Pseudorandom Sequences I: Linear Complexity and Related Measures

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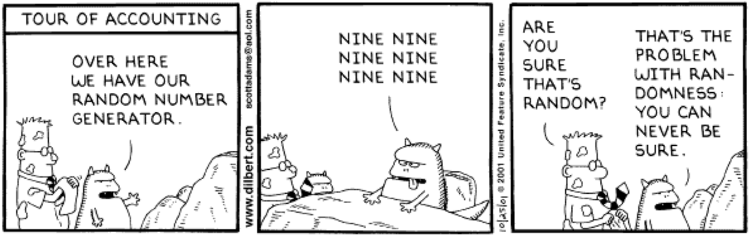
Why pseudorandom and not 'truly' random sequences?



Pseudorandom Sequences

Sequences which are generated by a deterministic algorithm and 'look random' are called pseudorandom.

Desirable 'randomness properties' depend on the application!

cryptography: unpredictability
numerical integration (quasi-Monte Carlo): uniform distribution
radar: distinction from reflected signal
gambling: a good lawyer

Linear Complexity

The linear complexity $L(s_n)$ of a periodic sequence (s_n) over a field $\mathbb{F}$ is the smallest positive integer L such that there are constants $c_0, \ldots, c_{L-1} \in \mathbb{F}$ with

$$s_{n+L} = c_{L-1}s_{n+L-1} + \ldots + c_0 s_n, \ \ n \geq 0.$$

For a positive integer N the Nth linear complexity $L(s_n, N)$ of a sequence (s_n) over $\mathbb{F}$ is the smallest positive integer L such that there are constants $c_0, \ldots, c_{L-1} \in \mathbb{F}$ satisfying

$$s_{n+L} = c_{L-1}s_{n+L-1} + \ldots + c_0 s_n,$$

$0 \leq n \leq N - L - 1.$

Cryptographic Background

A low Nth linear complexity has turned out to be undesirable for cryptographical applications as stream ciphers.

Example (Stream Cipher)

We consider a message $m_0, m_1, \dots$ represented as a sequence over $\mathbb{F}$. In a *stream cipher* each message symbol m_j is enciphered with an element x_j of another sequence $x_0, x_1, \dots$ over $\mathbb{F}$, the *key stream*, by

$$c_j = m_j + x_j.$$

The cipher text $c_0, c_1, \dots$ can be deciphered by subtracting the key stream

$$m_j = c_j - x_j.$$

Relation to Quasi-Monte Carlo Methods

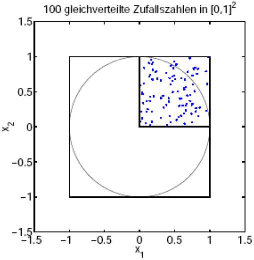
Sequences with low linear complexity are shown to be unsuitable for some applications using quasi-Monte Carlo methods as well. The following example describes a typical quasi-Monte Carlo application.

Example (Quasi-Monte-Carlo Calculation of π)

1. Choose N pairs of a sequence (x_n) in $[0, 1)$

$$(x_n, x_{n+1}) \in [0, 1)^2, \quad n = 0, \ldots, N-1.$$

2. Count the number K of pairs (x_n, x_{n+1}) in the unit circle.
3. Approximate π by $\frac{4K}{N}$.



k	π
10	3.2000000
100	2.9600000
1000	3.2040000
10000	3.1196000
100000	3.1390000
1000000	3.1460440
10000000	3.1416592

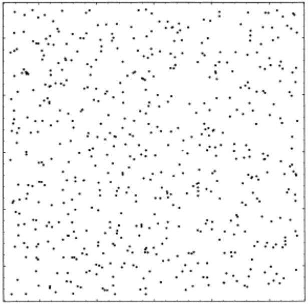
Marsaglia's Lattice test, 1972

(η_n) T-periodic sequence over $\mathbb{F}_p$
For $s \geq 1$ we say that (η_n) passes the s-dimensional lattice test if the vectors $\{\mathbf{u}_n - \mathbf{u}_0 : 1 \leq n < T\}$ span $\mathbb{F}_p^s$, where

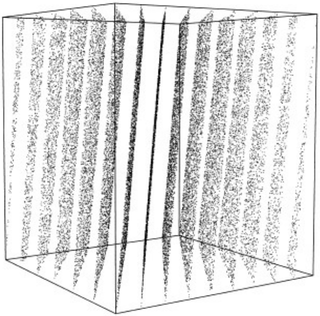
$$\mathbf{u}_n = (\eta_n, \eta_{n+1}, \dots, \eta_{n+s-1}), \quad 0 \leq n < T.$$

$$S(\eta_n) = \max\left\{s : \langle \mathbf{u}_n - \mathbf{u}_0,\ 1 \leq n < T\rangle = \mathbb{F}_p^s\right\}$$

$s = 2$:



$s = 3$:



Niederreiter/W., 2002: $L(\eta_n) = S(\eta_n)$ or $= S(\eta_n) + 1$

Dorfer/W., 2003: Lattice test for parts of the period, $S(\eta_n, N)$
We have either

$$S(\eta_n, N) = \min(L(\eta_n, N), N + 1 - L(\eta_n, N))$$

or

$$S(\eta_n, N) = \min(L(\eta_n, N), N + 1 - L(\eta_n, N)) - 1.$$

Relation to Information and Coding Theory

The Kolmogorov complexity of a sequence over $\mathbb{F}$ is the length of a shortest Turing machine that generates it.

Beth/Dai, 1989: $\mathbb{F} = \mathbb{F}_2$: Linear complexity and Kolmogorov complexity are the same for almost all binary sequences.

In general there is no algorithm for calculating the Kolmogorov complexity.
In contrast we have the Berlekamp-Massey Algorithm for calculating the linear complexity.
This algorithm stems from coding theory.

A Consequence of the Berlekamp-Massey Algorithm

Theorem

If $L(s_n, N) > N/2$, *then we have*

$$L(s_n, N+1) = L(s_n, N).$$

If $L(s_n, N) \leq N/2$, *then we have either*

$$L(s_n, N+1) = L(s_n, N)$$

or

$$L(s_n, N+1) = N + 1 - L(s_n, N).$$

The Expected Value

$\mathbb{F} = \mathbb{F}_q$

Theorem

The expected value for $L(s_n, N)$ *is*

$$\begin{cases} \frac{N}{2} + \frac{q}{(q+1)^2} - q^{-N}\frac{N(q+1)+q}{(q+1)^2} & \text{for even } N, \\ \frac{N}{2} + \frac{q^2+1}{2(q+1)^2} - q^{-N}\frac{N(q+1)+q}{(q+1)^2} & \text{for odd } N. \end{cases}$$

Lower Bounds

In case of a p-periodic sequence (ξ_n) over $\mathbb{F}_p$, where p is a prime, linear complexity is related to the degree of the polynomial $g(X) \in \mathbb{F}_p[X]$ representing the sequence (ξ_n), i.e., $g(X)$ is the unique polynomial which satisfies $\deg g \leq p-1$ and

$$\xi_n = g(n), \quad 0 \leq n \leq p-1.$$

These sequences are called *explicit nonlinear congruential generators* and we have

$$L(\xi_n) = \deg g + 1.$$

High linear complexity but low Nth linear complexity

Example:

$$\xi_n = 1 - (n+1)^{p-1}, \quad 0 \le n \le p-1$$

$$(\xi_0, \xi_1, \ldots, \xi_{p-2}, \xi_{p-1}) = (0, 0, \ldots, 0, 1)$$

$$L(\xi_n) = p$$

$$L(\xi_n, N) = 0, \quad 1 \le N \le p-1$$

highly predictable

The *explicit inversive congruential generator* (z_n) is produced by the relation

$$z_n = (an+b)^{p-2}, \quad n = 0, \ldots, p-1, \quad z_{n+p} = z_n, \; n \geq 0,$$

with $a, b \in \mathbb{F}_p$, $a \neq 0$, and $p \geq 5$. We have

$$L(z_n, N) \geq \begin{cases} (N-1)/3, & 1 \leq N \leq (3p-7)/2, \\ N-p+2, & (3p-5)/2 \leq N \leq 2p-3, \\ p-1, & N \geq 2p-2. \end{cases}$$

$c_L = -1$, $N \le p$

$$\sum_{l=0}^{L} c_l z_{n+l} = 0, \quad 0 \le n \le N - L - 1$$

$a(n+l) + b \ne 0$, $0 \le l \le L$:

$$\sum_{l=0}^{L} c_l (a(n+l)+b)^{-1} = 0$$

$$F(X) = \sum_{l=0}^{L} c_l \prod_{\substack{j=0 \\ j \ne L}}^{L} (a(X+j)+b)$$

has at least $N - L - (L+1)$ zeros and degree at most L.

$$F(-a^{-1}b - L) = c_L \prod_{j=0}^{L-1} (a(j-L)) \ne 0$$

$N - 2L - 1 \le L$ and thus $L \ge (N-1)/3$

$z_n = (an + b)^{p-2}$ is still highly predictable since inversion is cheap and $a = z_{n+1}^{-1} - z_n^{-1}$ for all but two n.

Open problem: Define a modified linear complexity with inversions and analyze it.

Let $p > 2$ be a prime. The *Legendre-sequence* (l_n) is defined by

$$l_n = \begin{cases} 1, & \left(\frac{n}{p}\right) = -1, \\ 0, & \text{otherwise}, \end{cases} \qquad n \geq 0,$$

where $\left(\frac{\cdot}{p}\right)$ is the Legendre-symbol.

Theorem

The linear complexity of the Legendre sequence is

$$L(l_n) = \begin{cases} (p-1)/2, & p \equiv 1 \bmod 8, \\ p, & p \equiv 3 \bmod 8, \\ p-1, & p \equiv 5 \bmod 8, \\ (p+1)/2, & p \equiv 7 \bmod 8. \end{cases}$$

Theorem

The Nth linear complexity of the Legendre sequence satisfies

$$L(l_n, N) > \frac{\min\{N, p\}}{1 + p^{1/2}(1 + \log p)} - 1, \quad N \geq 1.$$

Weil:

Let $f(X) \in \mathbb{F}_p[X]$ a (monic) polynomial which is not a square and $a \in \mathbb{F}_p^*$ then we have

$$\left| \sum_{x \in \mathbb{F}_p} \left(\frac{af(x)}{p} \right) \right| \leq (\deg(f) - 1)p^{1/2}.$$

$$\sum_{k=0}^{L} c_k l_{n+k} = 0 \in \mathbb{F}_2, \quad 0 \leq n \leq N - L - 1, \quad (c_L = 1)$$

$$(-1)^{l_n} = \left(\frac{n}{p} \right), \quad n \neq 0,$$

$$(-1)^{\sum_{k=0}^{L} c_k l_{n+k}} = \left(\frac{\prod_{l=0}^{L} (n+l)^{c_l}}{p} \right) = 1$$

for at least $\min\{N, p\} - (L+1)$ different n

Summing over n:

$$\min\{N, p\} - (L+1) \le \sum_{n=0}^{N-1} \left(\frac{\prod_{l=0}^{L} (n+l)^{c_l}}{p} \right)$$

$$< (L+1)p^{1/2}(1 + \log p)$$

Relation to Wireless Communication

The correlation measure of order k of a binary sequence (s_n) is introduced as

$$C_k(s_n) = \max_{M,D} \left| \sum_{n=1}^{M} (-1)^{s_{n+d_1}} \cdots (-1)^{s_{n+d_k}} \right|, \quad k \geq 1,$$

where the maximum is taken over all $D = (d_1, d_2, \ldots, d_k)$ with non-negative integers $d_1 < d_2 < \cdots < d_k$ and M such that $M - 1 + d_k \leq T - 1$.

$$L(s_n, N) \geq N - \max_{1 \leq k \leq L(s_n,N)+1} C_k(s_n), \quad 2 \leq N \leq t - 1.$$

Examples.

a) $b_n = 0$, $0 \leq n \leq t-2$, $b_{t-1} = 1$
$L(b_n) = t$, one change $L(b'_n) = 0$

b) $c_{n+4} = c_n$, $n \geq 0$, with $c_0 = c_1 = c_2 = 1$, $c_3 = 0$
over $\mathbb{F}_2$: $L(c_n) = 4$
over $\mathbb{F}_3$: $L(c_n) = 3$ since $c_{n+3} = 2c_{n+2} + 2c_{n+1} + 2c_n$, $n \geq 0$

Desirable:

1. high linear complexity even if we change a few elements
2. high linear complexity over different fields

Let (s_n) be a sequence over $\mathbb{F}$, with period t. The *k-error linear complexity* $L_k(s_n)$ of (s_n) is defined as

$$L_k(s_n) = \min_{(y_n)} L(y_n),$$

where the minimum is taken over all t-periodic sequences (y_n) over $\mathbb{F}$, for which the Hamming distance of the vectors $(s_0, s_1, \ldots, s_{t-1})$ and $(y_0, y_1, \ldots, y_{t-1})$ is at most k.

Theorem

Let $L_k(l_n)$ denote the k-error linear complexity over $\mathbb{F}_p$ of the Legendre sequence (l_n). Then,

$$L_k(l_n) = \begin{cases} p, & k = 0, \\ (p+1)/2, & 1 \le k \le (p-3)/2, \\ 0, & k \ge (p-1)/2. \end{cases}$$

$$l_n = 2^{-1}(n^{p-1} - n^{(p-1)/2}) \in \mathbb{F}_p, \quad n \geq 0$$

$$l_n = 2^{-1}(1 - n^{(p-1)/2}) =: h(n), \quad n \neq 0$$

$l_n = f(n)$ implies $L(l_n) = \deg(f) + 1$
Let (y_n) be obtained from (l_n) by at most k changes.
Case I: $(y_n) = (l_n)$: $L(y_n) = p$
Case II: $(y_n) = h(n)$: $L(y_n) = (p+1)/2$
Case III: $y_n = g(n)$, $g(X) \neq h(X)$

$$\deg(g - h) \geq p - k - 1 \geq (p+1)/2 \quad \text{if } k \leq (p-3)/2$$

Shparlinski/W.,2006: linear complexity over $\mathbb{F}_k$, k prime:

$$L(l_n) \geq \frac{1}{2\log k}\min\left\{\frac{p}{p^{1/2}\log p+2}-1, 2k-1\right\}.$$

Open Problem

Find more sequences with high (Nth, k-error) linear complexity.

For example, study recursive sequences.

Linear Pseudorandom Number Generators

$\mathbb{F}_q$ finite field of q elements, $a, b, x_0 \in \mathbb{F}_q$, $a \neq 0$

$$\boxed{x_{n+1} = ax_n + b, \quad n \geq 0}$$

$q = p$ prime, $\mathbb{F}_p = \{0, 1, \ldots, p-1\}$: $y_n = x_n/p \in [0, 1)$, $n \geq 0$

Nice features:
– long period can be easily obtained
– uniform distribution in dimension 1
flaws:
– predictable ($L(x_n) \leq 2$)
– coarse structure

Nonlinear Pseudorandom Numbers

$f \in \mathbb{F}_q[X]$, $2 \leq \deg(f) \leq q-1$, $x_0 \in \mathbb{F}_q$

$$\boxed{x_{n+1} = f(x_n), \quad n \geq 0}$$

(purely) periodic with period $t \leq q$
$q = p$ prime: $y_n = x_n/p \in [0,1)$

Lower Bound on the Linear Complexity Profile

Gutierrez/Shparlinski/W., 2003:
The linear complexity profile of a nonlinear sequence (x_n) defined by

$$x_{n+1} = f(x_n), \quad n = 0, 1, \ldots,$$

with a polynomial $f \in \mathbb{F}_q[X]$ of degree $d \geq 2$, purely periodic with period t, satisfies

$$L(x_n, N) \geq \min \left\{ \lceil \log_d(N - \lfloor \log_d N \rfloor) \rceil, \lceil \log_d t \rceil \right\}.$$

Proof.
$F_0(X) := X, \quad F_i(X) := F_{i-1}(f(X)),\ i \geq 1$
$\deg(F_i) = d^i$, $x_{n+j} = F_j(x_n)$
$x_{n+L} = a_{L-1}x_{n+L-1} + \ldots + a_0 x_n$,
$0 \leq n \leq N - L - 1$

$$F(X) := -F_L(X) + a_{L-1}F_{L-1}(X) + \ldots + a_0 F_0(X)$$

has degree d^L and at least $\min\{N - L, t\}$ zeros, namely, x_n with $0 \leq n \leq \min\{N - L - 1, t - 1\}$.

$$d^L \geq \min\{N - L, t\}$$

□

Inversive Generators

$a, b, y_0 \in \mathbb{F}_q$, $a \neq 0$

$$y_{n+1} = ay_n^{q-2} + b = \begin{cases} ay_n^{-1} + b, & y_n \neq 0, \\ b, & y_n = 0. \end{cases}$$

Gutierrez/Shparlinski/W., 2003:

$$L(y_n, N) \geq \min\left\{ \left\lceil \frac{N-1}{3} \right\rceil, \left\lceil \frac{t-1}{2} \right\rceil \right\}$$

Reason for better result:
$f(X) = \frac{bX+a}{X}$, $F_j(X) = \frac{a_jX+b_j}{c_jX+d}$

Dickson and Power Generator

The Dickson polynomial $D_e(X, a) \in \mathbb{F}_q[X]$ is defined by the following recurrence relation

$$D_e(X, a) = XD_{e-1}(X, a) - aD_{e-2}(X, a), \quad e = 2, 3, \ldots,$$

with initial values

$$D_0(X, a) = 2, \quad D_1(X, a) = X,$$

where $a \in \mathbb{F}_q$. Obviously, the degree of D_e is e. Moreover, if $a \in \{0, 1\}$ then we have $D_e(D_f(X, a), a) = D_{ef}(X, a)$.

$a = 0$:
$D_e(X, 0) = X^e$, $e \geq 2$

$$p_{n+1} = p_n^e, \quad n \geq 0$$

power generator
Griffin/Shparlinski, 2000: ($q = p$ prime)

$$L(p_n, N) \geq \min\left\{\frac{N^2}{4(p-1)}, \frac{t^2}{p-1}\right\}, \quad N \geq 1.$$

Reason for better result: $F_k(X) = X^{e^k \bmod p-1}$

$a = 1$:

$$D_e(x + x^{-1}, 1) = x^e + x^{-e}, \quad x \in \mathbb{F}_{q^2}$$

$$u_{n+1} = D_e(u_n, 1), \quad n \geq 0,$$

with some initial value u_0 and $e \geq 2$.
Dickson generator

Aly/W., 2006:

$$L(u_n, N) \geq \frac{\min\{N^2, 4t^2\}}{16(p+1)} - (p+1)^{1/2}$$

Redéi generator

Suppose that

$$r(X) = X^2 - \alpha X - \beta \in \mathbb{F}_p[X]$$

is an irreducible quadratic polynomial with the two different roots ξ and $\zeta = \xi^p$ in $\mathbb{F}_{p^2}$. Then any polynomial $b(X) \in \mathbb{F}_{p^2}[X]$ can uniquely be written in the form $b(X) = g(X) + h(X)\xi$ with $g(X), h(X) \in \mathbb{F}_p[X]$. We consider the elements

$$(X + \xi)^e = g_e(X) + h_e(X)\xi.$$

e is the degree of the polynomial $g_e(X)$, and $h_e(X)$ has degree at most $e - 1$. The Rédei function $f_e(X)$ of degree e is then given by

$$f_e(X) = \frac{g_e(X)}{h_e(X)}.$$

$$u_{n+1} = f_e(u_n), \quad n \geq 0,$$

with a Rédei permutation $f_e(X)$ and some initial element $u_0 \in \mathbb{F}_p$.

Meidl/W., 2007:

$$L(u_n, N) \geq \frac{\min\{N^2, 4t^2\}}{20(p+1)^{3/2}}, \quad N \geq 2.$$

p-Weight Degree

n nonnegative integer
p-weight of n:

$$\sigma_p\left(\sum_{i=0}^{l} n_i p^i\right) = \sum_{i=0}^{l} n_i, \quad 0 \le n_i < p.$$

$0 \le e_1 < e_2 < \cdots < e_l$ integers, $q = p^r$, $f(X) = \sum_{i=1}^{l} \gamma_i X^{e_i} \in \mathbb{F}_q[X]$ nonzero polynomial over $\mathbb{F}_q$ with $\gamma_i \neq 0$, $i = 1, \ldots, l$
p-weight degree of f:

$$\mathrm{w}_p(f) = \max\{\sigma_p(e_i) : 1 \le i \le l\}.$$

$\mathrm{w}_p(f) \le \deg(f)$

If $g(X) \in \mathbb{F}_q[X]$ and $\{\beta_1, \ldots, \beta_r\}$ is a fixed ordered $\mathbb{F}_p$-basis of $\mathbb{F}_q$, we define

$$G(X_1, \ldots, X_r) = \mathrm{Tr}(g(X_1\beta_1 + \ldots + X_r\beta_r)),$$

where $\mathrm{Tr}(X) = X + X^p + \ldots + X^{p^{r-1}}$ is the absolute trace function of $\mathbb{F}_q$. Then the transformed polynomial $G_R(X_1, \ldots, X_r)$ of $g(X)$ is the unique polynomial with all local degrees smaller than p such that

$$G_R(X_1, \ldots, X_r) \equiv G(X_1, \ldots, X_r) \mod (X_1^p - X_1, \ldots, X_r^p - X_r).$$

The interest of this construction relies on the fact that, under certain assumptions, the total degree of $G_R(X_1, \ldots, X_r)$ coincides with the p-weight degree of $g(X)$.

$$f(X) = \alpha X^d + \tilde{f}(X) \in \mathbb{F}_q[X] \text{ with } \alpha \neq 0, \quad \mathrm{w}_p(\tilde{f}) < \sigma_p(d). \qquad (1)$$

If the sequence (x_n) given by $x_{n+1} = f(x_n)$, $n \geq 0$, with a polynomial $f(X) \in \mathbb{F}_q[X]$ of the form (1) satisfying

$$\gcd\left(d, \frac{q-1}{p-1}\right) \leq \sigma_p(d)^{r/2},$$

with p-weight degree $w = \sigma_p(d) > 1$, is purely periodic with period t, then for $N \geq 1$,

$$L(x_n, N) \geq \frac{\min\left\{\log(N/p^{r-1} - \log(N/p^{r-1})/\log w), \log(t/p^{r-1})\right\}}{\log w}.$$

(Ibeas/W., 2010)

Polynomial Systems

Let $\{F_1, \ldots, F_r\}$ be a system of $r \geq 2$ polynomials $F_i \in \mathbb{F}_q[X_i, \ldots, X_m],\ i = 1, \ldots, r$, defined in the following way:

$$\begin{aligned}
F_1(X_1, \ldots, X_r) &= X_1 G_1(X_2, \ldots, X_r) + H_1(X_2, \ldots, X_r),\\
F_2(X_1, \ldots, X_r) &= X_2 G_2(X_3, \ldots, X_r) + H_2(X_3, \ldots, X_r),\\
&\ldots\\
F_{r-1}(X_1, \ldots, X_r) &= X_{r-1} G_{r-1}(X_r) + H_{r-1}(X_r),\\
F_r(X_1, \ldots, X_r) &= g_r X_r + h_r.
\end{aligned}$$

Using the following vector notation

$$\vec{F} = (F_1(X_1, \ldots, X_r), \ldots, F_r(X_1, \ldots, X_r)),$$

we define the following vector sequence

$$\vec{w}_{n+1} = \vec{F}(\vec{w}_n), \quad n = 0, 1, \ldots.$$

Identifying the r dimensional vectors over $\mathbb{F}_q$ with elements of $\mathbb{F}_{q^r}$ we get

$$L(\vec{w}_n, N) \gg \frac{N^{1/(r-1)}}{q}, \qquad 1 \le N \le t.$$

(Ostafe, Shparlinski, W., 2010)

Open Problem

Find more good nonlinear generators.

Thank you for your attention.