Stochastic Models for Movements of Equities which Employ the Power-Variance Family

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The First Group of Models pertains to the case where logarithmic returns on equities either belong to the class of *Hougaard processes* or are represented as their difference. A comprehensive review is given in V (2002, 2004ab).

It appears that they could be interpreted as those *modelling market behavior near equilibrium* (see below).

Hougaard processes $\{X_{p,\mu,\lambda}(t),\ t \geq 0\}$ are specific univariate Lévy processes generated by the members of the power-variance family (PVF) of probability laws (3)–(7). The terminology goes back to Lee & Whitmore (1993). Namely,

$$X_{p,\mu,\lambda}(1) \stackrel{\mathrm{d}}{=} Tw_p(\mu, \lambda),$$

where the collection of r.v.'s

$$\{Tw_p(\mu, \lambda),\ p \in \Delta,\ \mu \in \Omega_p,\ \lambda \in \mathbf{R}^1_+\}$$

constitutes PVF. Hereinafter, $\mathbf{R}^1_+ := (0, \infty)$. The reason for using **PVF** acronym is because

$$\begin{aligned} \mathbf{Var}\, Tw_p(\mu, \lambda) &= \lambda^{-1} \cdot \mu^p \\ &= \lambda^{-1} \cdot (\mathbf{E}\, Tw_p(\mu, \lambda))^p. \end{aligned} \tag{1}$$

Scaling Property:

for each fixed real $c > 0$,

$$c^{-1} \cdot \mathrm{Tw}_p(c \cdot \mu, c^{p-2} \cdot \lambda) \stackrel{\mathrm{d}}{=} \mathrm{Tw}_p(\mu, \lambda).$$

Domain of the *power parameter*:

$$\Delta = (-\infty, 0] \cup [1, +\infty).$$

λ & $\mu \in \Omega_p$ are scaling & location parameters.

$$\Omega_p = \begin{cases} [0,\infty) & \text{if } p \in (-\infty, 0); \\ \mathbf{R}^1 & \text{if } p = 0; \\ \mathbf{R}^1_+ & \text{if } p \in [1,2]; \\ (0,\infty] & \text{if } p \in (2,\infty). \end{cases}$$

$\forall p \in \Delta \setminus \{1\}$, define the *exponential tilting* parameter

$$\theta(p,\mu,\lambda) := \frac{1}{|1-p|} \cdot \lambda \cdot \mu^{1-p}. \qquad (2)$$

$\forall p \in \Delta \setminus \{1;2\}$, set

$$B_{p,\lambda} := \frac{|1-p|^{(2-p)/(1-p)}}{|2-p|} \cdot \lambda^{1/(p-1)}.$$

The law of r.v. $Tw_p(\mu,\lambda)$ is characterized by its c.g.f. $\zeta_{p,\mu,\lambda}(\cdot)$.

For $p \in (-\infty, 0]$,

$$\zeta_{p,\mu,\lambda}(s) = B_{p,\lambda} \cdot \{(\theta(p,\mu,\lambda) + s)^{\frac{2-p}{1-p}} - \theta(p,\mu,\lambda)^{\frac{2-p}{1-p}}\}. \tag{3}$$

$$\zeta_{1,\mu,\lambda}(s) = \mu \cdot \lambda \cdot (e^{s/\lambda} - 1). \tag{4}$$

$$\zeta_{2,\mu,\lambda}(s) = -\lambda \cdot \log\ (1 - s/\theta(p,\mu,\lambda)). \tag{5}$$

For $p \in (1, 2)$,

$$\zeta_{p,\mu,\lambda}(s) = B_{p,\lambda} \cdot \{(\theta(p,\mu,\lambda) - s)^{\frac{2-p}{1-p}} - \theta(p,\mu,\lambda)^{\frac{2-p}{1-p}}\}. \tag{6}$$

If $p \in (2, +\infty)$,

$$\zeta_{p,\mu,\lambda}(s) = B_{p,\lambda} \cdot \{\theta(p,\mu,\lambda)^{\frac{2-p}{1-p}} - (\theta(p,\mu,\lambda) - s)^{\frac{2-p}{1-p}}\}. \tag{7}$$

Composition of the Family $\{X_{p,\mu,\lambda}(t)\}$:

$p = 0$ corresponds to the *scaled Brownian* *motion with a drift* that starts from the origin:

$$X_{0,\mu,\lambda}(t) \stackrel{\mathrm{d}}{=} \mu \cdot t + \lambda^{-1/2} \cdot \mathcal{B}_t.$$

$X_{1,\mu,\lambda}(t)$ is scaled Poisson process with intensity $\mu\lambda$ and magnitude of jumps $1/\lambda$. $X_{2,\mu,\lambda}(t)$ is gamma process.

Compound *Poisson-gamma* Hougaard processes correspond to $p \in (1,2)$. Poisson-exponential member is characterized by $p = 3/2$.

Poisson-exponential law has some importance in quantitative finance. See Yor et al (1994) or Chacko & Viceira (2003). See Hochberg & V (2009) for a review of this class.

Hougaard processes with $p \in (-\infty, 0) \cup (2, \infty)$ are Esscher transforms of spectrally negative extreme stable processes with skewness $\beta = -1$ & $\alpha(p) := (2-p)/(1-p) \in (1,2)$, and positive stable processes with skewness $\beta = 1$ & $\alpha(p) \in (0,1)$, respectively.

Fluctuation Property:

$\forall p \in (-\infty, 0]$, define first-passage time

$$\tau_u := \inf\{t : X_{p,\mu,\lambda}(t) > u\}.$$

Theorem 1 $\forall p \in (-\infty, 0)$, *Lévy process* $X_{p,\mu,\lambda}(t)$ *is spectrally negative.* $\forall b > 0$,

$$\tau_b \overset{\mathrm{d}}{=} Tw_{3-p}(b/\mu, b^{2-p} \cdot \lambda). \tag{8}$$

For $p = 0$, (8) coincides with the result by Schrödinger (1915) & Smoluchowsky (1915).

The marginals of Hougaard processes: $\forall t > 0$,

$$X_{p,\mu,\lambda}(t) \stackrel{\mathrm{d}}{=} Tw_p(\mu \cdot t, \lambda \cdot t^{p-1}). \quad (9)$$

Define an important *shape parameter*

$$\phi_p := \lambda \cdot \mu^{2-p}. \quad (10)$$

Definition 2 *Lévy process which corresponds to the generating triplet* (b, a, ν) *is said to be of*
(i*)* type $\mathbf{A}$ *if* $a = 0$ *and* $\nu(\mathbf{R}^1) < \infty$*;*
(ii*)* type $\mathbf{B}$ *if* $a = 0$*,* $\nu(\mathbf{R}^1) = \infty$*, and*

$$\mathcal{I}_\nu := \int_{|x|\le 1} |x| \cdot \nu(dx) < \infty;$$

(iii*)* type $\mathbf{C}$ *if* $a \neq 0$ *or* $\mathcal{I}_\nu = \infty$*.*

Theorem 3 *(i)* $\forall p \le 0$, $X_{p,\mu,\lambda}(t)$ *is of type* $\mathbf{C}$.
(ii) $\forall\ p \in [1, 2)$, $X_{p,\mu,\lambda}(t)$ *is of type* $\mathbf{A}$.
(iii) $\forall\ X_{p,\mu,\lambda}(t)$ *with* $p \in [2, \infty)$ *is of type* $\mathbf{B}$.

All the Hougaard processes have a lot in common. See (9) and the next property of the *linear growth* of shape parameter (10): $\forall t > 0$,

$$\phi_p^{(t)} := \lambda_t \cdot \mu_t^{2-p} = \lambda \cdot \mu^{2-p} \cdot t = \phi_p \cdot t. \quad (11)$$

Here, $\mu_t := \mu \cdot t$; $\lambda_t := \lambda \cdot t^{p-1}$ (see (9)).

Linear growth property (11) is quite remarkable. It is closely related to additivity of shape parameter (see V (2004b)) and to Cramér's, Raikov's and Cochran's theorems. Nevertheless, in view of Th. 3, typical trajectories of Hougaard processes exhibit a great diversity (which is good for modelling). This is because path properties of Lévy processes belonging to types **A**, **B**, **C** are *qualitatively* different.

If generic Lévy process X_t is of type $\mathbf{A}$ then its typical trajectory $X_t(\omega)$ performs *finite* number of jumps on each finite time interval.
If $\nu(\mathbf{R}^1) = \infty$ then the set of jump times

$$\mathcal{D}(\omega) := \{s > 0 : \ X_s(\omega) - X_{s-}(\omega) \neq 0\}$$

is a *countable* and *dense* subset of $[0, \infty)$.
If X_t is of type $\mathbf{A}$ or $\mathbf{B}$ then its typical trajectories have a bounded variation on each finite time interval. In contrast, the typical sample paths of any Lévy process of type $\mathbf{C}$ are functions of unbounded variation on any time interval. Suppose that $p \in (-\infty, 0]$ and $S_0 > 0$ is non-random initial price of stock. Define

geometric Hougaard process as

$$S_{p,\mu,\lambda}(t) \\ := S_0 \cdot \exp\{X_{p,\mu,\lambda}(t) - \zeta_{p,0,\lambda}(1) \cdot t\}. \tag{12}$$

Hereinafter, $S_{p,\mu,\lambda}(t)$ is interpreted as price of stock at time t. GBM corresponds to $p = 0$:

$$S_{0,\mu,\lambda}(t) = S_0 \cdot \exp\{X_{0,\mu,\lambda}(t) - \zeta_{0,0,\lambda}(1)t\} \\ \overset{\mathrm{d}}{=} S_0 \exp\{(\mu - 1/(2\lambda))t + \lambda^{-1/2}\mathcal{B}_t\}.$$

For this special case, it is more common to denote the reciprocal $1/\lambda$ of scaling parameter λ by σ^2, while σ is called the *volatility* of equity.

Rational Price for European Call Option:

Let T be time of option to maturity, $\mathrm{C}_{T,p}$ denote rational price of European call option that corresponds to price process $S_{p,\mu,\lambda}(t)$. This option may be exercised at time T. Assume that K is exercise price of this option & short-term interest rate $r \geq 0$ remains constant. Our approach relies on constructing risk-neutral measure by using *Esscher transform*. Set $\mu(\theta) := ((1-p)\cdot\theta/\lambda)^{1/(1-p)}$, which is obtained by solving (2) for θ. One can show that $\exists$ unique solution $\theta = \theta_{r,\lambda}(p) \in [0, +\infty)$ to next equation:

$$F(r, \lambda, p, \theta) := \zeta_{p,0,\lambda}(1) \cdot ((\theta + 1)^{\frac{2-p}{1-p}} - \theta^{\frac{2-p}{1-p}} - 1) - r = \log\ \mathbf{E}(S_{p,\mu(\theta),\lambda}(1)) - r = 0.$$

Set

$$m_1 := \mu_{r,\lambda}(p) := \mu(\theta_{r,\lambda}(p))$$
$$= ((1-p)\cdot\theta_{r,\lambda}(p)/\lambda)^{1/(1-p)};$$

$$m_2 := ((1-p)/\lambda)^{1/(1-p)} \cdot T^{1/(2-p)}$$
$$\cdot (1+\theta_{r,\lambda}(p))^{1/(1-p)};$$

$$L := L(K, S_0, T, \lambda, p)$$
$$= \frac{\log\ (K/S_0)}{T^{(1-p)/(2-p)}} + \zeta_{p,0,\lambda}(1)T^{1/(2-p)}.$$

Theorem 4 *Let price process for risky asset follow geometric Hougaard process $S_{p,\mu,\lambda}(t)$ (12) with $p \in (-\infty, 0]$. For this price process, consider European call option with time to maturity T & exercise price K. Assume that short-term interest rate $r \geq 0$ is constant. Then its* *rational price* $\mathbf{C}_{T,p}$ *is as follows:*

$$\begin{aligned}\mathbf{C}_{T,p} &= S_0 \cdot \mathbf{P}\{Tw_p(m_1, \lambda) > L\} \\ &- K \cdot e^{-r \cdot T} \cdot \mathbf{P}\{Tw_p(m_2, \lambda) > L\}.\end{aligned} \tag{13}$$

For $p = 0$, (13) becomes the Black-Scholes formula.

Theorem 5 *For arbitrary admissible* μ *&* λ*, and* $\forall T > 0,$

$$(i) \quad S_{p,\mu,\lambda}(\cdot) \overset{\mathrm{D[0,T]}}{\longrightarrow} S_{0,\mu,\lambda}(\cdot) \text{ as } p \uparrow 0.$$

(ii*) Analogous result with respect to the risk-neutral martingale measure holds as* $p \uparrow 0$*:*

$$S_{p,m_1,\lambda}(\cdot) \overset{\mathrm{D[0,T]}}{\longrightarrow} S_{0,r,\lambda}(\cdot).$$

By Th. 5.ii, $\forall$ fixed $T > 0$ and as $p \uparrow 0$,

$$\mathbf{C}_{T,p} \to \mathbf{C}_{T,0}. \tag{14}$$

Near-equilibrium-type results stipulated by Th. 5.i & formula (14) justify using both continuous GBM & Black-Scholes formula as reasonably accurate approximations for prices of stocks and options, respectively. This is due to low magnitude of downward jumps of the underlying Hougaard processes. Accurate approximations for probabilities which emerge in (13) can be derived by employing Vinogradov (2004b).

Large Deviations of Non-Cramér Type:

Theorem 6 *Fix* $p \in (-\infty, 0) \cup (2, \infty)$, $\mu \in \mathbf{R}^1_+$, $\lambda \in \mathbf{R}^1_+$, *and consider* $X_{p,\mu,\lambda}(t)$. *Suppose that* $t \to +\infty$ *&* $y/t^{(1-p)/(2-p)} \to -\infty$ *if* $p < 0$, *or*

$y/t^{(1-p)/(2-p)} \to +\infty$ *if* $p > 2$. *Then*

(i) $\forall$ *fixed* $p \in (-\infty, 0)$,

$$\mathbf{P}\{X_{p,\mu,\lambda}(t) < y\}$$
$$\sim t \cdot \mathbf{P}\{X_{p,\mu,\lambda}(1) < y\} \cdot e^{(t-1)\cdot\phi_p/(p-2)}.$$

(ii) $\forall$ *fixed* $p \in (2, \infty)$,

$$\mathbf{P}\{X_{p,\mu,\lambda}(t) > y\}$$
$$\sim t \cdot \mathbf{P}\{X_{p,\mu,\lambda}(1) > y\} \cdot e^{(t-1)\cdot\phi_p/(p-2)}.$$

Th.6 follows from V(1994,Ch.5&2004a,Th.1). The results of V (1994,Ch. 5) stipulate the existence of specific critical point (denoted by u_0) in the formation of large deviations. Under the conditions of Th. 6.i, $u_0 \equiv 0$ $\forall p < 0$, since it coincides with the expectation w.r.t. the extreme conjugate measure & is given by $\zeta'_{p,\mu,\lambda}(s)$ at the threshold of its domain. In the case where $y \sim -\mathcal{C}t$ with constant $\mathcal{C} > 0$, mechanism of occurrence of a large deviation is as follows:

$$\mathbf{P}\{\inf_{0<s\le t} \Delta X_{p,\mu,\lambda}(s) < -\mathcal{C}t | X_{p,\mu,\lambda}(t) < -\mathcal{C}t\} \to 1$$

as $t \to +\infty$, where

$$\Delta X_{p,\mu,\lambda}(s) := X_{p,\mu,\lambda}(s) - X_{p,\mu,\lambda}(s-).$$

Alternatively, for $p \in (-\infty, 0)$ & $y \in [\epsilon \cdot t, (\mu - \epsilon) \cdot t]$, or $p \in (2, \infty)$ and $y \sim Const \cdot t$, the exact asymptotics of the probabilities of large deviations are given by Cramér's formula. Hence, in the case where $p \in (-\infty, 0)$, critical point $u_0 \equiv 0$ serves as a threshold between two different mechanisms of formation of the probabilities of large deviations. This phenomenon could be related to the occurrence of financial bubbles and downfalls.

J.H. McCulloch (OSU) provided the following feedback on the model described above: "Tilted stable distributions have already been used in the context of option pricing by Vinogradov (2002), who points out that for $\alpha \in (1, 2]$, they are a special case of the *Tweedie distributions*, and generate what are known as *Hougaard processes*.

He notes that exponential tilting of a density is known as the *Esscher Transformation*. If I understand his option pricing model correctly, he is valuing options as their discounted

expected payoff under an exponentially tilted stable distribution like the RNM of Figure 2, under the assumption that S_0 equals the discounted expectation of S_T under this distribution. According to the present model, this is the correct procedure for valuing options when the FM itself is *stable* with $\beta = 1$, provided the tilting coefficient λ is unity. It would not, however, be the correct procedure if the FM were *tilted stable*, unless $\log U_2$ were for some reason tilted stable and $\log U_1$ were nonstochastic, so that assets were priced as if investors were risk-neutral."

Generalization *of the model that allows both up- & downward jumps in the prices of equities*.

Consider price process

$$S(t) = S(0) \cdot \exp\left\{X(t) - A \cdot t\right\}. \qquad (15)$$

Here, $S(0) > 0$ is the initial price of an equity, $A \in \mathbf{R^1}$ is a constant, and $X(t)$ is a driftless Lévy process with no Gaussian component.

We investigate properties of the geometric Lévy process (15) in the case where the marginal distributions are those of the difference between two independent r.v.'s from PVF. To this end, assume that Lévy measure of infinitely divisible r.v. $X(1)$ possesses the density

$$\pi(x) = \begin{cases} d_+ x^{-\alpha_+-1} e^{-\theta_+ x} & \text{if } x > 0, \\ d_- |x|^{-\alpha_- -1} e^{\theta_- x} & \text{if } x < 0. \end{cases} \quad (16)$$

Here, $d_\pm$ and $\theta_\pm$ are fixed non-negative real,

$$d_+ + d_- > 0;$$

$$\alpha_\pm \in [0, 1) \cup (1, 2).$$

Evidently, Lévy process $X(t)$ performs noticeable up- & downward jumps. The case where

$$\alpha_+ = \alpha_- = 0 \text{ and } \theta_\pm > 0$$

corresponds to the *variance-gamma* model (see Madan (2001)). The case where

$$d_+ = d_-, \quad \alpha_+ = \alpha_-,$$

and $\theta_\pm > 0$ characterizes *CGMY process* (see [CGMY]). In contrast to [CGMY], we drop their condition $d_+ = d_-$. In particular, suppose that

$$\alpha_+ = \alpha_- =: \alpha \in (0,1) \cup (1,2),$$

and define

$$\theta := \theta_+ + \theta_- .$$

In the case where $d_+ > 0$, assume that

$$\theta_+ \geq 1. \tag{17}$$

Empirical data are often consistent with (17) (compare to [CGMY, Table 2, Th. 1 &(13)]).

Our set of constraints provides a reasonable level of generality which enables one to rigorously derive the properties consistent with the distinct features exhibited by the movements of overleveraged equities. Let

$$A := \frac{\Gamma(1-\alpha)}{\alpha} \cdot (d_+(\theta^\alpha - (\theta-1)^\alpha) - d_-).$$

Under the above conditions, the price process $\{S(t), t \geq 0\}$ given by (15)-(17) is a *positive submartingale*. Also, one can demonstrate that for each real $t \geq 0$,

$$\mathbf{E}S(t) = S(0) \cdot \exp\{g(\theta_\pm) \cdot t\} \quad \uparrow$$

Here, positive growth rate $g(\theta_\pm)$ of an equity can be written down explicitly.

In the case where $\alpha \in (1, 2)$, set

$$u_0^\pm := \pm \frac{\Gamma(2-\alpha)}{\alpha - 1} d_\mp \cdot \theta^{\alpha-1}.$$

One can show that

$$u_0^- \le \mathbf{E}X(1) \le u_0^+.$$

Also, define

$$H(u_0^\pm) := \frac{\Gamma(2-\alpha)}{\alpha-1} \cdot \{\pm\theta_\pm \cdot d_\mp \cdot \theta^{\alpha-1}$$
$$+(d_\pm \cdot \theta_\pm^\alpha - d_\mp \quad \cdot(\theta^\alpha - \theta_\mp^\alpha))/\alpha\} \ge 0.$$

(compare to V (1994, Ch. 5)).

The next result reveals the existence of **critical points** in the formation of large deviations thus generalizing Th. 6. For simplicity, suppose that $\alpha \in (1, 2)$.

Theorem 7 *Consider a driftless Lévy process* $X(t)$ *that satisfies (16). Assume that* $t \to \infty$

takes on integer values. Then

$$\mathbf{P}\{X(t) > x\} \sim t\,\mathbf{P}\{X(1) > x - (t-1)u_0^+\}$$
$$\cdot \exp\{-(t-1)\cdot H(u_0^+)\}$$

as $(x - t\cdot u_0^+)/t^{1/\alpha} \to +\infty$;

$$\mathbf{P}\{X(t) < x\} \sim t\,\mathbf{P}\{X(1) < x - (t-1)u_0^-\}$$
$$\cdot \exp\{-(t-1)H(u_0^-)\}$$

as $(x - t\cdot u_0^-)/t^{1/\alpha} \to -\infty$.

One can show that u_0^- constitutes a *critical point* in the formation of large deviations of $X(t)$ to $\mathbf{R}^{\mathbf{1}}_-$ (downfall). Similarly, u_0^+ constitutes a *critical point* in the formation of large deviations of $X(t)$ to $\mathbf{R}^{\mathbf{1}}_+$ (bubble).

Marginal distributions of $X(t)$ can be represented in terms of the **difference** of two independent r.v.'s which belong to PVF with common $p = p(\alpha) := (2-\alpha)/(1-\alpha) \in (-\infty, 0) \cup (2, +\infty)$. Subsequently, we can determine a risk-neutral measure by using the Esscher transforms, and employ it for pricing European call options. The closed-form formula involves the tails of the difference between these members of this family. It can be evaluated numerically and naturally generalizes Th. 4 to a more general setting. Similar to Th. 5, Black-Scholes formula can be derived by taking limit as

$$\alpha \uparrow 2 \text{ or } p(\alpha) \uparrow 0.$$

In view of Th. 5 and the analogous convergence result as $\alpha \uparrow 2$, which is valid in more general setting, both our models considered above could be regarded as those modelling market behavior near equilibrium.

Also, the second model (15)–(17) complies with three features exhibited by the stock price movements & their implied distributions, which are associated with the *leverage effect*. One of them is the fact that the volatility and the growth rate are usually *negatively correlated*. This is related to (1) with $p < 0$.

The tails of spectral densities which have the form (16) are usually termed *semi-heavy*. It is known that the tails of both the original and implied distributions are frequently semi-heavy. Also, our results yield that the risk-neutral Escher transform is more skewed to the left than the original distribution.

This is typical for chaotic movements of over-leveraged equities and their option prices (compare to Madan (2001)). 3^{rd} feature is an increased likelihood of financial crashes. Voit (1999, Ch. 9) proposes their investigation by employing the critical state theory. This is consistent with the existence of critical points for our model (see Th.'s 6 & 7).

G. Castellacci (Interactive Data Co.) provided a valuable feedback on the model described above in his MathSciNet review of Vinogradov (2004a). See http://www.ams.org/mathscinet "The author studies a model for the stochastic evolution of stock prices $S(t)$ given by the following exponential Lévy process:

$$S(t) = S(0)\exp\left(X(t) - At\right),$$

where X is a pure-jump Lévy process with density

$$\pi(x) = \begin{cases} d_+ x^{-(\alpha+1)} e^{\theta_+ x} & \text{if } x > 0, \\ d_- |x|^{-(\alpha+1)} e^{\theta_- x} & \text{if } x < 0, \end{cases}$$

with the parameters $d_\pm$, $\theta_\pm$ and $\alpha_\pm$ real and nonnegative such that at least one $d_\pm$ is positive and $\alpha_\pm \in [0,1) \cup (1,2)$. The case when $\alpha_\pm = 0$ and $\theta_\pm > 0$ corresponds to the well-known gamma-variance model while for the case when $d_+ = d_-$, $\alpha_+ = \alpha_-$, and $\theta_\pm > 0$ we obtain the CGMY process of [P. P. Carr et al., J. Business 75 (2002), no. 2, 305–332]. Therefore the above is a generalization of these earlier models.

It is further assumed that if $d_+ > 0$, then $\theta_+ \geq 1$, and the drift is defined as

$$A := \Gamma(1-\alpha)\frac{d_+(\theta^\alpha - (\theta-1)^\alpha) - d_-}{\alpha},$$

where $\Gamma(1-\alpha) := \frac{\Gamma(2-\alpha)}{1-\alpha}$ for $\alpha \in (1,2)$ and $\theta := \theta_+ + \theta_-$.

After deriving asymptotics for the tails of the distribution of $X(t)$ in terms of the parameters $d_\pm$, $\theta_\pm$ and α, the author focuses on showing how, in this setting, option pricing generalizes the standard Black-Scholes formula. The arbitrage pricing of a call with expiry T and strike K is

$$C_{T,\alpha} := e^{-rT} E_Q \left[\max[S(T) - K, 0] \right].$$

Here the risk-free interest rate r is assumed constant and bounded as $r \in [0, r_*(\alpha)]$,

$(r_*(\alpha) := (d_+ + d_-)((\theta-1)^\alpha + 1 - \theta^\alpha)\Gamma(1-\alpha)/\alpha)$

and Q is some risk-neutral measure, which is not unique. The ensuing market model is incomplete and the Esscher transform is used to choose Q.

The author obtains a pricing formula for $C_{T,\alpha}$ that is formally analogous to Black-Scholes and replaces exercise probabilities with tail probabilities of differences of certain power-variance random variables. This is based on the expression of X as the difference of two so-called Hougaard processes, whose properties the author previously investigated in [C. R. Math. Acad. Sci. Soc. R. Can. 24 (2002), no. 4, 152–159; MR1940554].

In the same vein, the author proves that both the process $S(\cdot)$ and its risk-neutral (discounted) counterpart converge to suitable geometric Brownian motions in the Skorokhod topology of the càdlàg space $\mathbb{D}[0,T]$ as

$$\frac{2-\alpha}{1-\alpha} =: p(\alpha) \to 0^-.$$

Similarly, the arbitrage call price $C_{T,\alpha}$ converges to an actual Black-Scholes value. Geometric Brownian motion and the Black-Scholes formula can therefore be assumed to approximate underlying prices that follow the proposed class of processes and options thereon, respectively. The former can be interpreted as modelling market behavior near equilibrium,

whereas the latter account for several features related to the leverage effect, including negative correlation between volatility and growth rates and left skewness of the pricing density (obtained by the Esscher transform).

The paper is an interesting generalization of the results of [P. P. Carr et al., op. cit.] and similar work. However, unlike that article, the present one does not test its model empirically and implicitly relies on that and previous work for its financial raison d$\hat{e}$tre."

A Different Model which could have potential applications in portfolio selection. Suppose that $\exists$ one bond with price process

$$S^b(t) = S^b(0) \cdot e^{r \cdot t}.$$

There are also k equities satisfying SDE's

$$\frac{dS^{(i)}_{p,m_i,\lambda_i,r}(t)}{S^{(i)}_{p,m_i,\lambda_i,r}(t-)} = r \cdot dt + dX^{(i)}_{p,m_i - r,\lambda_i}(t). \quad (18)$$

Here, $1 \leq i \leq k$, all $m_i{}'s > r$, and all Hougaard processes $X^{(i)}_{p,m_i-r,\lambda_i}(t)$ are independent.

$\forall p \in \mathbf{R}^1 \setminus (0,1)$, $\exists$ solution to (18). But it constitutes price process for an equity only for $p \in \{0\} \cup [1, \infty)$. The solution is called the *stochastic exponential*

$$S^{(i)}_{p,m_i,\lambda_i,r}(t) = \mathcal{SE}\{rt + X^{(i)}_{p,m_i-r,\lambda_i}(t)\} > 0.$$

In the continuous case ($p = 0$), solution to (18) is the following GBM:

$$S^{(i)}_{0,m_i,\lambda_i,r}(t) := S^{(i)}(0)$$

$$\cdot \exp\left\{ \left(m_i - \frac{1}{2\lambda_i}\right) t + \lambda_i^{-1/2} \cdot B_i(t) \right\}.$$

Merton's weights in continuous case maximize the expected logarithmic utility. Denote them by $W = (w_0, w_1, ..., w_k)$ such that $\forall\ 1 \leq i \leq k$,

$$w_i := (m_i - r)/\sigma_i^2 = \lambda_i \cdot (m_i - r), \qquad (19)$$

whereas

$$w_0 := 1 - \sum_{i=1}^{k} w_i.$$

The *expected logarithmic growth rate* is

$$r + \frac{1}{2} \sum_{i=1}^{k} SR_i^2 = r + \frac{1}{2} \sum_{i=1}^{k} \lambda_i (m_i - r)^2. \quad (20)$$

Here, SR_i is the *Sharpe measure* (or *Sharpe ra-*

tio) for i^{th} asset; $SR_i := (m_i - r)/\sigma_i$.

$r \geq 0$ stands for constant *risk-free rate*. Let $R_{port} - r$ and σ_{port} denote the expected excess return and risk for portfolio, which is assumed to be comprised of *independent* assets.

Equation (20) implies the *Pythagorean theorem for Sharpe portfolio performance measure:*

$$SR^2_{port} = \sum_{i=1}^{k} SR^2_i. \tag{21}$$

If $p \neq 0$ in (18), optimal weights differ from Merton's weights (19). (see Kallsen (2000)). But the next result on price process holds for a *scalar multiple of Merton's weights*, where

$\forall\, 1 \leq i \leq k$ and $\forall$ fixed $\kappa > 0$,

$$\widehat{w}_i^{(\kappa)} := \kappa \cdot \lambda_i \cdot (m_i - r)^{1-p}.$$

Suppose that $\forall\, t \geq 0$, r.v.'s $\xi_0(t), \xi_1(t), ..., \xi_k(t)$ are the amounts of shares of the corresponding securities which are held in the portfolio at time $t-$. Let vector W correspond to weights $\widehat{w}_i^{(\kappa)}$'s.

Theorem 8 *Suppose that the movements of k equities satisfy (18). Then for cumulative price process*

$$S_p^{(W)}(t) := \xi_0(t) \cdot S^b(t) + \sum_{i=1}^{k} \xi_i(t) \cdot S^{(i)}_{p,m_i,\lambda_i,r}(t),$$

the following SDE is valid:

$$\frac{dS_p^{(W)}(t)}{S_p^{(W)}(t-)} = r \cdot dt + dX_{p,m-r,\lambda}(t). \qquad (22)$$

Here, m are λ are obtained from the equations

$$m - r = \sum_{i=1}^{k} \widehat{w}_i^{(\kappa)} \cdot (m_i - r);$$

$$\lambda \cdot (m-r)^{2-p} = \sum_{i=1}^{k} \lambda_i \cdot (m_i - r)^{2-p}. \qquad (23)$$

Denote the expected excess growth rates by $g_i^{(\kappa)}$ and $g^{(\kappa)}$, respectively. (23) is closely related to the Pythagorean theorem for Sharpe ratio, which follows from additivity of shape parameter ϕ_p (10), that is established in V (2004b).

$$\sum_{i=1}^{k} SR_i^2 = \sum_{i=1}^{k} (g_i^{(\kappa)} - r)$$
$$= g^{(\kappa)} - r = SR_{port}^2$$

(compare to (21)).

The cumulative price process $S_p^{(W)}(t)$ given by (22) is a geometric Lévy process. Hence, it is the *ordinary exponential* of some Lévy process, whose generating triplet is found explicitly.

Observe that under fulfillment of (18), Merton's allocation of weights leads to the same structure of the cumulative price process as those of the individual assets.

The fact that the additivity of the shape parameter ϕ_p is an if-and-only-if condition stipulates that in the continuous case ($p = 0$), Merton's selection of weights is also necessary for deriving the optimality of the expected logarithmic growth rate, which is given by (20).

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