The Harrison-Pliska Story (and a little bit more)

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Key References We Used

- J.M. Harrison and D.M. Kreps, Martingales and arbitrage in multiperiod securities markets, *J. Economic Theory* 20 (1979), 381-408.
- J. Jacod, *Calcul Stochastique et Problèmes de Martingales,* Lecture Notes in Mathematics 714, Springer, New York, 1979.
- P.-A. Meyer, Un cours sur les integrales stochastiques, *Seminaire de Probabilité X,* Lecture Notes in Mathematics 511, Springer, New York, 1979, 91-108.

1. Some Continuous Time Stock Price Models

1a. Geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dW$$

(and multi-stock extensions)

Reasonably satisfactory, but…

- Returns can be correlated at some frequencies
- Volatility is random
- Big price jumps like on October 19, 1987

1b. A point process model

$$S_t = S_0 \, exp\{bN_t - \mu t\},$$

where

N = Poisson process with intensity $\lambda > 0$

b = positive scalar

μ = positive scalar

J. Cox and S. Ross, The valuation of options for alternative stochastic processes, *J. Financial Economics* 3 (1976), 145-166.

1c. Some generalizations and variations

- Diffusions (μ and σ are time and state dependent)
- Ito processes (μ and σ are stochastic)
- General jump processes
- Jump-diffusion processes
- Fractional Brownian motion
- Etc.

But we want to proceed in a unified, very general way, so …

1d. Semi-martingale models

$$S_t = M_t + A_t$$

where

$$M = \text{local martingale}$$
$$A = \text{adapted bounded variation process}$$

- M is a *martingale* if $E[M_{t+s}|\mathfrak{F}_t] = M_t$ for all $s,t \geq 0$ (here $\{\mathfrak{F}_t : t \geq 0\}$ is the filtration for underlying probability space)
- M is a *local martingale* if there exists a sequence $\{\tau_n\}$ of stopping times increasing to infinity such that each process $M(t \wedge \tau_n)$ is a martingale
- *A* is a *bounded variation process* if it is the difference of two non-decreasing processes

2. Alternative Justification of Black-Scholes Formula

Stock:	$dS = \mu S dt + \sigma S dW$
Bank Account;	$B_t = \exp\{rt\}$
Stock position:	H^1
Bank position:	H^0
Trading strategy:	$H_t = (H^0(t), H^1(t))$
Value of portfolio:	$V_t = H^0(t)B_t + H^1(t)S_t$
Black-Scholes formula:	

$$f(S,t) = SN[d(S,t)] - K\exp\{-rt\}N[d(S,t) - \sigma t^{1/2}]$$

$$d(S,t) = [\ln(S/K) + (r + \tfrac{1}{2}\sigma^2)t] / [\sigma t^{1/2}]$$

Idea: Choose the trading strategy so $V_t = f(S_t,T-t)$ for all t (thus, in particular, $V_T = \max\{0, S_T - K\}$), in which case $f(S_t,T-t)$ must be the time-*t* price of the call or there would be an arbitrage opportunity.

The replicating trading strategy:

$H^1(t) = f_1(S_t,T\text{-}t)$ ← partial with respect to first argument

$H^0(t) = [\, f(S_t,T\text{-}t) – H^1(t)S_t \,]/B_t$

Thus

$$V_t = H_0(t)B_t + H_1(t)S_t = f(S_t,T\text{-}t)$$

$$V_t = V_0 + \int_0^t H_u^0 dB_u + \int_0^t H_u^1 dS_u$$

This last equation, called the *self financing condition*, was derived with the help of Ito's lemma and says that all changes in portfolio value are due to trading in the stock and bank account. But this stochastic integral representation of the *gains process* requires justification.

3. The Preliminary Security Market Model

T	planning horizon
$(\Omega, \mathfrak{I}, P)$	probability space
$\{\mathfrak{I}_t : 0 \le t \le T\}$	filtration (information structure)
$B_t = S^0(t)$	value of one unit in bank account (continuous VF process, $B_0 = 1$, and usually nondecreasing – see below)
$S^k(t)$	price of kth security, $k = 1, \ldots, K$ (adapted, non-negative)
$H^k(t)$	units of security k held at time t, $k = 0, 1, \ldots, K$
H_t	trading strategy = $(H^0(t), H^1(t), \ldots, H^K(t))$

Value Process:

$$V_t = H^0(t)B_t + H^1(t)S^1(t) + \ldots + H^K(t)S^K(t)$$

Predictable, Piecewise Constant (that is, Simple) Trading Strategies

For each security k there is a number $N < \infty$, a sequence of times $0 = t_0 < t_1 < \ldots < t_N = T$, and a sequence $\{\theta_n\}$ of scalars such that

$$H^k(t) = \theta_n \text{ for } t_{n-1} < t \le t_n,\ n = 1, 2, \ldots, N.$$

Note that H^k is left-continuous with right-hand limits and thus is a *predictable* stochastic process

Note that $\theta_n(S^k(t_n) - S^k(t_{n-1}))$ is the gain or loss during $(t_{n-1},t_n]$ due to trading in security k

Cumulative net trading profit in security k through time t, $t_{n-1} < t \le t_n$:

$$G^k(t) \equiv \sum_{i=1}^{n-1} \theta_i^k [S^k(t_i) - S^k(t_{i-1})] + \theta_n^k [S^k(t) - S^k(t_{n-1})]$$

$$= \int_0^t H^k(u)\, dS^k(u)$$

The Gains Process – A Stochastic Integral Representation of Net Profits Due to Trading

$$G_t = G^0(t) + G^1(t) + \ldots + G^K(t)$$

- We would like to consider more general trading strategies
- We shall require S to be a semimartingale and then use stochastic calculus to provide suitable requirements on the trading strategies (predictable + some technical conditions + ?)
- Important issue: we want the trading strategies to be sensible from the economic point of view

4. Economic Considerations

A trading strategy H is said to be <u>self financing</u> if

$$V_t = V_0 + G_t, \quad \text{all } t \geq 0$$

Thus no money is added to or withdrawn from the portfolio

An <u>arbitrage opportunity</u> is a self financing trading strategy H such that

1. $V_T = 0$
2. $V_T \geq 0$
3. $P(V_T > 0) > 0$

A <u>martingale measure</u> (aka a <u>risk neutral probability measure</u>) is a probability measure such that

1. Q is equivalent to the real-world probability measure P
2. Each discounted price process S^k/B, $k = 1, \ldots, K$, is a martingale under Q

Some More Economic Criteria (all for self financing trading strategies)

<u>Law of One Price</u>

If the time-T values of two portfolios are (almost surely) identical, then their time-t values must be the same for all $t < T$

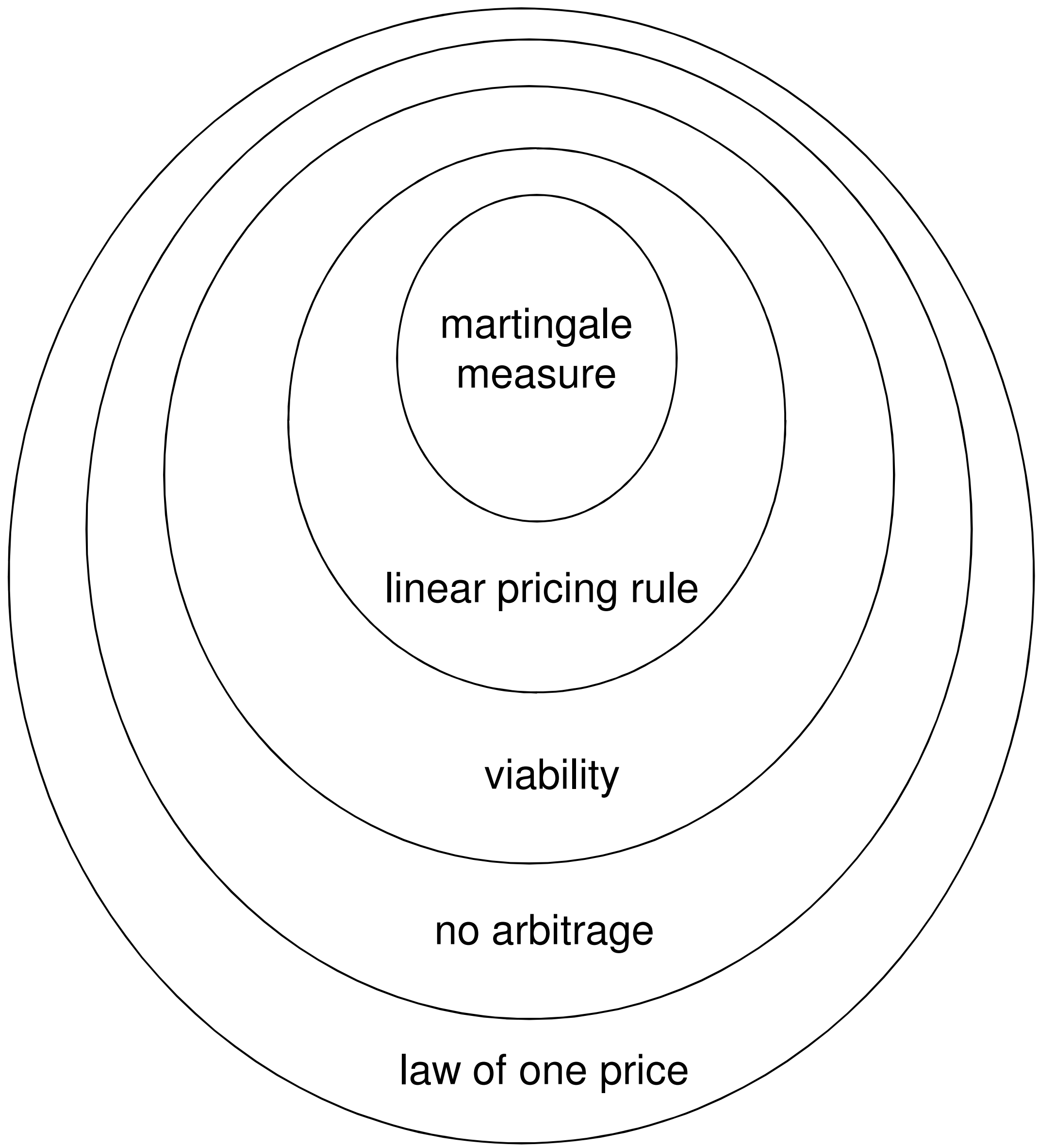
<u>Viability</u>

There exists an optimal trading strategy for some agent who prefers more to less

<u>Existence of a Linear Pricing Rule</u>

For any random variable X the function $\pi(X)$ is linear, where $\pi(X) \equiv V_0$, the time-0 value under any trading strategy H satisfying $V_T = X$

Relationships Between the Economic Criteria



Remarks About the Economic Relationships

- It’s not hard to establish the Venn diagram
- One needs to be creative to find examples where one set is a proper subset of another (such examples are rare)
- Depending upon the circumstances (e.g., nature of the security processes, restrictions on trading strategies,…), it can be shown that various sets coincide
- For example, Harrison and Kreps used a separating hyperplane theorem to show that viability coincides with the existence of a martingale measure
- With discrete time models having finitely many trading periods (also various continuous time models) absence of arbitrage coincides with existence of martingale measure

5. The General Security Market Model

<u>Assumptions</u>

Price process is a semimartingale

There exists a martingale measure

<u>Admissible Strategies</u>

Predictable

Technical requirement(s)

<u>Gains Process</u>

$$G_t(H) = \sum_{k=0}^{K} \int_0^t H_u^k \, dS_u^k$$

- G(H) is a good model of trading gains for piecewise constant H
- G(H) is well defined for all admissible H

Link and Final Justification

For any admissible H, there exists a sequence $\{H_n\}$ of piecewise constant trading strategies such that (in suitable topologies)

$$H_n \rightarrow H \qquad \text{and} \qquad G(H_n) \rightarrow G(H)$$

<u>Variations</u>

- Unrestricted trading strategies
- Nonnegative terminal wealth ($V_T(H) \geq 0$)
- Nonnegative wealth throughout ($V_t(H) \geq 0, \ 0 \leq t \leq T$)
- No borrowing from bank ($H^0(t) \geq 0, \ 0 \leq t \leq T$)
- No short sales of stock ($H^k \geq 0, \ k = 1, \ldots, K$)

6. Computing the martingale measure

First Some Probability Theory (Girsanov Transformation)

Given b, an arbitrary bounded, real-valued function on the real line

Define
$$M_t = \exp\left\{\int_0^t b(W_s)\,dW_s - \tfrac{1}{2}\int_0^t b^2(W_s)\,ds\right\}$$

where W is Brownian motion on the probability space $(\Omega, \mathfrak{F}, P)$.

Then M is a positive martingale under P and one can define a new probability measure Q for arbitrary t > 0 by

$$Q(A) = E[M_t 1_A],$$

where 1_A is the indicator function for the event $A \in \mathfrak{F}_t$

Moreover $$\hat{W}_t \equiv W_t - \int_0^t b(W_s)\,ds$$

defines a process which is a Brownian motion under Q.

Using the Girsanov transformation

1. Choose the function b in a clever way
2. Replace W in the security model using top equation
3. Verify that each discounted price process S^k/B is a martingale under Q

Example: S = Geometric Brownian Motion

1. $b(W) = (r - \mu)/\sigma$

2. $dS/S = \mu dt + \sigma d\hat{W} + \sigma[(r - \mu)/\sigma]dt = rdt + \sigma d\hat{W}$

3. Some probability calculations verify that S/B is a martingale under Q

Note: the function b can have a second argument, namely, time t, as seen in the following example.

Example: Diffusion Process

$$dS^k / S^k = \mu_k(t)dt + \sum_{j=1}^{K} \sigma_{kj}(t)dW^j, k = 1,..., K$$

$$B_t = \exp\left\{\int_0^t r(s)ds\right\}$$

where r, μ_k, and σ_{kj}, j,k = 1, …, K, are all suitable bounded functions of time.

Set $\quad b(W,t) = [\sigma(t)]^{-1} [r(t)1 - \mu(t)]$

so $\quad dS^k/S^k = r(t)dt + \sum_j \sigma_{kj}(t)d\hat{W}^j, \quad k = 1, \dots, K$

and, by further calculations, each S^k/B is a martingale under Q

7. Pricing Contingent Claims (European options)

Given X = time-T payoff = $\mathcal{F}_T$–measurable random variable

Suppose $V_T(H) = X$ for some self-financing H

Then $V_t(H)$ is the time-t price of X (by Law of One Price)

But V(H)/B is a martingale under Q (easy to prove)

So $V_t(H) = B_t E_Q[V_T(H)/B_T | \mathcal{F}_t] = B_t E_Q[X/B_T | \mathcal{F}_t]$

Solution approach Compute value process $V_t = B_t E_Q[X/B_T | \mathcal{F}_t]$ and then the replicating strategy via the martingale representation problem:

$$V_t = V_0 + \sum_{k=0}^{K} \int_0^t H_u^k \, dS_u^k$$

Example: Derivation of Black & Scholes

$X = (S_T - K)^+$ $\quad$ $B_t = \exp\{rt\}$ $\quad$ $dS/S = \mu dt + \sigma dW$

Under Q, $dS/S = rdt + \sigma d\hat{W}$ and S_T has the log-normal distribution with

$$E_Q[X/B_T] = \exp\{-rt\}\, E_Q[(S_0 \exp\{\sigma \hat{W}_T + (r - \sigma^2/2)T\} - K)^+]$$

Under Q, $\hat{W}_T$ is a normal RV with mean zero and variance T, so express the preceding expectation in terms of its density function f(w):

$$\exp\{-rt\} \int (S_0 \exp\{\sigma w + (r - \sigma^2/2)T\} - K)^+ f(w)dw$$

Write the integral in 2 parts depending on whether $S_0\exp\{\sigma w + (r - \sigma^2/2)T\}$ exceeds K, and then grind through calculations to get B-S formula.

8. Return Processes

<u>Definition</u> A process is said to be VF (*variation finie*) if it is adapted, RCLL, and sample paths are of finite variation.

<u>Assumption</u> The bank account process B is VF and, in fact, continuous.

<u>Definition</u> Setting $\alpha_t = \log(B_t)$, $0 \le t \le T$, we call α the *return process* for B. Note $\alpha_0 = 0$ since, by assumption, $B_0 = 1$. Moreover, we also have $dB = Bd\alpha$.

If B is actually *absolutely continuous*, then

$$\alpha_t = \int_0^t r_s \, ds$$

for some process r, in which case r_s should be interpreted as the time-s interest rate with continuous compounding.

Return Process for a Stock Price Process *S*

Analogous to $dB = B\,d\alpha$ we would like to consider the equation $dS = SdR$ for any semimartingale price process S. Since S might have jumps, it would be better to consider the equation

$$dS = S_- \, dR.$$

<u>Questions</u> Given a semimartingale S, does this always have a solution *R?* Conversely, given a semimartingale *R*, does this always have a semimartingale solution *S?*

Note this equation is equivalent to (to be denoted equation #1):

$$S_t = S_0 + \int_0^t S_u dR_u$$

as well as to:

$$R_t = \int_0^t (1/S_{u-}) dS_u$$

So given a (reasonable) price process *S*, this last equation defines the corresponding *return process R*. What about the converse?

The Semimartingale Exponential

Given S_0 and a semimartingale R, equation #1 always has a semimartingale solution S. It is unique and given by

$$S_t \; = \; S_0 \, \Psi_t(R), \quad 0 \le t \le T,$$

where

$$\Psi_t(R) \equiv \exp(R_t - R_0 - .5[R,R]_t) \prod_{s \le t} (1 + \Delta R_s) \exp(-\Delta R_s + .5(\Delta R_s)^2)$$

$$\Delta R_s \equiv R_s - R_{s-}$$

$$[R,R]_t \equiv R_t R_t - 2\int_0^t R_{s-} dR_s = \text{quadratic variation process}$$

Note that R is such that $1+\Delta R>0$ for any and all jumps if and only if $\Psi_t(R) > 0$ for all t. Similarly with weak inequalities.

Note also that $\Psi_0(R) = 1$.

Some Probabilistic Properties

Given two semimartingales X and Y,

$$\Psi(X)\Psi(Y) = \Psi(X + Y + [X,Y]),$$

where

$$[X,Y]_t \equiv X_t Y_t - \int_0^t X_{s-}\,dY_s - \int_0^t Y_{s-}\,dX_s$$

is called the *joint variation process.*

If X is also continuous and VF (such as is assumed to be the case for our bank account process B), then for any semimartingale Y

$$[X,Y] = 0$$

in which case

$$\Psi(X)\Psi(Y) = \Psi(X + Y).$$

Discounted Return Processes

$$Z \equiv S/B = S_0\Psi(R)/\exp(\alpha) = S_0\Psi(R)\exp(-\alpha)$$

is called the *discounted price process.*

But $-\alpha$ is continuous, $\alpha_0 = 0$, and $[-\alpha,-\alpha] = 0$, so by the semimartingale exponential expression

$$Z = S_0\Psi(R)\Psi(-\alpha).$$

Since also $[R,-\alpha] = 0$, we then have by an earlier property

$$Z = S_0\Psi(R - \alpha) = Z_0\Psi(R - \alpha).$$

Thus $Y \equiv R - \alpha$ should be interpreted as the return process for the discounted price process Z.

Example: Black-Scholes Model

If $dS/S = \mu dt + \sigma dW$, then the return process

$$R_t = \int_0^t (1/S_u)\,dS_u = \mu t + \sigma W_t$$

With a constant interest rate *r* one has $\alpha_t = rt$, and the return process for the discounted price process $Z = S/B$ is

$$Y = R - \alpha$$

9. Complete Markets

<u>Problem</u> How do you know there is some H such that $V_T(H) = X$?

<u>Answer</u> Maybe not, unless market is *complete*

<u>Key Results</u>

Complete $\Leftrightarrow$ martingale representation property $\Rightarrow$ Q is unique

Q is unique + orthogonality condition $\Rightarrow$ Complete

If Q is not complete, then X can be *replicated* (i.e., $V_T(H) = X$ for some H), if and only if $E_Q[X/B_T]$ takes the same value for all risk neutral Q

Proof: martingale representation property $\Rightarrow$ complete

For arbitrary X define the martingale $M_t = E_Q[X/B_T|\mathcal{F}_t]$

Choose $H^1, \ldots, H^K$ so $M_t = M_0 + \sum_k \int H^k d(S^k/B)$

Set $H^0(t) = M_t - H^1(t)S^1(t)/B_t - \ldots - H^K(t)S^K(t)/B_t$

so $V_t(H) = H^0(t)S^0(t) + H^1(t)S^1(t) + \ldots + H^K(t)S^K(t) = B_t M_t$

Therefore $V_T(H) = B_T M_T = B_T E\,[X/B_T|\mathcal{F}_T] = X$

and hence the contingent claim X can be replicated.

Proof: complete $\Rightarrow$ martingale representation property

Consider the contingent claim $X = B_T M_T$, where M is an arbitrary martingale

Let trading strategy H be such that $V_T(H) = X$

Then $M_t = E_Q[M_T | \mathcal{F}_t] = E_Q[X/B_T | \mathcal{F}_t] = E_Q[V_T(H)/B_T | \mathcal{F}_t]$

$= V_t(H)/B_t$ ← the discounted value process

$= V_0(H)/B_0 + \int H^1 d(S^1/B) + \ldots + \int H^K d(S^K/B)$

Hence the martingale M can be represented as a stochastic integral with respect to the discounted price process martingales

10. American Options

An *American option* is a financial instrument with the following features:

- An expiration date $T < \infty$
- A payoff process F, an adapted, nonnegative stochastic process
- The holder of this option can choose one *stopping time* $\tau \leq T$, thereby receiving the payoff $F(\tau)$

Note: the challenge is to determine the optimal *early exercise strategy* (that is, stopping time τ) and the price of this option.

<u>Example</u> American put: $F(\tau) = (K - S(\tau))^{+}$

<u>Notation</u> $\mathcal{T}(t,T)$ = all stopping times τ with $t \leq \tau \leq T$

The adapted process X is a *super-martingale* if $E[X_{t+s} | \mathcal{F}_t] \leq X_t$, all $s,t \geq 0$

Denote $\quad X_t = \text{ess sup}\{E_Q[F(\tau)/B(\tau)|\mathfrak{I}_t] : \tau \in \mathcal{T}(t,T)\}$

Thus X is the *Snell envelope*, that is, the smallest Q-super-martingale majorizing the discounted payoff process.

<u>Main Result</u> For a complete market

$$V_t = B_t X_t$$

is the value of the American option. The optimal early exercise time for the holder of this option is to choose the stopping time given by

$$\tau = \inf\{ t \geq 0 : V_t = F(t)\}.$$

The seller of this option can hedge by using the trading strategy corresponding to V.

<u>Early Exercise</u> If $-F(t)/B_t$ is a super-martingale, then one should never exercise early. In particular, for an American call on a non-dividend paying stock, $-(S_t - K)^+/B_t$ is a super-martingale, and so such American calls should never be exercised early.

References

- J. M. Harrison and S. R. Pliska, Martingales and stochastic integrals in the theory of continuous trading, *Stochastic Processes and Their Application* 11 (1981), 215-260.
- J. M. Harrison and S. R. Pliska, A stochastic calculus model of continuous trading: complete markets, *Stochastic Processes and Their Application* 15 (1983), 313-316.
- Numerous more recent books, including those listed in the course syllabus