EPIDEMIC MODELS I

REPRODUCTION NUMBERS
AND
FINAL SIZE RELATIONS

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THE SIR MODEL

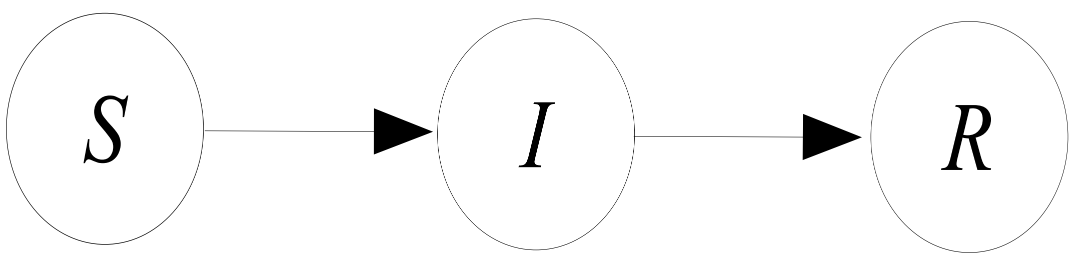
Start with simple SIR epidemic model

$$\begin{aligned} S' &= -\beta SI \\ I' &= (\beta S - \alpha)I \ , \end{aligned}$$

with initial conditions

$$S(0) = S_0, \quad I(0) = I_0, \quad S_0 + I_0 = N.$$

Flow chart.



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Integration gives

$$\alpha \int_0^\infty [(S(t)+I(t)]' dt = S_0 + I_0 - S_\infty = N - S_\infty$$

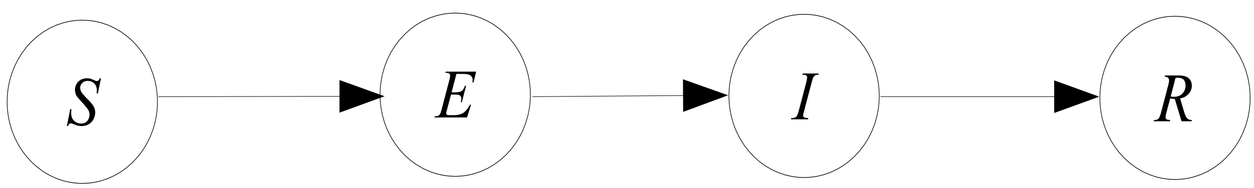
and

$$\begin{aligned} \ln \frac{S_0}{S_\infty} &= \beta \int_0^\infty I(t) dt \\ &= \frac{\beta}{\alpha}[N - S_\infty] \\ &= \mathcal{R}_0 \left[1 - \frac{S_\infty}{N}\right]. \end{aligned}$$

Generalize to $SEIR$ model

$$\begin{aligned} S' &= -\beta SI & S(0) &= S_0 \\ E' &= \beta SI - \kappa E & E(0) &= E_0 \\ I' &= \kappa E - \alpha I & I(0) &= I_0 \\ R' &= \alpha I & R(0) &= 0. \end{aligned}$$

Flow chart.



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Basic reproduction number is

$$\mathcal{R}_0 = \frac{\beta N}{\alpha}.$$

Final size relation is

$$\ln \frac{S_0}{S_\infty} = \mathcal{R}_0 \left[1 - \frac{S_\infty}{N} \right].$$

Generalize to $SEIR$ model with infectivity in exposed stage

$$\begin{aligned} S' &= -\beta S(I + \varepsilon E) & S(0) &= S_0 \\ E' &= \beta S(I + \varepsilon E) - \kappa E & E(0) &= E_0 \\ I' &= \kappa E - \alpha I & I(0) &= I_0 \\ R' &= \alpha I & R(0) &= 0. \end{aligned}$$

Basic reproduction number is

$$\mathcal{R}_0 = \frac{\beta N}{\alpha} + \frac{\varepsilon \beta N}{\kappa}.$$

Final size relation is

$$\ln \frac{S_0}{S_\infty} = \mathcal{R}_0 \left[1 - \frac{S_\infty}{N}\right] - \frac{\varepsilon \beta}{\kappa} I_0.$$

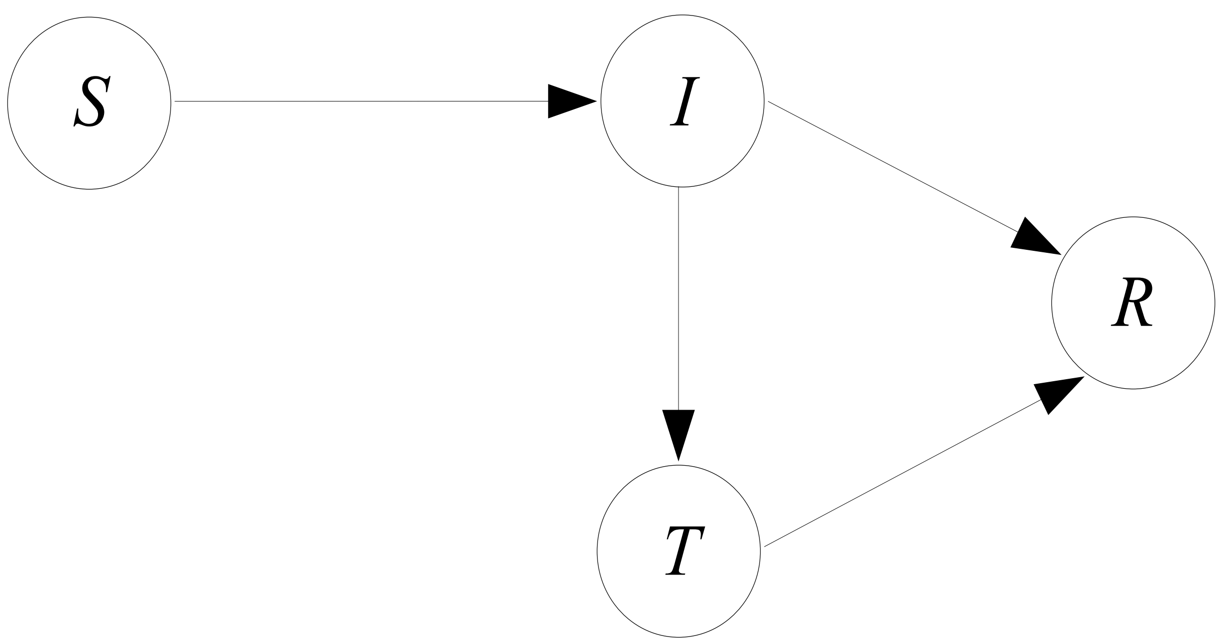
A SIMPLE TREATMENT MODEL

Now add treatment at a rate γ to the basic model. Assume

- treatment moves infectives to a class T with infectivity decreased by a factor δ and with a recovery rate η
- treatment continues so long as an individual remains infective.
- Treatment is beneficial,

$$\eta > \delta\alpha.$$

Flow chart.



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Model is

$$\begin{aligned} S' &= -\beta S(I+\delta T), & S(0) &= S_0 \\ I' &= \beta S(I+\delta T) - (\alpha+\gamma)I, & I(0) &= I_0 \\ T' &= \gamma I - \eta T, & T(0) &= 0. \end{aligned}$$

Integration of the first equation, the sum of the first two equations, and the third equation gives

$$\ln \frac{S_0}{S_\infty} = \mathcal{R}(\gamma)\left[1 - \frac{S_\infty}{N}\right].$$

The quantity

$$\mathcal{R}(\gamma) = \frac{\beta N}{\alpha+\gamma}\left[1 + \frac{\delta\gamma}{\eta}\right]$$

again represents the mean number of secondary infections caused by a single infective introduced into a fully susceptible population and is a decreasing function of γ if $\eta > \delta\alpha$.

THE AGE OF INFECTION MODEL

Let $S(t)$ denote the number of susceptibles at time t and $\varphi(t)$ the total infectivity at time t, and $\varphi_0(t)$ the total infectivity at time t of those individuals who were already infected at time $t = 0$. Let $A(\tau)$ be the total infectivity of members of the population with infection age τ.

Age of infection epidemic model is

$$\begin{aligned} S' &= -\beta S \varphi \\ \varphi(t) &= \varphi_0(t) + \int_0^t \beta S(t-\tau)\varphi(t-\tau)A(\tau)d\tau \\ &= \varphi_0(t) + \int_0^t [-S'(t-\tau)]A(\tau)d\tau. \end{aligned}$$

Basic reproduction number is

$$\mathcal{R}_0 = \beta N \int_0^\infty A(\tau)d\tau.,$$

and final size relation is

$$\ln \frac{S_0}{S_\infty} = \mathcal{R}_0 \left(1 - \frac{S_\infty}{N}\right).$$

EXAMPLE:
THE STAGED PROGRESSION EPIDEMIC

Consider an epidemic with progression from S (susceptible) through k infected stages $I_1, I_2, \cdots, I_k$ with the distribution of stay in stage i given by P_i, meaning that the fraction of infectives who enter stage i and are still in stage i a time τ after entering the stage is $P_i(\tau)$, with $P_i(0) = 1$, $\int_0^\infty P(t)dt < \infty$, and P_i non-negative and monotone non-decreasing.

Assume that in stage i the relative infectivity is ε_i. Then $S'(t) = -\beta S(t)\varphi(t)$ and the infectivity $\varphi(t)$ is

$$\varphi(t) = \sum_{i=1}^{k} \varepsilon_i I_i(t).$$

The basic reproduction number is

$$\mathcal{R}_0 = \beta N \sum_{i=1}^{k} \varepsilon_i \int_0^\infty P_i(t)dt,$$

General final size relation is

$$\ln \frac{S_0}{S_\infty} = \mathcal{R}_0 \left[1 - \frac{S_\infty}{N}\right] - \beta \int_0^\infty [(N - S_0)A(t) - \varphi_0(t)]dt.$$

Initial term satisfies

$$\int_0^\infty [(N - S_0)A(t) - \varphi_0(t)]dt \geq 0.$$

If all initial infectives have infection-age zero at $t = 0$, $\varphi_0(t) = [N - S_0]A(t)$, and

$$\int_0^\infty [\varphi_0(t) - (N - S_0)A(t)]dt = 0.$$

Then final size relation is

$$\ln \frac{S_0}{S_\infty} = \mathcal{R}_0 \left(1 - \frac{S_\infty}{N}\right).$$

If initial infectives are outside the population under study, $I_0 = 0$, and final size relation is

$$\ln \frac{S_0}{S_\infty} = \mathcal{R}_0 \left(1 - \frac{S_\infty}{S_0}\right).$$

INITIAL EXPONENTIAL GROWTH RATE

For the simple SIR model, if t is small, $S \approx N$, and the equation for I is approximately

$$I' = (\beta N - \alpha)I = (\mathcal{R}_0 - 1)\alpha I,$$

and solutions grow exponentially with growth rate $(\mathcal{R}_0 - 1)\alpha$. The exponential initial growth rate r can be measured (?), and then we have an estimate

$$\mathcal{R}_0 = 1 + \frac{r}{\alpha}.$$

More complicated models are approximated for small t by linear systems, whose solutions have an exponential growth rate given by the largest eigenvalue of the coefficient matrix. Thus for the $SEIR$ model, the initial exponential growth rate $r < \alpha(\mathcal{R}_0 - 1)$ is the (unique if $\mathcal{R}_0 > 1$) positive eigenvalue of

$$\begin{bmatrix} -\kappa & \beta N \\ \kappa & -\alpha \end{bmatrix}.$$

For the age of infection model, an epidemic means that the disease-free equilibrium, with $S = N$ and all infected variables zero is unstable. An epidemic means that the equilibrium $S = N, \varphi = 0$ is unstable To find equilibria, we need to use the limit equation

$$\begin{aligned} S' &= -\beta S \varphi \\ \varphi(t) &= \int_0^\infty \beta S(t-\tau)\varphi(t-\tau)A(\tau)d\tau \end{aligned}$$

to find equilibria.

The linearization at the equilibrium $S = N, \varphi = 0$ is

$$\begin{aligned} u'(t) &= -\beta N v(t) \\ v(t) &= \beta N \int_0^\infty v(t-\tau)A(\tau d\tau. \end{aligned}$$

The characteristic equation is the condition on λ that the linearization have a solution $u = u_0 e^{\lambda t}, v = v_0 e^{\lambda t}$, and this is just

$$\beta N \int_0^\infty e^{-\lambda\tau} A(\tau)d\tau = 1.$$

The initial exponential growth rate is the solution λ of this equation.

EPILOGUE

The deeper knowledge Faust sought
Could not from the Devil be bought
But now we are told
By theorists bold
That all you need is R naught.

- R.M.May (Lord May of Oxford)