M. Tanter, and M. Fink, SIAP, 68 (2008)

B. Gebauer and O. Scherzer, SIAM J. Applied Math., 69 (2009)

Y. Capdeboscq, J. Fehrenbach, F. de Gournay, and O. Kavian, SIIS, 2 (2009)

G. Bal and J. C. Schotland, Phys. Rev. Letters, 104 (2010)

F. Triki, Inverse Problems, 26 (2010)

P. Kuchment and L. Kunyansky, *2D and 3D reconstructions in AET*, 2011

G. Bal, E. Bonnetier, F. Monard, and F. Triki, *Inverse diffusion from knowledge of power densities*, (2011); also Monard & Bal, (2011).

G. Bal, *Cauchy problem and Ultrasound modulated EIT*, (2011)

O. Scherzer, *Handbook of Mathematical Methods in Imaging*, Springer, 2011

Solutions to HIP: a Roadmap

HIP starts with *unknown* coefficients, *unknown* elliptic solutions for *known* (elliptic) models, and *known* internal functionals.

1. We **eliminate** unknowns to focus on one.

2. **IF** some **qualitative** properties of elliptic solutions are satisfied, then we obtain **unique** and **stable** reconstructions for *some* coefficients.

3. We verify the **IF** for **well-chosen** illuminations g. Typically done by means of CGO solutions in dimension $n \geq 3$.

QPAT (and MRE/TE) with two/more measurements

$$-\nabla\cdot\gamma(x)\nabla u+\sigma(x)u=0 \text{ in } X,\quad u=g \text{ on } \partial X,\quad H(x)=\Gamma(x)\sigma(x)u(x).$$

Let (g_1,g_2) providing (H_1,H_2). Define $\beta=H_1^2\nabla\frac{H_2}{H_1}$. **IF:** $|\beta|\geq c_0>0$, then

Theorem[B.-Uhlmann'10, B.-Ren'11] (i) (H_1,H_2) uniquely determine the whole measurement operator $g\in H^{\frac{1}{2}}(\partial X)\mapsto\mathcal{H}(g)=H\in H^1(X)$.
(ii) The measurement operator $\mathcal{H}$ uniquely determines

$$\chi(x):=\frac{\sqrt{\gamma}}{\Gamma\sigma}(x),\qquad q(x):=-\Big(\frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}}+\frac{\sigma}{\gamma}\Big)(x).$$

(iii) (χ,q) uniquely determine (H_1,H_2).

Two well-chosen measurements suffice to reconstruct (χ,q) and thus (γ,σ,Γ) up to transformations leaving (χ,q) invariant.

Quantitative PAT, transport, and diffusion

The proof of (i) & (ii) is based on the *elimination* of σ to get

$$-\nabla\cdot\chi^2\Big[H_1^2\nabla\frac{H}{H_1}\Big]=0 \text{ in } X \quad (\chi,H) \text{ known on } \partial X.$$

$$\text{Then we verify that } q:=-\Big(\frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}}+\frac{\sigma}{\gamma}\Big)(x)=-\frac{\Delta(\chi H_1)}{\chi H_1}.$$

$$\text{(iii)} \quad \text{Finally, define } (\Delta+q)v_j=0 \text{ to get } H_j=\frac{v_j}{\chi}.$$

The **IF** implies that **vector field** $H_1^2\nabla\frac{u_2}{u_1}\neq 0$. This is a qualitative statement on the absence of critical points of elliptic solutions.

Theorem[B.-Ren'11] When *one* coefficient in (γ,σ,Γ) is known, then the other two are **uniquely** determined by the two measurements (H_1,H_2).

Stability of the reconstruction

Assuming **IF** satisfied, then the reconstruction of (e.g.) χ is **stable**.

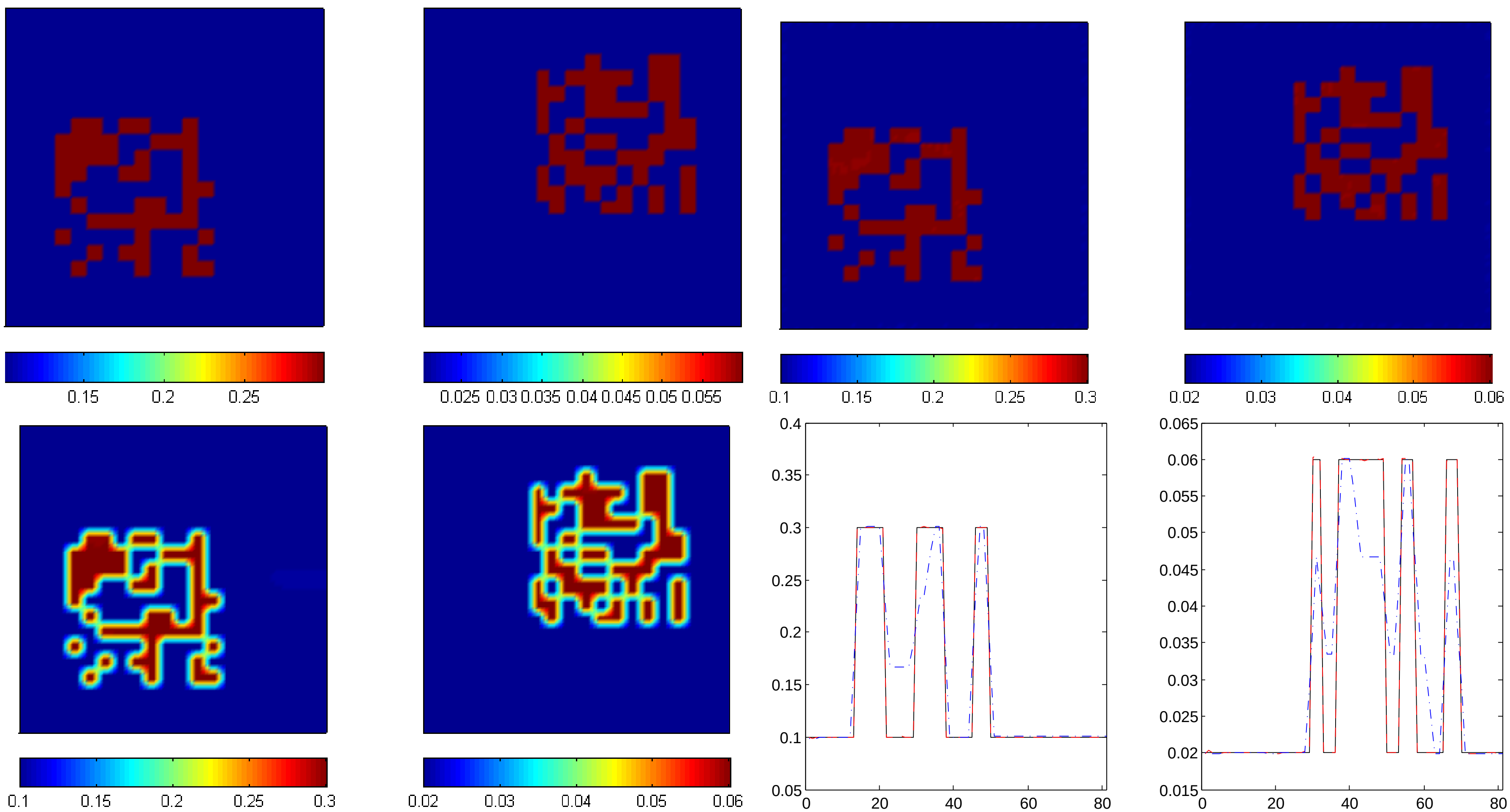
CGO method. Analyzing the transport equation by the method of characteristics and using CGO solutions, we show that for appropriate illuminations (and for $k \geq 3$):

$$\|\chi - \tilde{\chi}\|_{C^{k-1}(X)} \leq C\|H - \tilde{H}\|_{(C^k(X))^2}.$$

Transport method. Analyzing the transport equation directly and the renormalization property ($\varphi(\rho)$ satisfies a transport equation when ρ does) we obtain under appropriate regularity assumptions that

$$\|\chi - \tilde{\chi}\|_{L^\infty(X)} \leq C\|H - \tilde{H}\|_{(L^{\frac{p}{2}}(X))^2}^{\frac{p}{3(n+p)}}, \qquad \text{for all } 2 \leq p < \infty.$$

Reconstruction of two discontinuous parameters



Stability result for QTAT

$$\Delta u + k^2 u + i\sigma(x)u = 0 \text{ in } X, \quad u = g \text{ on } \partial X, \quad H(x) = \sigma(x)|u|^2.$$

Theorem [B.,Ren,Uhlmann,Zhou'11] Let σ and $\tilde{\sigma}$ be uniformly bounded functions in $Y = H^p(X)$ for $p > n$ with X the bounded support of the unknown conductivity.

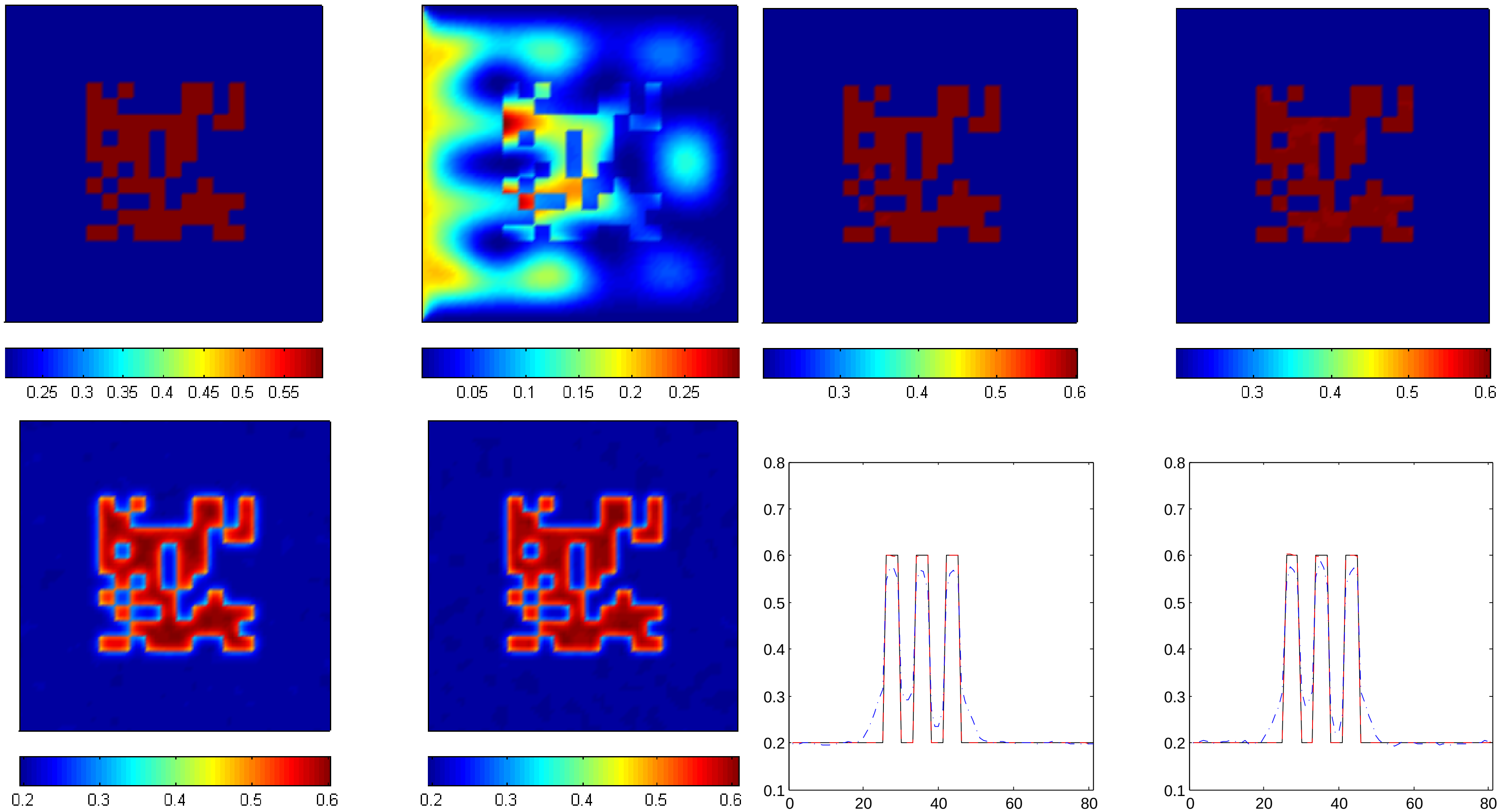
Then there is an **open set of illuminations** g such that

$$H(x) = \tilde{H}(x) \text{ in } Y \qquad \text{implies that} \qquad \sigma(x) = \tilde{\sigma}(x) \text{ in } Y.$$

Moreover, there exists C such that $\boxed{\|\sigma - \tilde{\sigma}\|_Y \leq C\|H - \tilde{H}\|_Y.}$

The **inverse scattering problem with internal data** is **well posed**. We apply a Banach fixed point **IF** appropriate functional is a contraction.

Discontinuous conductivity in TAT



UMEIT and the 0−Laplacian

$$-\nabla\cdot\gamma(x)\nabla u = 0 \text{ in } X,\quad u = g \text{ on } \partial X, \qquad H(x) = \gamma(x)|\nabla u|^2(x).$$

The elimination of γ is straightforward and yields the 0-Laplace equation

$$-\nabla\cdot\frac{H(x)}{|\nabla u|^{2-p}}\nabla u = 0 \text{ in } X, \qquad u = g \text{ on } \partial X, \qquad p = 0.$$

For $1 < p < \infty$, the above problem is **elliptic** and associated to the strictly convex functional $J(x) = \int_X H(x)|\nabla u|^p dx$. When $p = 1$ (with applications in the HIP: CDII and MREIT), the problem is **degenerate elliptic**.

For $p < 1$, or $p = 0$ as in UMEIT, the problem is **hyperbolic**. We thus modify the HIP. and assume that the current $j = \partial_\nu u$ is also known. This becomes a **0-Laplacian with Cauchy data**.

Nonlinear Hyperbolic Problem

The above equation may be transformed as

$$(I-2\widehat{\nabla u}\otimes\widehat{\nabla u}) : \nabla^2 u+\nabla \ln H\cdot\nabla u = 0 \ \text{ in } X, \qquad u=f \ \text{ and } \ \frac{\partial u}{\partial \nu}=j \ \text{ on } \partial X.$$

Here $\widehat{\nabla u}=\frac{\nabla u}{|\nabla u|}$. With

$$g^{ij}=g^{ij}(\nabla u)=-\delta^{ij}+2(\widehat{\nabla u})_i(\widehat{\nabla u})_j \quad \text{ and } \quad k^i=-(\nabla \ln H)_i,$$

the above equation is in coordinates

$$g^{ij}(\nabla u)\partial^2_{ij}u+k^i\partial_i u=0 \ \text{ in } X, \qquad u=f \ \text{ and } \ \frac{\partial u}{\partial \nu}=j \ \text{ on } \partial X.$$

Here g^{ij} is a definite matrix of signature $(1,n-1)$ so that we have quasilinear strictly hyperbolic equation with $\widehat{\nabla}u(x)$ the "time" direction. Stable Cauchy data must be on "space-like" part of ∂X for the *metric* g.

Stability on domain of influence

Let u and $\tilde{u}$ be two solutions of the hyperbolic equation and $v = u - \tilde{u}$.

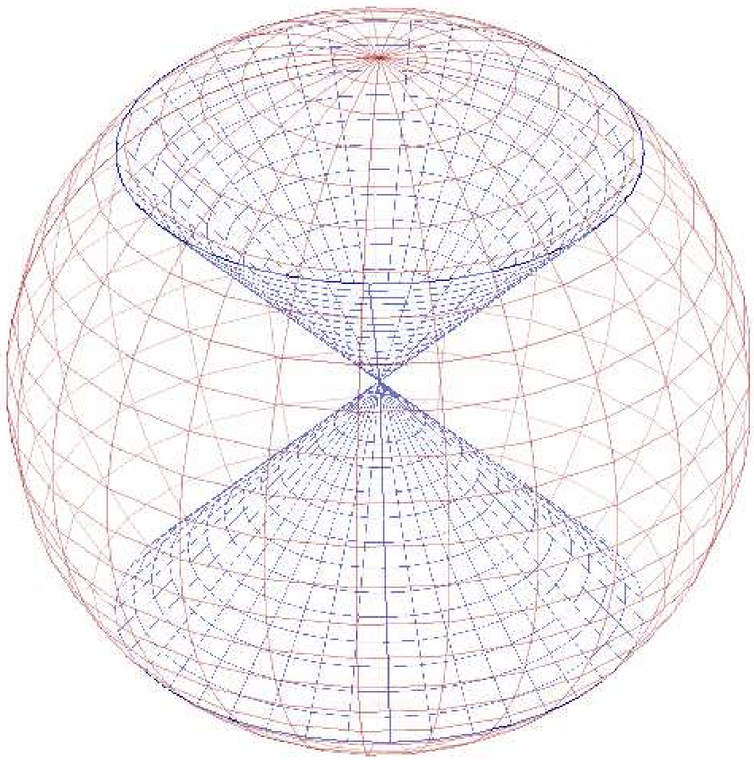
IF (appropriate) Lorentzian metric is strictly hyperbolic, then:

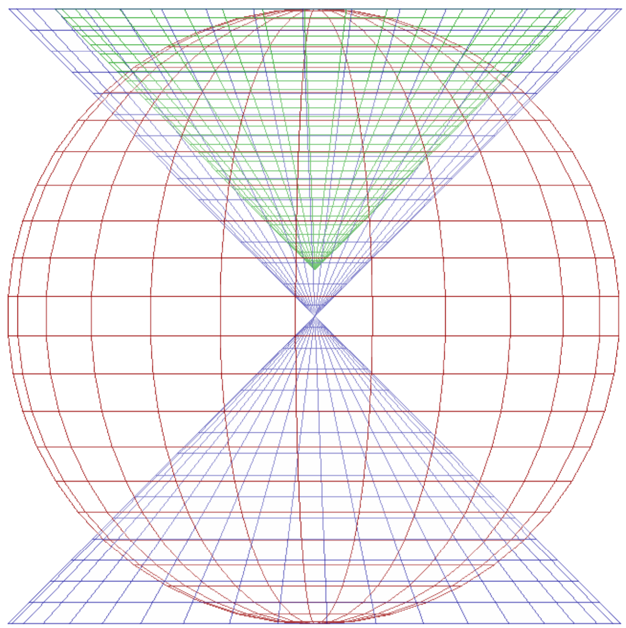
Theorem [B.'11]. Let $\Sigma_1 \subset \Sigma_g$ the space-like component of ∂X and $\mathcal{O}$ the **domain of influence** of Σ_1. For θ the distance of $\mathcal{O}$ to the boundary of the domain of influence of Σ_g, we have the local stability result:

$$\int_{\mathcal{O}} |v^2| + |\nabla v|^2 + (\gamma - \tilde{\gamma})^2 \, dx \le \frac{C}{\theta^2} \Big(\int_{\Sigma_1} |f - \tilde{f}|^2 + |j - \tilde{j}|^2 \, d\sigma + \int_{\mathcal{O}} |\nabla \delta H|^2 \, dx \Big),$$

where $\gamma = \frac{H}{|\nabla u|^2}$ and $\tilde{\gamma} = \frac{\tilde{H}}{|\nabla \tilde{u}|^2}$ are the reconstructed conductivities.

Domain of Influence





Domain of influence (blue) for metric $g = 2e_z \otimes e_z - I$ on sphere (red). Null-like vectors (surface of cone) generate instabilities. Right: Sphere (red), domains of uniqueness (blue) and with controlled stability (green).

Multiple Measurement UMEIT

$$-\nabla\cdot\gamma(x)\nabla u_j = 0 \;\; X, \quad u_j = g_j \;\; \partial X, \quad H_{ij}(x) = \gamma(x)^{2\alpha}\nabla u_i\cdot\nabla u_j(x), \;\; 1 \le i,j \le J.$$

Reconstructions in UMEIT ($\alpha = \frac{1}{2}$) are obtained by acquiring **redundant** *internal functionals* $H_{ij} = S_i \cdot S_j(x)$ with $S_i(x) = \gamma^\alpha \nabla u_i(x)$. Then

$$\nabla\cdot S_j = (\alpha-1)F\cdot S_j, \qquad dS_j^\flat = \alpha F^\flat \wedge S_j^\flat, \qquad 1 \le j \le J, \qquad F = \nabla(\log\gamma).$$

Strategy: (i) *Eliminate* F and find closed-form equation for $S = (S_1|\ldots|S_n)$ or equivalently for the $SO(n;\mathbb{R})$-valued matrix $R = H^{-\frac{1}{2}}(S_1|\ldots|S_n)$.
(ii) Solve for the redundant system of ODEs for S or R.

Works **IF** H is invertible in $\mathcal{M}(n;\mathbb{R})$, i.e., $\det(\nabla u_1,\ldots,\nabla u_n) \neq 0$. This qualitative property on elliptic solutions holds for well-chosen $\{g_j\}$.

Elimination and system of ODEs in UMEIT

Lemma [B.-Bonnetier-Monard-Triki'11; Monard-B.'11]. Let $\Omega \subset X$.
IF $\inf_{x\in\Omega} \det(S_1(x),\dots,S_n(x)) \geq c_0 > 0$, then with $D(x) = \sqrt{\det H(x)}$,

$$F(x) = \frac{c_F}{D} \sum_{i,j=1}^{n} \Big(\nabla(DH^{ij}) \cdot S_i(x)\Big) S_j(x),\ c_F = \frac{1}{1+(n-2)\alpha},\ H^{-1} = (H^{ij}).$$

Theorem [idem]. There exists an open set of illuminations g_j for $J = n$ in even dimension and $J = n+1$ in odd dimension such that for γ and γ' the conductivities corresponding to H and H', we have the following **global stability** result:

$$\|\log\gamma - \log\gamma'\|_{W^{1,\infty}(X)} \leq C\left(\varepsilon_0 + \|H - H'\|_{W^{1,\infty}(X)}\right)$$

$$\varepsilon_0 = |\log\gamma(x_0) - \log\gamma'(x_0)| + \textstyle\sum_{i=1}^{J} \|S_i - S_i'\|.$$

The IFs and the CGOs

Several HIPs require to verify qualitative properties of elliptic solutions:

- the absence of critical points in QPAT (and ET and UMOT)
- the contraction of appropriate functionals in QTAT
- the hyperbolicity of a given Lorentzian metric in UMOT
- the linear independence of gradients of elliptic solutions in UMOT.

In dimension $n = 2$, critical points of elliptic solutions are *isolated*. This greatly simplifies the analysis of the above statements.

In dimension $n \geq 3$, the existence of open sets of **illuminations** g_j such that these properties hold is obtained by means of **CGO** solutions.

Vector fields and complex geometrical optics

- Take $\rho \in \mathbb{C}^n$ with $\rho \cdot \rho = 0$. Then $\Delta e^{\rho\cdot x} = 0$. For $u_j = e^{\rho_j \cdot x}$, $j = 1,2$:

$$\Im\Big(e^{-(\rho_1+\rho_2)\cdot x} u_1^2 \nabla \frac{u_2}{u_1}\Big) = \Im(\rho_2 - \rho_1),$$

is a constant vector field $2k$ for $\rho_1 = k + ik^\perp$ and $\rho_2 = \bar\rho_1$.

- Let $u_\rho(x) = e^{\rho\cdot x}\big(1 + \psi_\rho(x)\big)$ solution of $\Delta u_\rho + q u_\rho = 0$.

Theorem[B.-Uhlmann'10]. For q sufficiently smooth and $k \geq 0$, we have

$$|\rho| \|\psi_\rho\|_{H^{\frac{n}{2}+k+\varepsilon}(X)} + \|\psi_\rho\|_{H^{\frac{n}{2}+k+1+\varepsilon}(X)} \leq C \|q\|_{H^{\frac{n}{2}+k+\varepsilon}(X)}.$$

- For **illuminations** g on ∂X close to *traces of* **CGO** *solutions* constructed in $\mathbb{R}^d$, we obtain "nice" vector fields $|\beta| \geq c_0 > 0$ and thus an open set of illuminations g for which stable reconstructions are guaranteed.

Conclusions

• Mathematically, many **hybrid imaging modalities** are stable inverse problems combining **high resolution** with **high contrast**.

• **Explicit** reconstructions for **one** or **several** coefficients are obtained by solving linear or nonlinear **transport**, **elliptic**, or **hyperbolic** equations or by using *Banach fixed point*. **Non-uniqueness** results exist.

• Reconstructions require qualitative properties of elliptic solutions. These properties hold true for appropriate **illuminations** constructed by means of **Complex Geometric Optics** solutions.