A Benchmark Approach to Quantitative Finance

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1 Asset Pricing

Law of One Price

"All replicating portfolios of a payoff have the same price!"

Debreu (1959), Sharpe (1964), Lintner (1965),

Merton (1973a, 1973b), Ross (1976), Harrison & Kreps (1979),

Cochrane (2001), ...

will be, in general, violated under the benchmark approach.

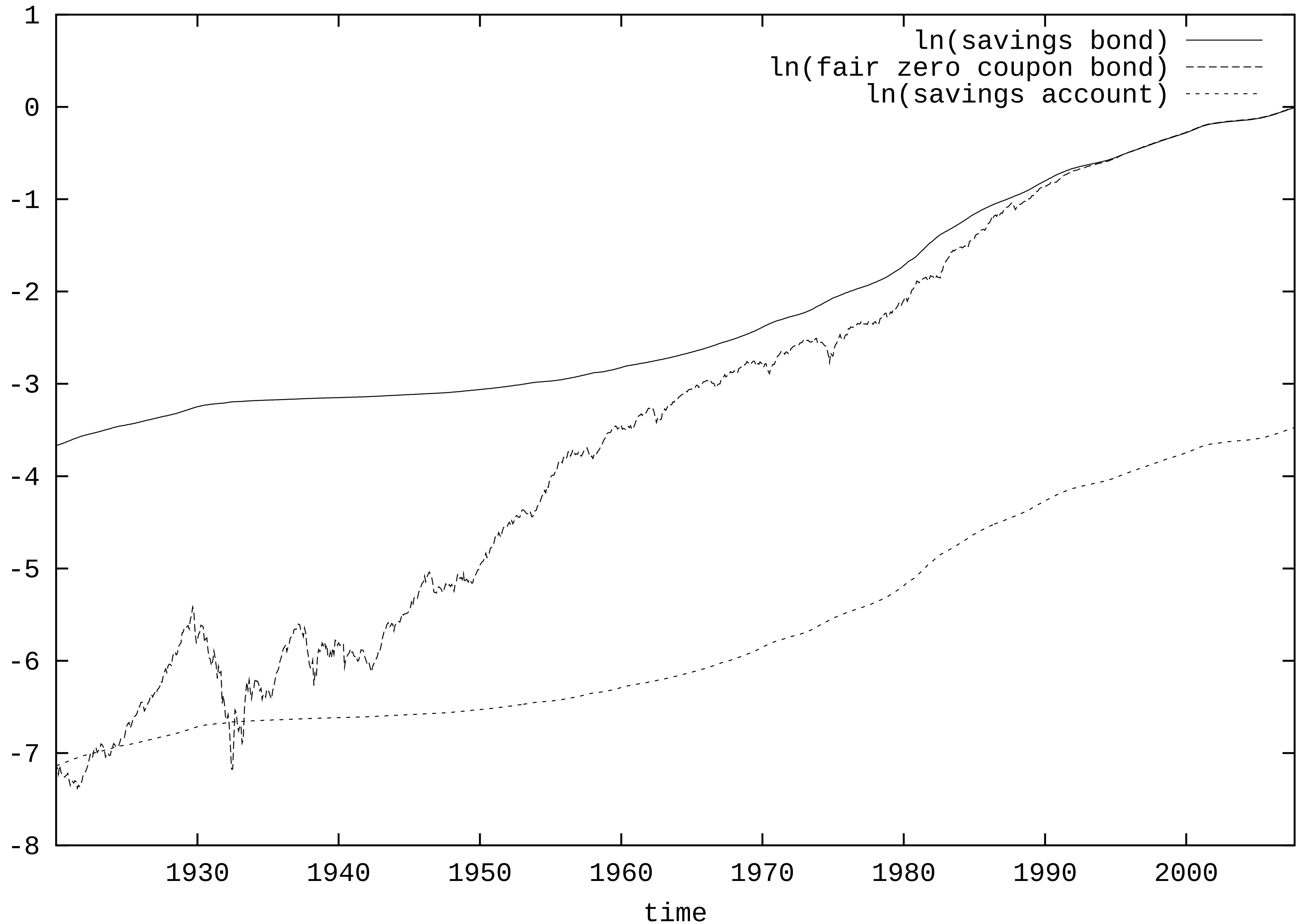

Figure 1.1: Logarithms of savings bond, fair zero coupon bond and savings account.

Financial Market

- jth primary security account

  S_t^j

  $j \in \{0, 1, \ldots, d\}$

- savings account

  S_t^0

  $t \geq 0$

- **strategy**

$$\delta = \{\delta_t = (\delta_t^0, \delta_t^1, \ldots, \delta_t^d)^\top, \; t \geq 0\}$$

predictable

- **portfolio**

$$S_t^\delta = \sum_{j=0}^{d} \delta_t^j \, S_t^j$$

- **self-financing**

$$dS_t^\delta = \sum_{j=0}^{d} \delta_t^j \, dS_t^j$$

- **strictly positive portfolio**

$$S_0^\delta = x > 0$$

$$S_t^\delta \in (0, \infty) \quad \text{for all} \quad t \in [0, \infty)$$

$$\Longrightarrow \quad S^\delta \in \mathcal{V}_x^+$$

Numeraire Portfolio

Definition 1.1 $S^{\delta_*} \in \mathcal{V}_x^+$ numeraire portfolio *if*

$$E_t\left(\frac{\frac{S^{\delta}_{t+h}}{S^{\delta_*}_{t+h}}}{\frac{S^{\delta}_t}{S^{\delta_*}_t}} - 1\right) \leq 0$$

for all nonnegative S^{δ} *and* $t, h \in [0, \infty)$.

- S^{δ_*} "best" performing portfolio

Long (1990), Becherer (2001), Pl. (2002, 2006),

Bühlmann & Pl. (2003), Goll & Kallsen (2003), Karatzas & Kardaras (2007)

Main Assumption of the Benchmark Approach

Assumption 1.2 *There exists a numeraire portfolio* $S^{\delta_*} \in \mathcal{V}_x^+$.

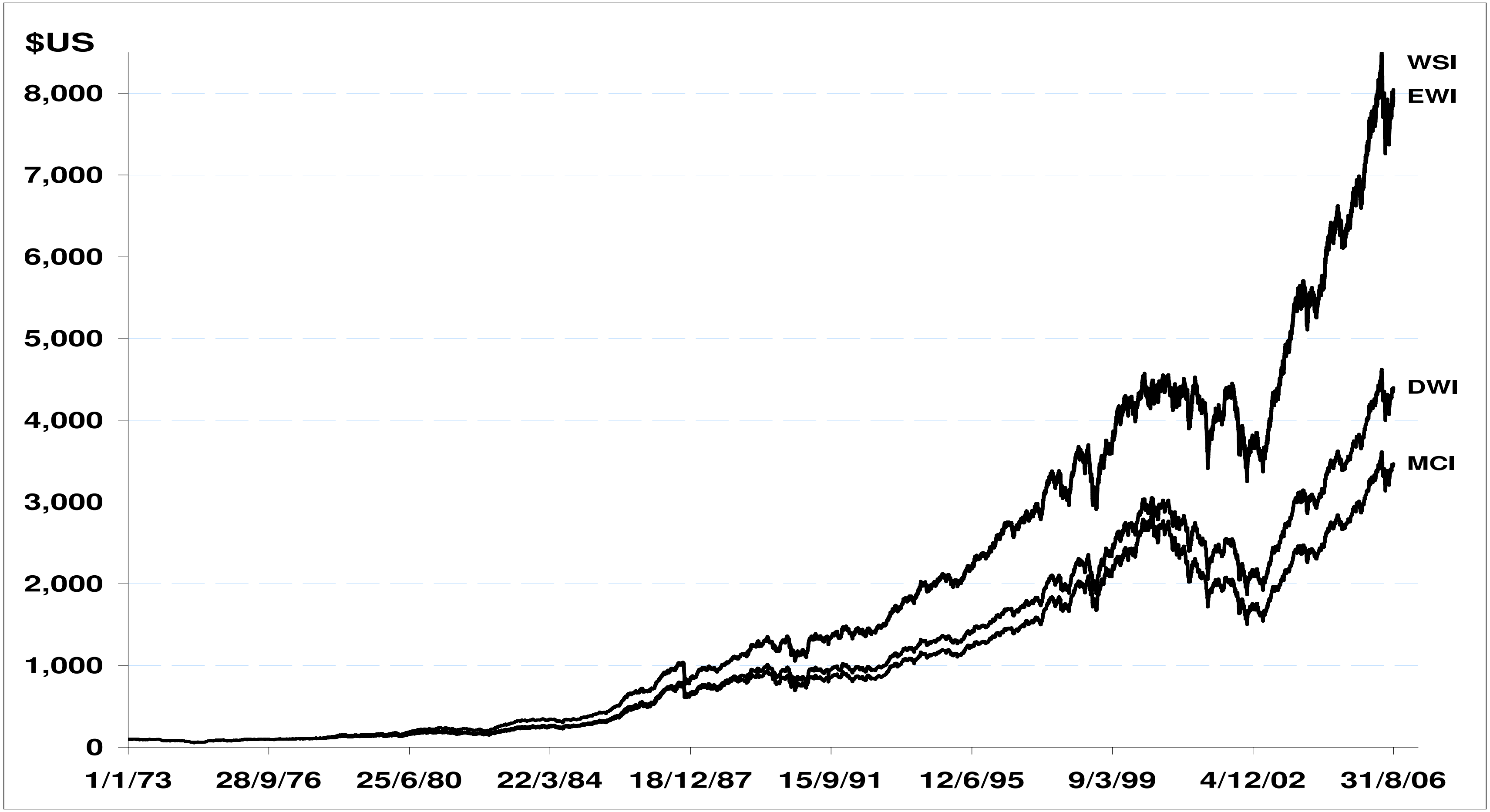

Figure 1.2: Constructed MCI, DWI, EWI and WSI. In the long term the WSI and EWI outperform all other indices, Le & Platen (2006).

- **benchmarked value**

$$\hat{S}_t^\delta = \frac{S_t^\delta}{S_t^{\delta_*}}$$

Supermartingale Property

Corollary 1.3 *For nonnegative* S^δ

$$\hat{S}_t^\delta \geq E_t\left(\hat{S}_s^\delta\right)$$

$0 \leq t \leq s < \infty$

$\hat{S}^\delta$ *supermartingale*

- **long term growth rate**

$$g^{\delta} = \limsup_{t \to \infty} \frac{1}{t} \ln\left(\frac{S^{\delta}_t}{S^{\delta}_0}\right)$$

Theorem 1.4 *For* $S^{\delta} \in \mathcal{V}^+_x$

$$g^{\delta} \leq g^{\delta_*}.$$

- "pathwise best" in the long run

  Karatzas & Shreve (1998), Pl. (2004a),

  Karatzas & Kardaras (2007)

Pl. (2004a)

Definition 1.5 *Nonnegative portfolio* S^{δ} **outperforms systematically** $S^{\tilde{\delta}}$ *if*

(i) $S_0^{\delta} = S_0^{\tilde{\delta}}$;

(ii) $P\left(S_t^{\delta} \geq S_t^{\tilde{\delta}}\right) = 1$

(iii) $P\left(S_t^{\delta} > S_t^{\tilde{\delta}}\right) > 0.$

- "relative arbitrage" Fernholz & Karatzas (2005)

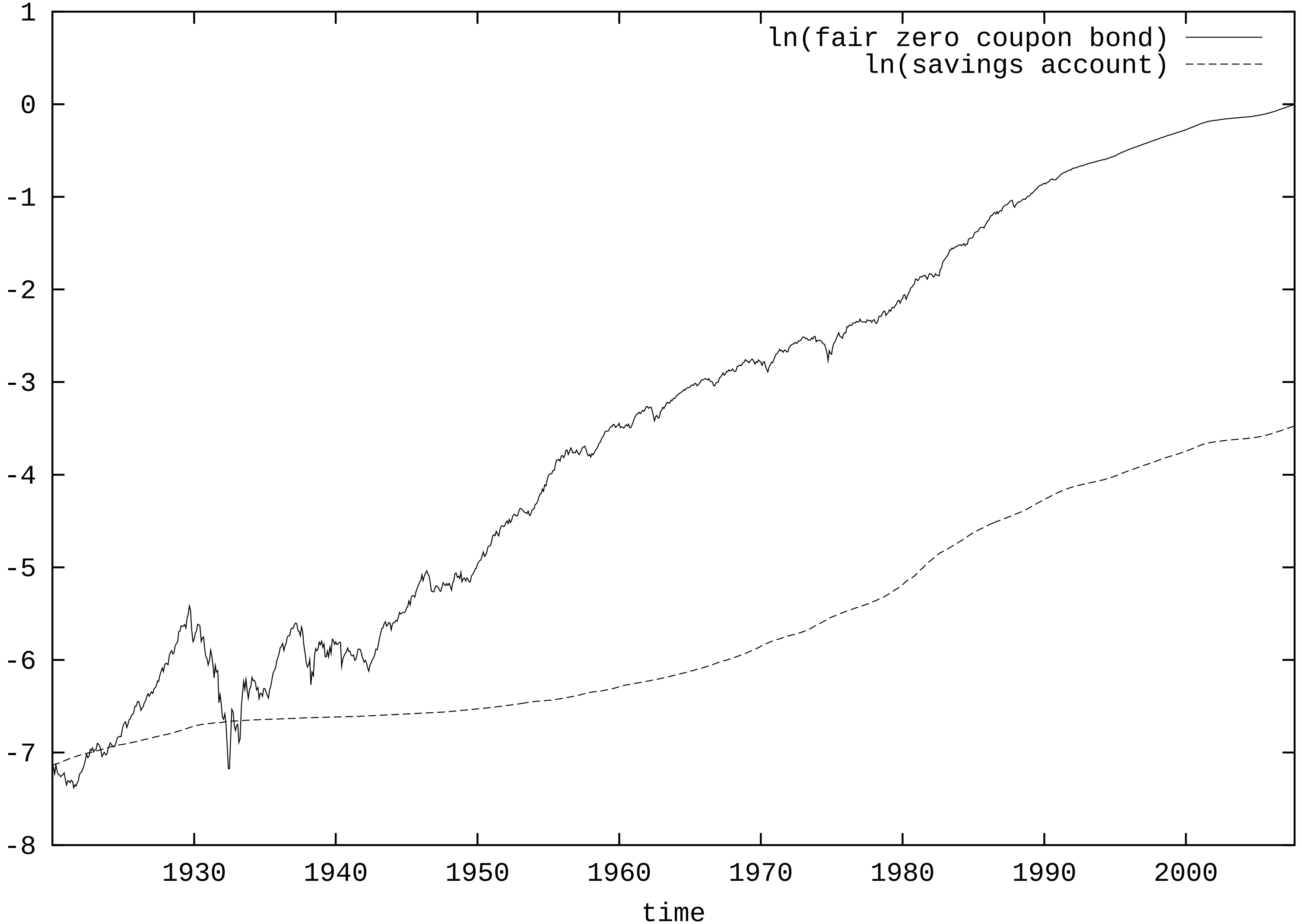

Figure 1.3: Logarithms of fair zero coupon bond and savings account.

Theorem 1.6 *Numeraire portfolio cannot be outperformed systematically.*

- **portfolio ratio**

$$A^{\delta}_{t,h} = \frac{S^{\delta}_{t+h}}{S^{\delta}_{t}}$$

- **expected growth**

$$g^{\delta}_{t,h} = E_t\left(\ln\left(A^{\delta}_{t,h}\right)\right)$$

$$t, h \geq 0$$

Definition 1.7 $S^{\underline{\delta}}$ growth optimal *if*

$$g^{\delta}_{t,h} \leq g^{\underline{\delta}}_{t,h}$$

for all $t, h \geq 0$ *and nonnegative* S^{δ}.

- **expected log-utility**

  Kelly (1956)

  Hakansson (1971a)

  Merton (1973a)

  Roll (1973)

  Markowitz (1976)

Theorem 1.8 *The numeraire portfolio is growth optimal.*

Strong Arbitrage

- market participants can only exploit arbitrage
- **limited liability**

$\Longrightarrow$ nonnegative total wealth of each market participant

Definition 1.9 *A nonnegative* S^δ *is a* strong arbitrage *if* $S_0^\delta = 0$ *and*

$$P\left(S_t^\delta > 0\right) > 0.$$

Pl. (2002)-mathematical arguments (super-martingale property)

Loewenstein & Willard (2000)-economic arguments

Theorem 1.10 *There is no strong arbitrage.*

- Delbaen & Schachermayer (1998)

  free lunches with vanishing risk (FLVR)

- Loewenstein & Willard (2000)

  free snacks & cheap thrills

$\Longrightarrow$ no-arbitrage pricing makes no sense under BA

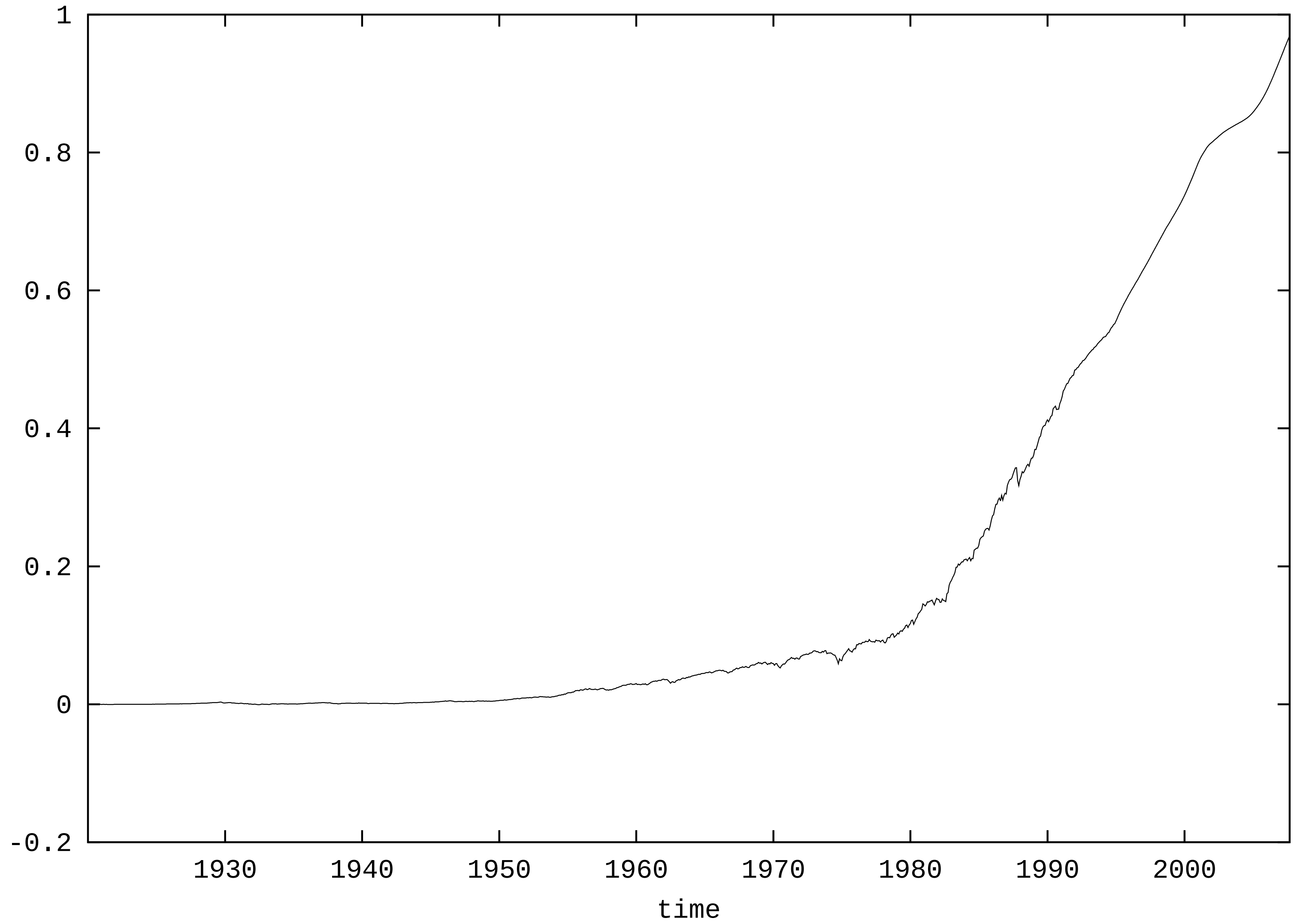

Figure 1.4: $P(t, T)$ minus savings account.

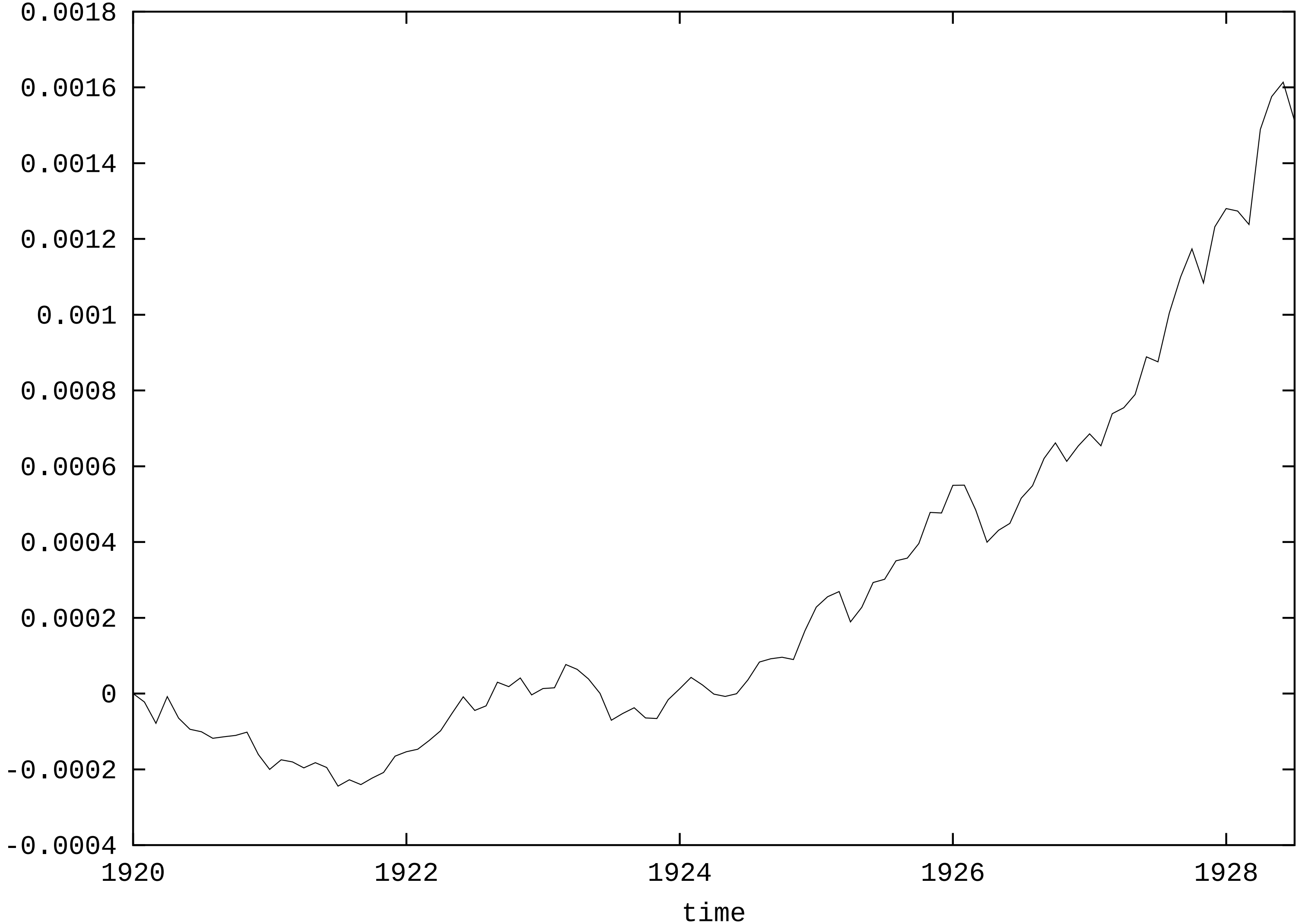

Figure 1.5: $P(t, T)$ minus savings account.

- Existence of equivalent risk neutral probability measure is more a **mathematical convenience** than an economic necessity.

- Basing quantitative methods on classical risk neutral pricing is restrictive.

- Trends are ignored under Fundamental Theorem of Asset Pricing

  Delbaen & Schachermayer (1998)

- **Equivalent to risk neutral pricing are pricing via:**

  stochastic discount factor, Cochrane (2001);

  deflator, Duffie (2001);

  pricing kernel, Constantinides (1992);

  state price density, Ingersoll (1987);

  numeraire portfolio as in Long (1990)

  All ignore trends !

- One needs consistent pricing concept that exploits trends

- All benchmarked nonnegative securities are supermartingales

  (no upward trend)

Definition 1.11 *Price is* fair *if, when benchmarked, forms martingale (no trend)*

$$\hat{S}^{\delta}_{t} = E_{t}\left(\hat{S}^{\delta}_{s}\right)$$

$0 \leq t \leq s < \infty.$

Lemma 1.12 *The* ***minimal nonnegative supermartingale*** *that reaches a given benchmarked contingent claim* ***is a martingale****.*

see Pl. & Heath (2006)

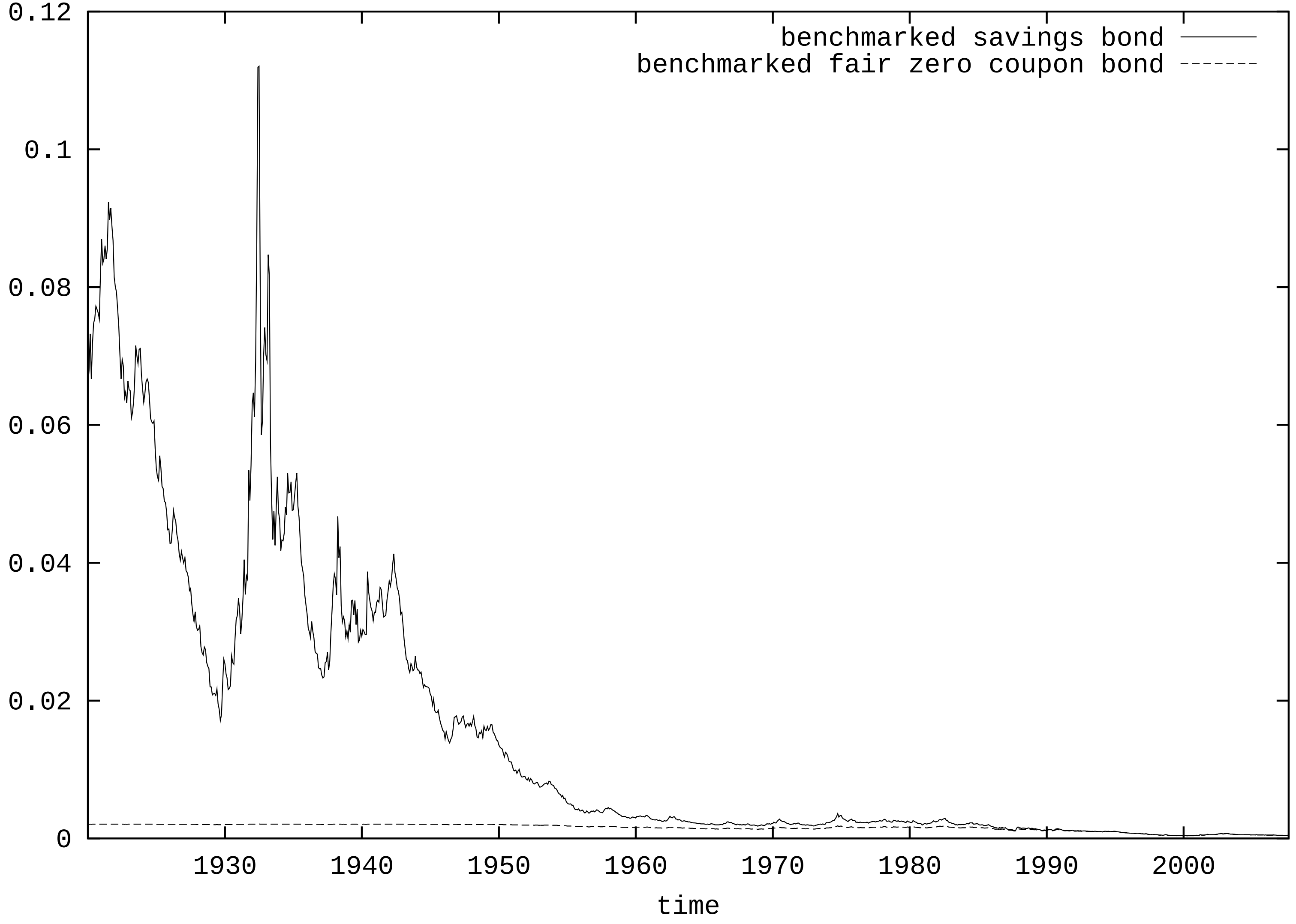

Figure 1.6: Benchmarked savings bond and benchmarked fair zero coupon bond.

Law of the Minimal Price

Pl. (2008)

Theorem 1.13 *If a fair portfolio replicates a nonnegative payoff, then this represents the minimal replicating portfolio.*

- least expensive
- minimal hedge
- economically correct price in a competitive market

- **contingent claim**

  $\boldsymbol{H_T}$

$$E_0\left(\frac{H_T}{S_T^{\delta_*}}\right) < \infty$$

- $S_t^{\delta_H}$ **fair** if

$$\hat{S}_t^{\delta_H} = \frac{S_t^{\delta_H}}{S_t^{\delta_*}} = E_t\left(\frac{H_T}{S_T^{\delta_*}}\right)$$

Link with real economy !

Law of the Minimal Price $\Longrightarrow$

Corollary 1.14

Minimal price for replicable H_T *is fair and given by*

real world pricing formula

$$S_t^{\delta_H} = S_t^{\delta_*} \, E_t \left(\frac{H_T}{S_T^{\delta_*}} \right) .$$

- most pricing concepts are unified and generalized by real world pricing

$\Longrightarrow$

actuarial pricing
risk neutral pricing
pricing with stochastic discount factor
pricing with numeraire change
pricing with deflator
pricing with state pricing density
pricing with pricing kernel
pricing with numeraire portfolio

When H_T independent of $S_T^{\delta_*}$

$\Longrightarrow$ rigorous derivation of

- **actuarial pricing formula**

$$S_t^{\delta_H} = P(t,T)\, E_t(H_T)$$

with **zero coupon bond**

$$P(t,T) = S_t^{\delta_*}\, E_t\left(\frac{1}{S_T^{\delta_*}}\right)$$

- **real world pricing formula** $\Longrightarrow$

$$S_0^{\delta_H} = E_0\left(\Lambda_T \frac{S_0^0}{S_T^0} H_T\right)$$

$$\Lambda_t = \frac{\hat{S}_t^0}{\hat{S}_0^0} \quad \text{supermartingale}$$

$$1 = \Lambda_0 \geq E_0(\Lambda_T)$$

$\Longrightarrow$

$$S_0^{\delta_H} \leq \frac{E_0\left(\Lambda_T \frac{S_0^0}{S_T^0} H_T\right)}{E_0(\Lambda_T)}$$

similar for any numeraire (here S^0 savings account)

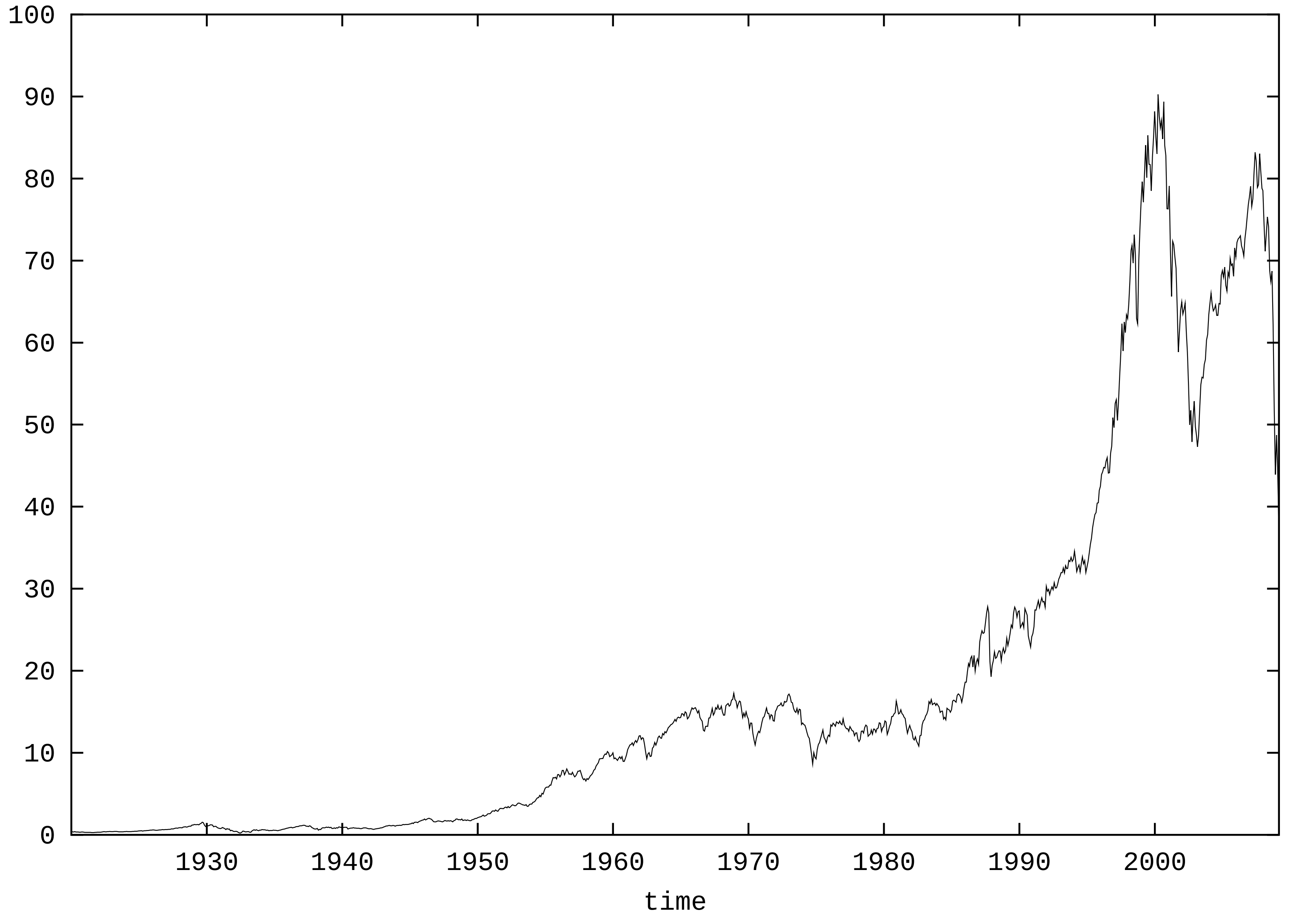

Figure 1.7: Discounted S&P500 total return index.

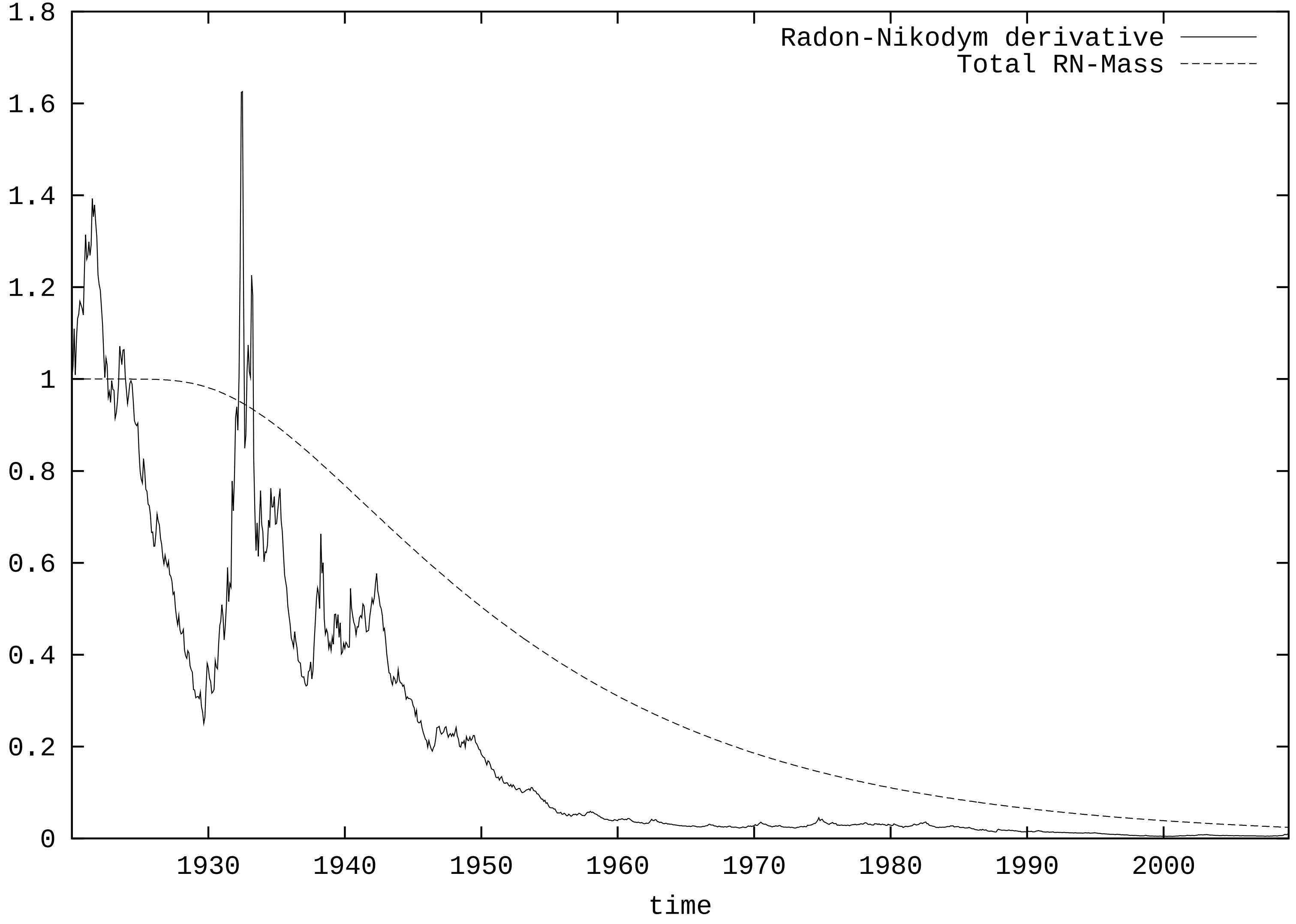

Figure 1.8: Radon-Nikodym derivative and total mass of putative risk neutral measure.

- **special case when savings account is fair:**

$$\Longrightarrow \qquad \Lambda_T = \frac{dQ}{dP} \text{ forms martingale;} \qquad E_0(\Lambda_T) = 1;$$

equivalent risk neutral probability measure Q exists;

Bayes' formula $\Longrightarrow$
risk neutral pricing formula

$$S_0^{\delta_H} = E_0^Q \left(\frac{S_0^0}{S_T^0} H_T \right)$$

Harrison & Kreps (1979), Ingersoll (1987),

Constantinides (1992), Duffie (2001), Cochrane (2001), ...

- otherwise "risk neutral price" $\geq$ real world price

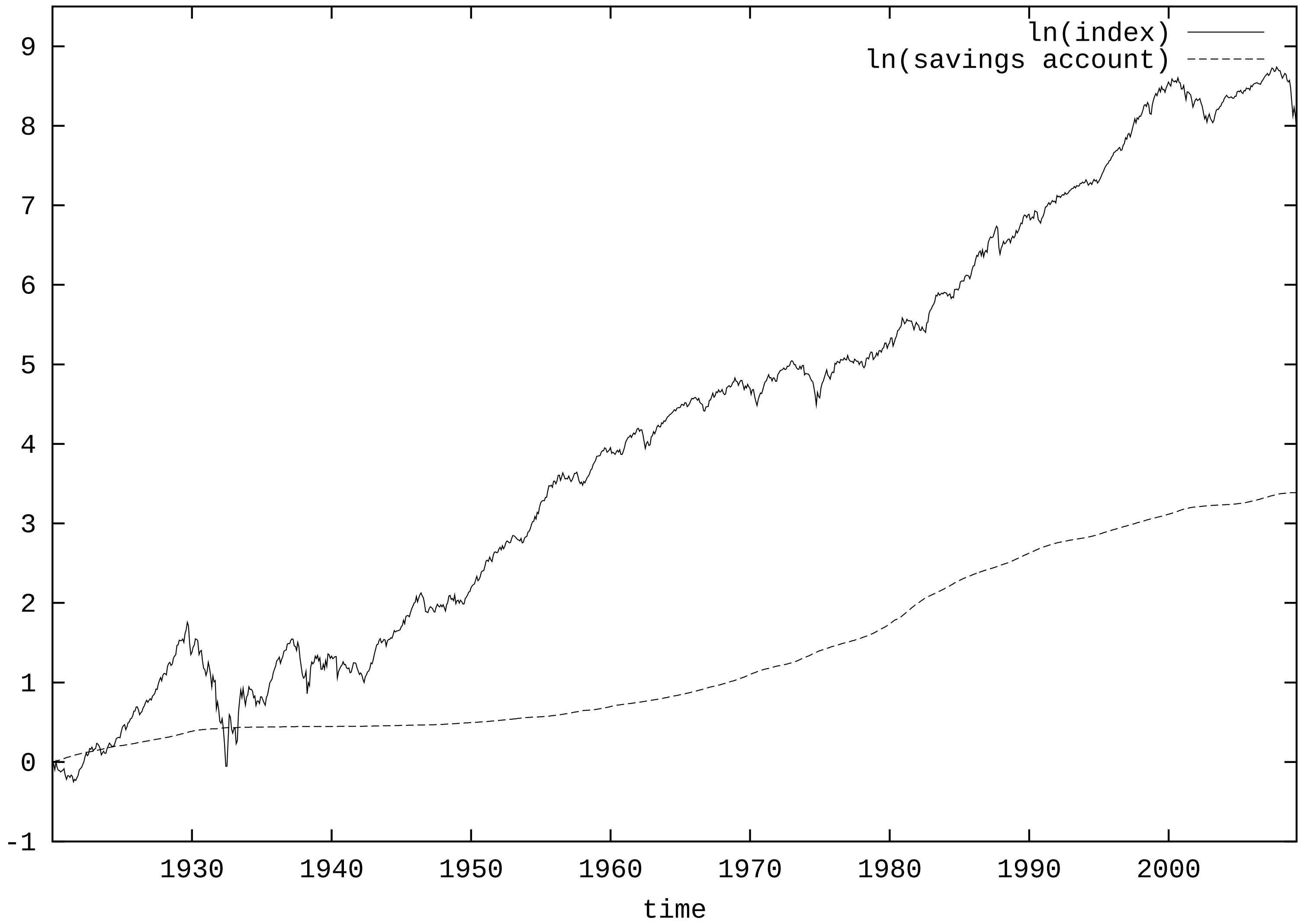

Figure 1.9: ln(S&P500 accumulation index) and ln(savings account).

Extreme Maturity Bond

- short rate deterministic

- **savings bond**

$$P^*(t,T) = \frac{S_t^0}{S_T^0}$$

- index S^1 as numeraire portfolio S^{δ_*}

- **fair zero coupon bond**

  Law of the Minimal Price

  $\Longrightarrow$ real world pricing formula

$$P(t,T) = S_t^{\delta_*} \, E_t\left(\frac{1}{S_T^{\delta_*}}\right)$$

- benchmarked fair zero coupon bond

$$\hat{P}(t,T) = \frac{P(t,T)}{S_t^{\delta_*}} \qquad \text{martingale}$$

- **discounted numeraire portfolio**

$$\bar{S}_t^{\delta_*} = \frac{S_t^{\delta_*}}{S_t^0}$$

numeraire portfolio for continuous market:

$$d\bar{S}_t^{\delta_*} = \alpha_t \, dt + \sqrt{\bar{S}_t^{\delta_*} \, \alpha_t} \, dW_t$$

time transformed squared Bessel process

- model the drift of discounted index as

$$\alpha_t = \alpha \exp\{\eta \, t\}$$

$\Longrightarrow$

$\bar{S}_t^{\delta_*}$ time transformed squared Bessel process of dimension four

$\Longrightarrow$ **Minimal Market Model**

MMM, see Pl. & Heath (2006)

- **net growth rate**

$\eta \approx 0.054$ with R^2 of 0.89

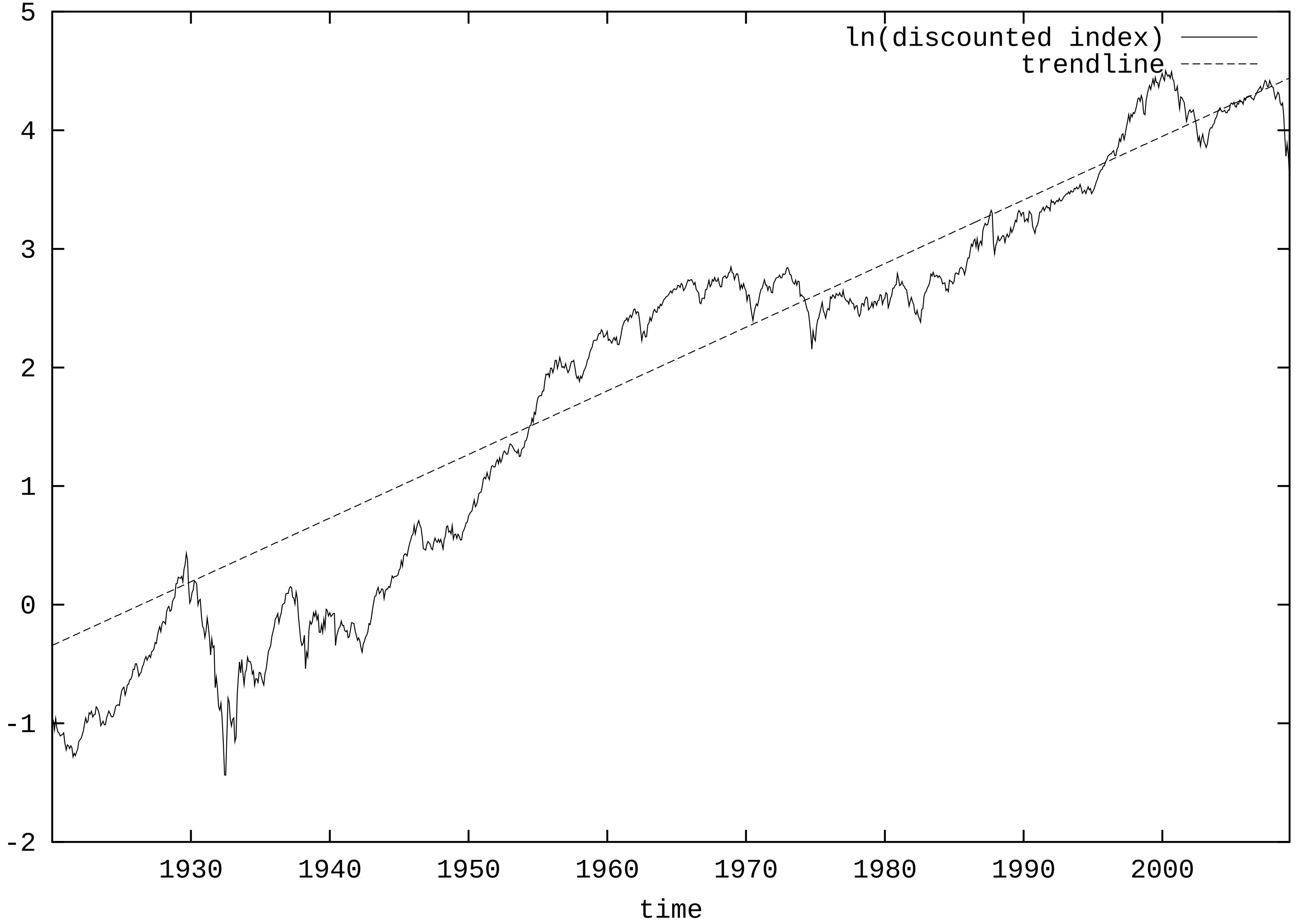

Figure 1.10: Logarithm of discounted S&P500 and trendline.

- **normalized index**

$$Y_t = \frac{\bar{S}_t^{\delta_*}}{\alpha_t}$$

$$dY_t = (1 - \eta\, Y_t)\, dt + \sqrt{Y_t}\, dW_t$$

- **quadratic variation of** $\sqrt{Y_t}$

$$d\sqrt{Y_t} = \ldots + \frac{1}{2}\, dW_t$$

$$\sum_{\ell=1}^{i_t} \left(\sqrt{Y_{t_\ell}} - \sqrt{Y_{t_{\ell-1}}} \right)^2 \approx \left[\sqrt{Y} \right]_t = \frac{t}{4}$$

- **scaling parameter** $\alpha \approx 0.01468$ with R^2 of 0.995

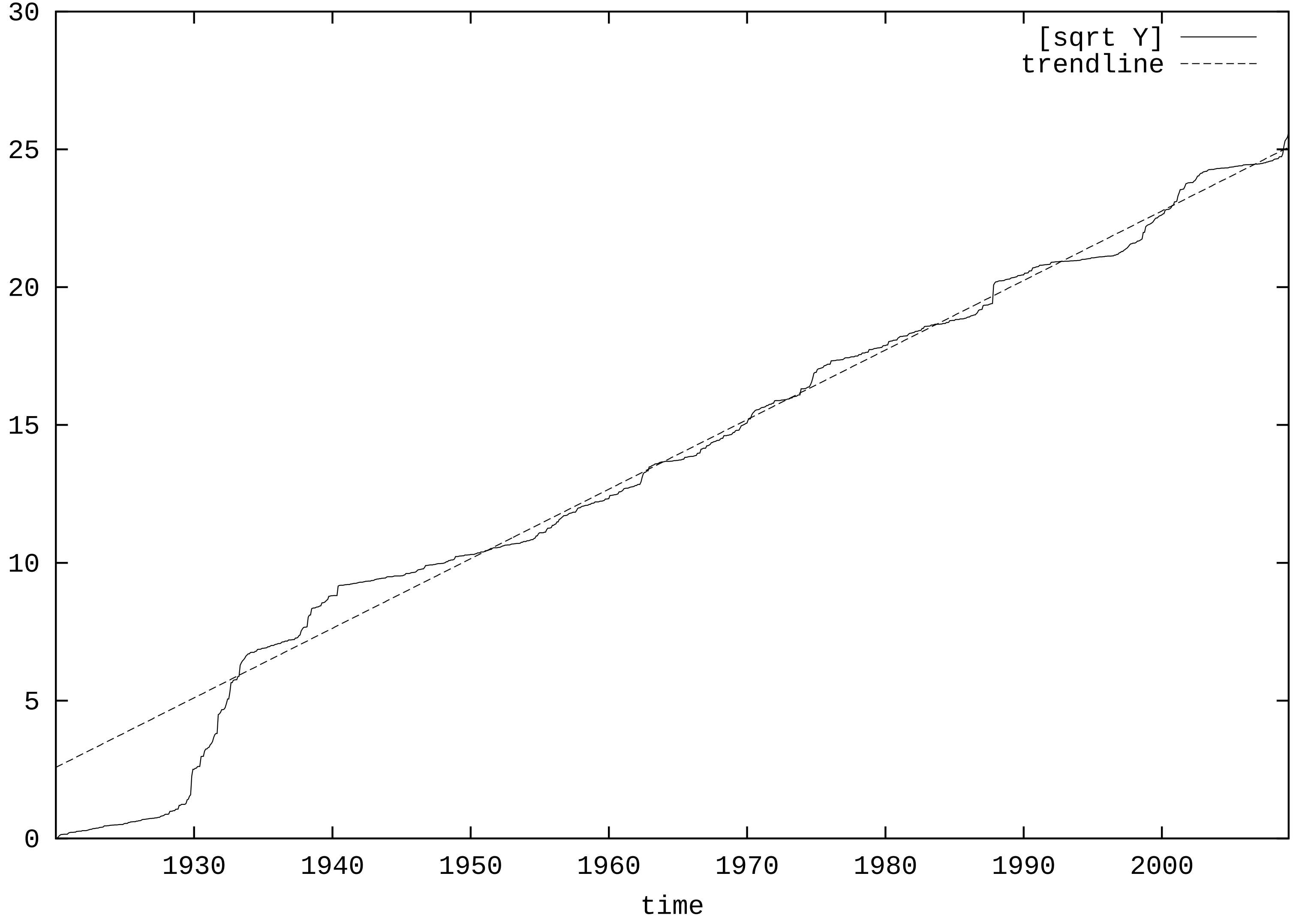

Figure 1.11: Quadratic Variation of $\sqrt{Y_t}$.

- **fair zero coupon bond**

$$P(t,T) = P^*(t,T)\left(1 - \exp\left\{-\frac{2\,\eta\,\bar{S}_t^{\delta_*}}{\alpha\,(\exp\{\eta\,T\} - \exp\{\eta\,t\})}\right\}\right)$$

- **initial prices**

$$P^*(0,T) = 0.0335$$

$$P(0,T) = 0.00077$$

$$\frac{P(0,T)}{P^*(0,T)} < 0.023 \qquad \Longrightarrow \qquad 2.3\%$$

no strong arbitrage

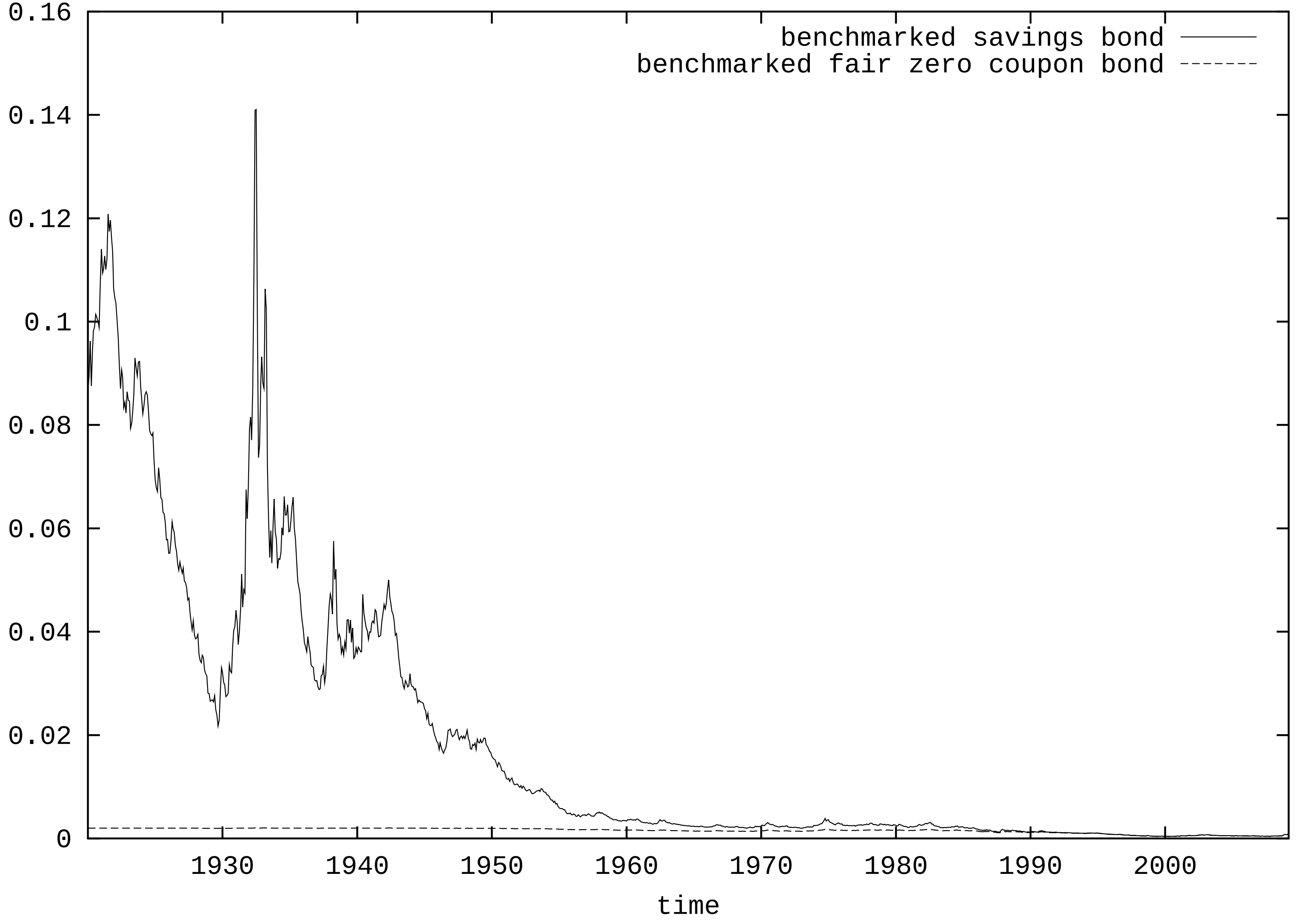

Figure 1.12: Benchmarked savings bond and benchmarked fair zero coupon bond.

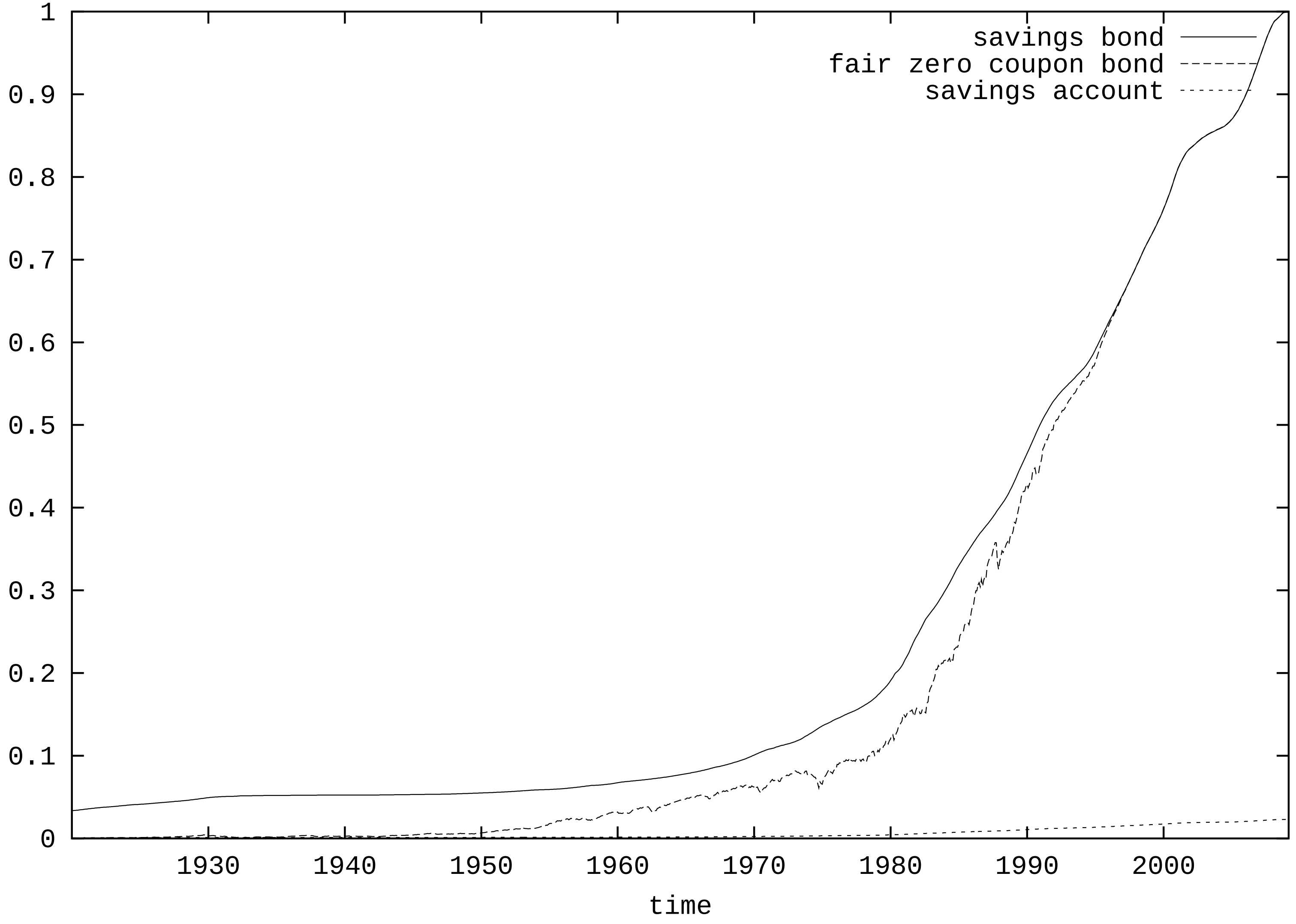

Figure 1.13: Savings bond, fair zero coupon bond and savings account.

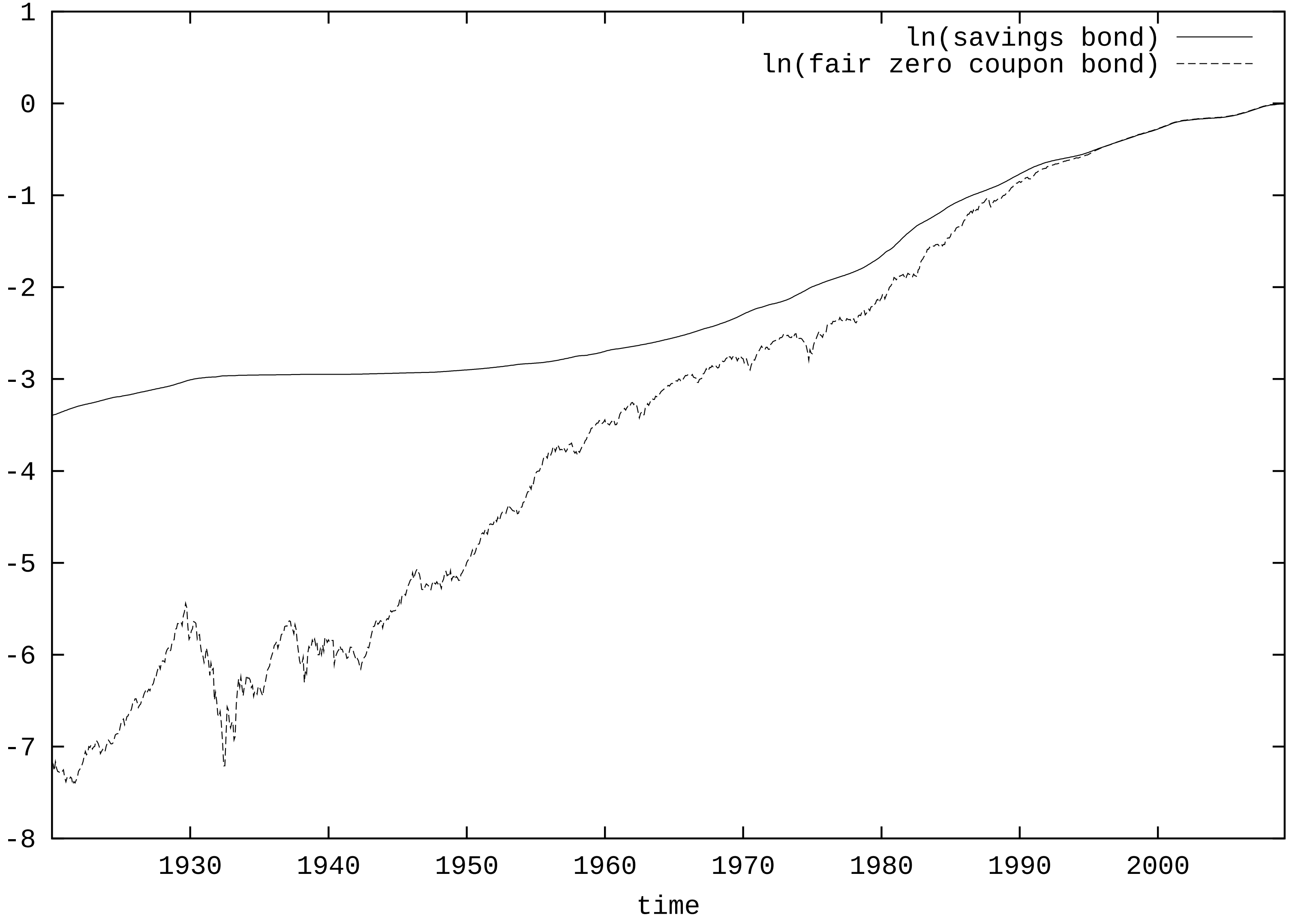

Figure 1.14: Logarithms of fair zero coupon bond and savings account.

- savings account is *outperformed systematically* by fair zero coupon bond

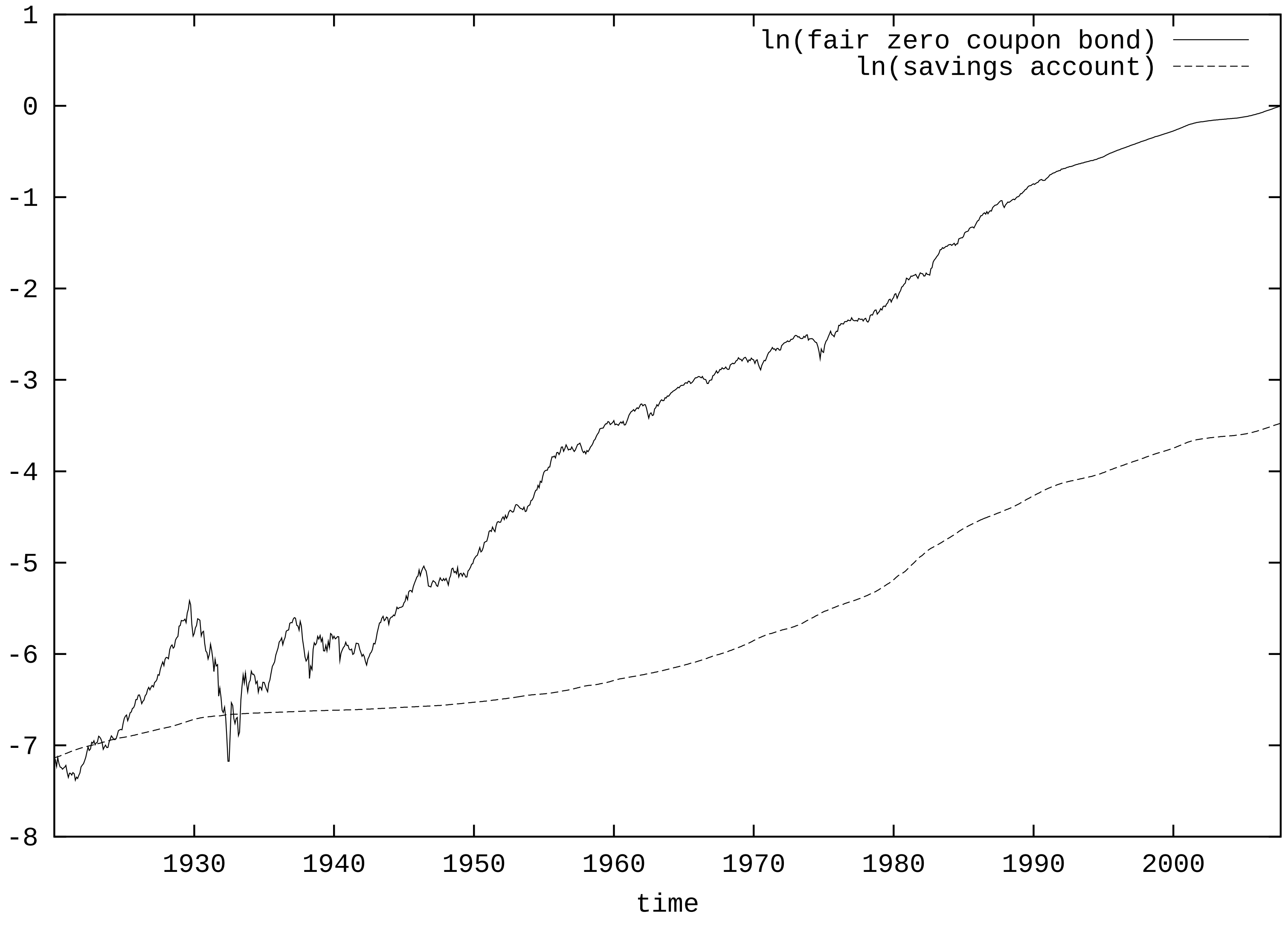

Figure 1.15: Logarithms of fair zero coupon bond and savings account.

- **self-financing hedge portfolio** hedge ratio

$$\delta_t^* = \frac{\partial \bar{P}(t,T)}{\partial \bar{S}_t^{\delta_*}}$$

$$= P^*(0,T) \exp\left\{\frac{-2\,\eta\,\bar{S}_t^{\delta_*}}{\alpha\,(\exp\{\eta\,T\} - \exp\{\eta\,t\})}\right\}$$

$$\times \frac{2\,\eta}{\alpha\,(\exp\{\eta\,T\} - \exp\{\eta\,t\})}$$

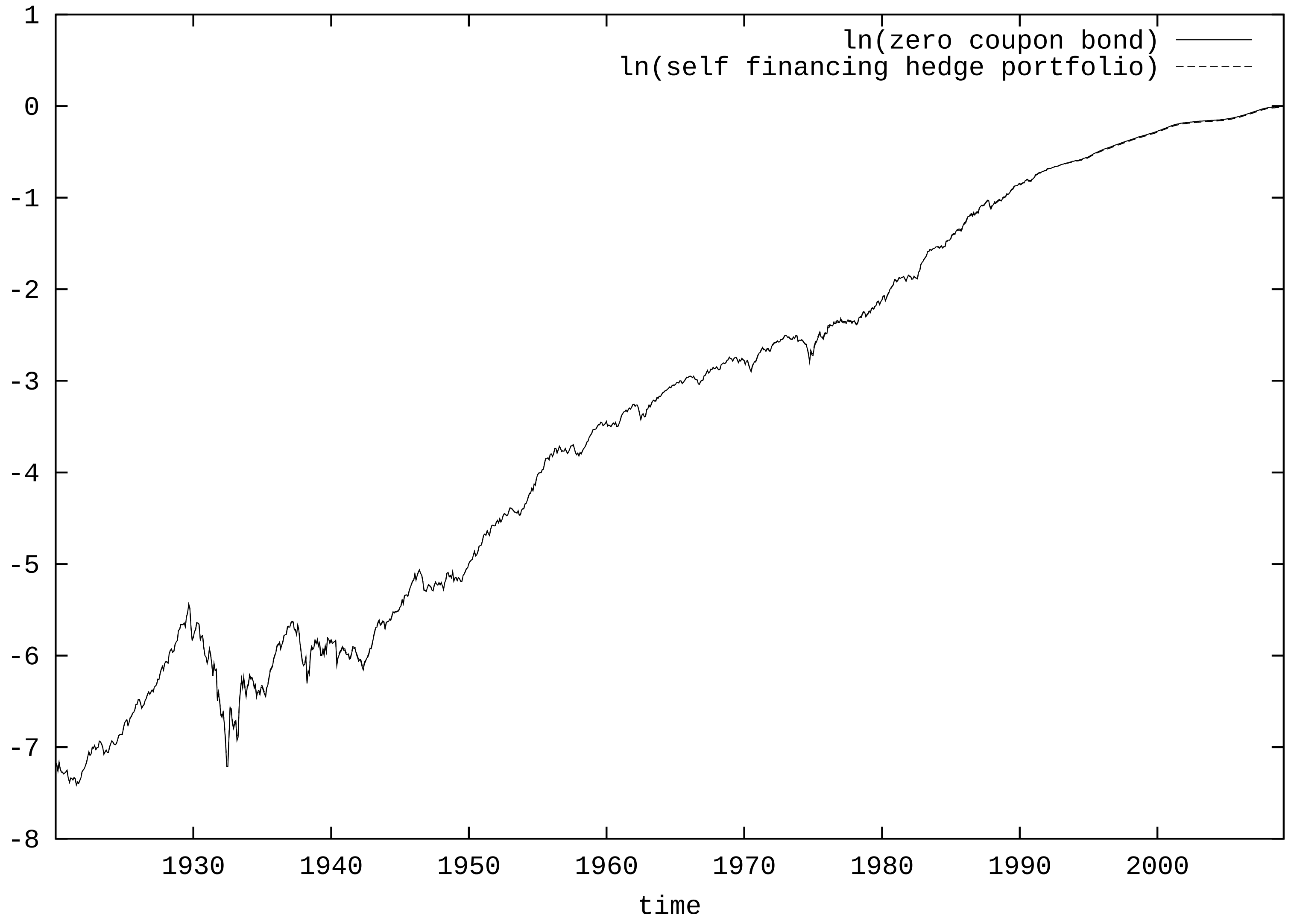

Figure 1.16: Logarithm of zero coupon bond and self-financing hedge portfolio.

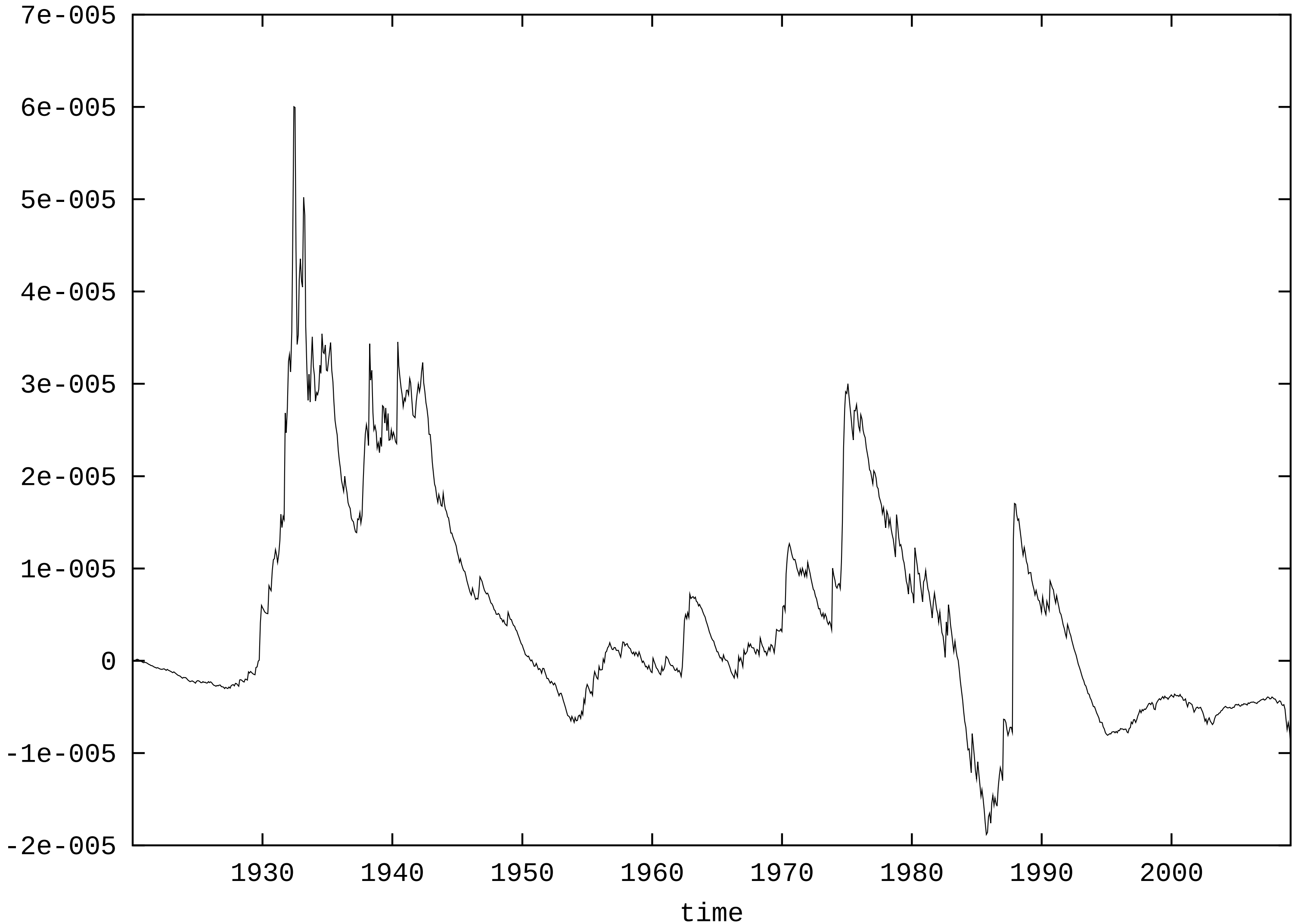

Figure 1.17: Benchmarked profit and loss.

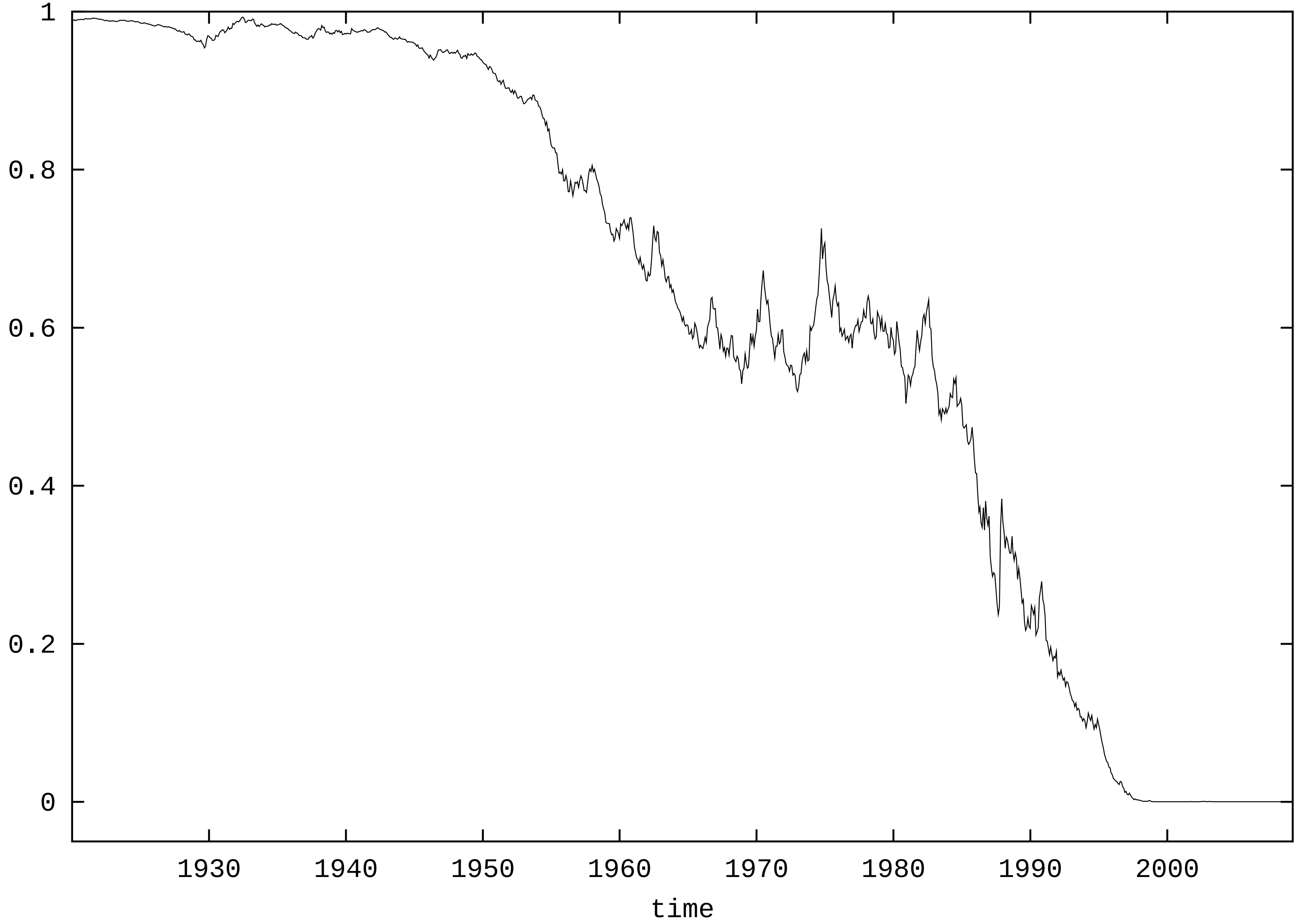

Figure 1.18: Fraction invested in the index.

2 Continuous Financial Markets

- d sources of continuous **traded uncertainty**

  independent standard Wiener processes

  $$W^1, W^2, \ldots, W^d, \quad d \in \{1, 2, \ldots\}$$

Primary Security Accounts

S_t^j - jth primary security account at time t,

$j \in \{0, 1, \ldots, d\}$

cum-dividend share value or savings account value,

dividends or interest are accumulated, reinvested

- vector process of primary security accounts

$$S = \{S_t = (S_t^0, \ldots, S_t^d)^\top,\ t \in [0, T]\}$$

- assume **unique strong solution** of SDE

$$dS_t^j = S_t^j \left(a_t^j \, dt + \sum_{k=1}^{d} b_t^{j,k} \, dW_t^k \right)$$

$t \in [0, \infty), \; S_0^j > 0, \; j \in \{1, 2, \ldots, d\}$

- predictable, integrable appreciation rate processes a^j

- predictable, square integrable volatility processes $b^{j,k}$

- **savings account**

$$S_t^0 = \exp\left\{\int_0^t r_s\, ds\right\}$$

predictable, integrable short rate process

$$r = \{r_t,\ t \in [0, \infty)\}$$

- **Market price of risk**

The unique invariant of the market

$$\theta_t = (\theta_t^1, \theta_t^2, \ldots, \theta_t^d)^\top$$

solution of relation

$$b_t \, \theta_t = a_t - r_t \, \mathbf{1}$$

where

$$a_t = (a_t^1, a_t^2, \ldots, a_t^d)^\top$$

$$\mathbf{1} = (1, 1, \ldots, 1)^\top$$

Assumption 2.1

Volatility matrix $\boldsymbol{b_t} = [b_t^{j,k}]_{j,k=1}^{d}$ *is* invertible

$\Longrightarrow$

- **market price of risk** equals

$$\boldsymbol{\theta_t} = \boldsymbol{b_t^{-1}} (\boldsymbol{a_t} - r_t \, \mathbf{1})$$

- rewrite jth primary security account SDE

$$dS_t^j = S_t^j \left\{ r_t \, dt + \sum_{k=1}^{d} b_t^{j,k} \, (\theta_t^k \, dt + dW_t^k) \right\}$$

$t \in [0, \infty), \; j \in \{1, 2, \ldots, d\}$

- **strategy**

$$\delta = \{\delta_t = (\delta_t^0, \dots, \delta_t^d)^\top, \, t \in [0, T]\}$$

predictable and S-integrable,

δ_t^j number of units of jth primary security account

- **portfolio**

$$S_t^\delta = \sum_{j=0}^{d} \delta_t^j \, S_t^j$$

- S^δ **self-financing** $\Longleftrightarrow$

$$dS_t^\delta = \sum_{j=0}^{d} \delta_t^j \, dS_t^j$$

changes in the portfolio value only due to gains from trading,

self-financing in each denomination

- **jth fraction**

$$\pi^j_{\delta,t} = \delta^j_t \, \frac{S^j_t}{S^\delta_t}$$

$$j \in \{0, 1, \ldots, d\}$$

$$\sum_{j=0}^{d} \pi^j_{\delta,t} = 1$$

- **portfolio SDE**

$$dS_t^\delta = S_t^\delta \left(r_t \, dt + \sum_{k=1}^{d} b_{\delta,t}^k \; (\theta_t^k \, dt + dW_t^k) \right)$$

with kth portfolio volatility

$$b_{\delta,t}^k = \sum_{j=1}^{d} \pi_{\delta,t}^j \, b_t^{j,k}$$

- **logarithm** of strictly positive portfolio

$$d\ln(S_t^\delta) = g_t^\delta\,dt + \sum_{k=1}^{d} b_{\delta,t}^k\,dW_t^k$$

with **growth rate**

$$g_t^\delta = r_t + \sum_{k=1}^{d} b_{\delta,t}^k \left(\theta_t^k - \frac{1}{2}\,b_{\delta,t}^k\right)$$

Numeraire Portfolio S^{δ_*} = Growth Optimal Portfolio

$\Longrightarrow$ **maximize the growth rate**

$$g_t^\delta = r_t + \sum_{k=1}^{d} \sum_{j=1}^{d} \pi_{\delta,t}^{j} \, b_t^{j,k} \left(\theta_t^k - \frac{1}{2} \sum_{\ell=1}^{d} \pi_{\delta,t}^{\ell} \, b_t^{\ell,k} \right)$$

$\Longrightarrow$ for each $j \in \{1, 2, \ldots, d\}$

$$0 = \sum_{k=1}^{d} b_t^{j,k} \left(\theta_t^k - \sum_{\ell=1}^{d} \pi_{\delta_*,t}^{\ell} \, b_t^{\ell,k} \right)$$

$\Longrightarrow$

$$\theta_t^\top = \pi_{\delta_*,t}^\top \, b_t$$

Since b_t invertible $\Longrightarrow$

$$\pi_{\delta_*,t} = (\pi^1_{\delta_*,t}, \ldots, \pi^d_{\delta_*,t})^\top$$
$$= (b_t^{-1})^\top \theta_t$$

$\Longrightarrow$ **NP**

$$dS_t^{\delta_*} = S_t^{\delta_*} \left(\left[r_t + \sum_{k=1}^{d} (\theta_t^k)^2 \right] dt + \sum_{k=1}^{d} \theta_t^k \, dW_t^k \right)$$

finite NP exists

Benchmarked Portfolios

$$\hat{S}_t^{\delta} = \frac{S_t^{\delta}}{S_t^{\delta_*}}$$

$$dS_t^{\delta} = S_t^{\delta} \left(r_t\, dt + \sum_{k=1}^{d} \sum_{j=0}^{d} \pi_{\delta,t}^{j}\, b_t^{j,k} \left(\theta_t^k\, dt + dW_t^k \right) \right)$$

$$dS_t^{\delta_*} = S_t^{\delta_*} \left(r_t\, dt + \sum_{k=1}^{d} \theta_t^k \left(\theta_t^k\, dt + dW_t^k \right) \right)$$

$$d\hat{S}_t^{\delta} = -\hat{S}_t^{\delta} \sum_{k=1}^{d} \sum_{j=0}^{d} \pi_{\delta,t}^{j}\, s_t^{j,k}\, dW_t^k$$

driftless $\Longrightarrow$ local martingale $\Longrightarrow$ supermartingale when nonnegative

Theorem 2.2 *Any nonnegative benchmarked portfolio is an*

$(\underline{\mathcal{A}}, P)$*-supermartingale* *(no upward trend).*

$\Longrightarrow$ **NP is numeraire portfolio**

use NP as **benchmark for investment** and as **numeraire for pricing**

- markets have thousands of primary security accounts

Example of a Multi-Asset BS Model

- jth primary security account

$$dS_t^j = S_t^j \left[\left(r + \sigma^2 \left(1 + \frac{1}{\sqrt{d}} \right) \right) dt + \frac{\sigma}{\sqrt{d}} \sum_{k=1}^{d} dW_t^k + \sigma \, dW_t^j \right]$$

$$t \in [0, T], j \in \{1, 2, \ldots, d\}, \sigma > 0$$

general market risk, specific market risk

- **NP fractions**

$$\pi^j_{\delta_*,t} = \left(\sqrt{d}\left(1+\sqrt{d}\right)\right)^{-1}$$

$j \in \{1, 2, \ldots, d\}$

$$\pi^0_{\delta_*,t} = \left(1+\sqrt{d}\right)^{-1}$$

- **NP SDE**

$$dS^{\delta_*}_t = S^{\delta_*}_t \left(\left(r+\sigma^2\right) dt + \frac{\sigma}{\sqrt{d}} \sum_{k=1}^{d} dW^k_t\right)$$

- simulate over $T = 32$ years $d = 50$ risky primary security accounts

$\sigma = 0.15, r = 0.05$

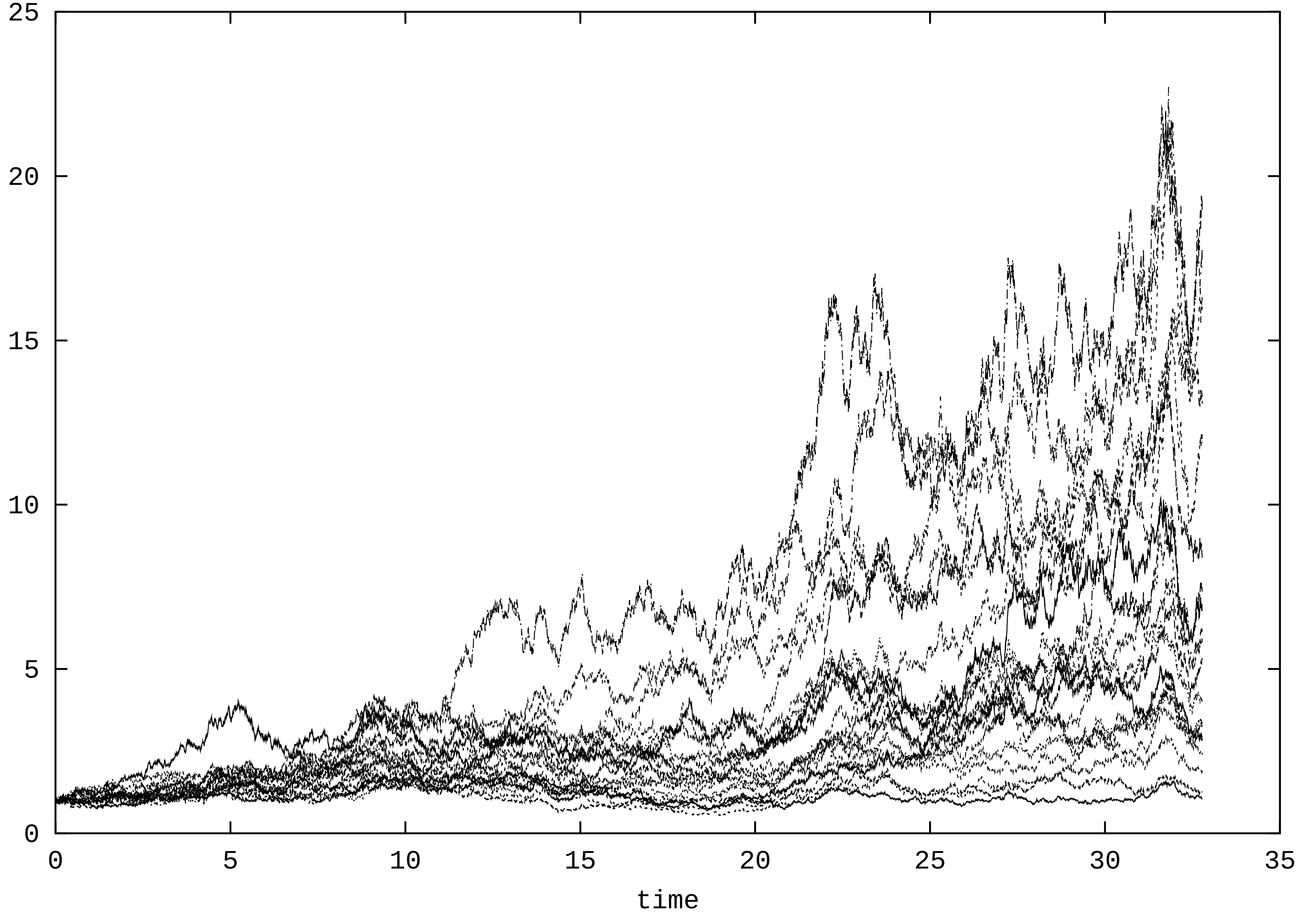

Figure 2.1: Simulated primary security accounts.

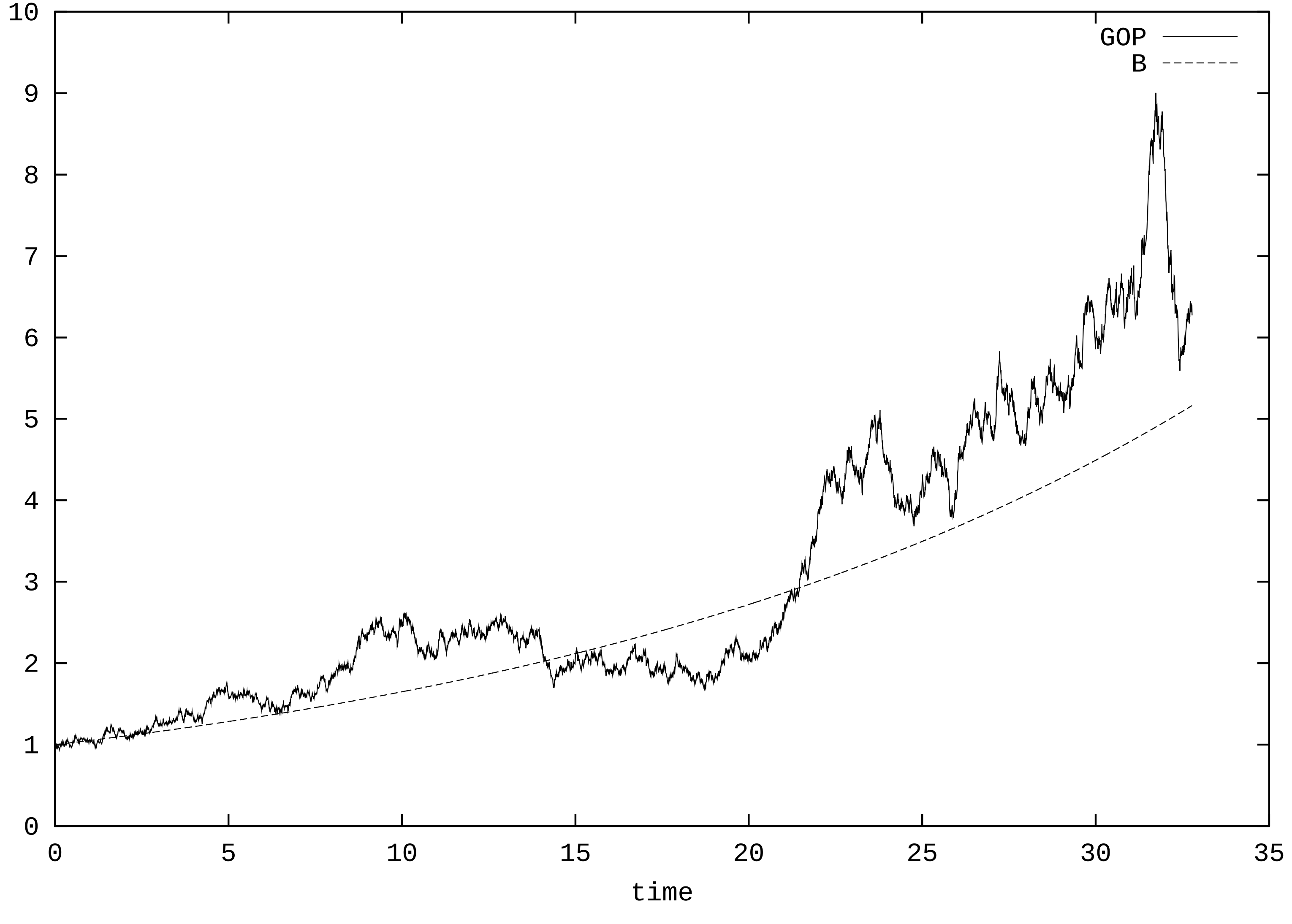

Figure 2.2: Simulated NP and savings account.

Diversified Portfolios

Fundamental phenomenon of **diversification** leads naturally to **NP**:

diversified $\Longrightarrow$ fractions "small"

Definition 2.3 *A strictly positive portfolio* S^{δ} *is a* **diversified portfolio** *if*

$$\left|\pi^{j}_{\delta,t}\right| \leq \frac{K_2}{d^{\frac{1}{2}+K_1}}$$

a.s. for all $j \in \{0, 1, \ldots, d\}$ *and* $t \in [0, T]$ *for* $K_1, K_2 \in (0, \infty)$ *and* $K_3 \in \{1, 2, \ldots\}$*, independent of* $d \in \{K_3, K_3 + 1, \ldots\}$.

- jth benchmarked primary security account

$$\hat{S}_t^j = \frac{S_t^j}{S_t^{\delta_*}}$$

$$d\hat{S}_t^j = -\hat{S}_t^j \sum_{k=1}^{d} s_t^{j,k}\, dW_t^k$$

- (j, k)th **specific volatility**

$$s_t^{j,k} = \theta_t^k - b_t^{j,k}$$

measures **specific market risk** of S^j with respect to W^k

- θ_t^k measures **general market risk** with respect to W^k

- **kth total specific absolute volatility**

$$\hat{s}_t^k = \sum_{j=0}^{d} |s_t^{j,k}|$$

regular $\Longrightarrow$ variance of specific returns “bounded”

Definition 2.4 *A market is called* **regular** *if*

$$E\left(\left(\hat{s}_t^k\right)^2\right) \leq K_5.$$

Tracking Rate

Variance of benchmarked portfolio returns

$$R^d_\delta(t) = \sum_{k=1}^{d} \left(\sum_{j=0}^{d} \pi^j_{\delta,t} \, s^{j,k}_t \right)^2$$

$R^d_\delta(t)$ - quantifies distance between S^δ and S^{δ_*}

benchmarked NP constant $\Longrightarrow$

$$R^d_{\delta_*}(t) = 0$$

Approximate NP

tracking rate "small"

Definition 2.5 *A strictly positive portfolio* S^δ *is an* **approximate NP** *if for all* $t \in [0,T]$ *and* $\varepsilon > 0$

$$\lim_{d\to\infty} P\left(R^d_\delta(t) > \varepsilon\right) = 0.$$

Diversification Theorem

Theorem 2.6 *In a regular market*

any diversified portfolio is an approximate NP.

- **model independent**
- similarity to CLT

Equal Weighted Index (EWI)

$S^{\delta_{\mathrm{EWI}}}$ holds equal fractions

$$\pi^j_{\delta_{\mathrm{EWI}},t} = \frac{1}{d+1}$$

$j \in \{0, 1, \ldots, d\}$

- tracking rate for multi-asset BS example:

$$R^d_{\delta_{\mathrm{EWI}}}(t) = \sum_{k=1}^{d} \left(\frac{1}{d+1} \left(\theta^k_t - b^{k,k}_t \right) \right)^2$$

$$= d \left(\frac{1}{d+1} \left(\frac{\sigma}{\sqrt{d}} - \sigma \right) \right)^2 \leq \frac{\sigma^2}{(d+1)} \to 0$$

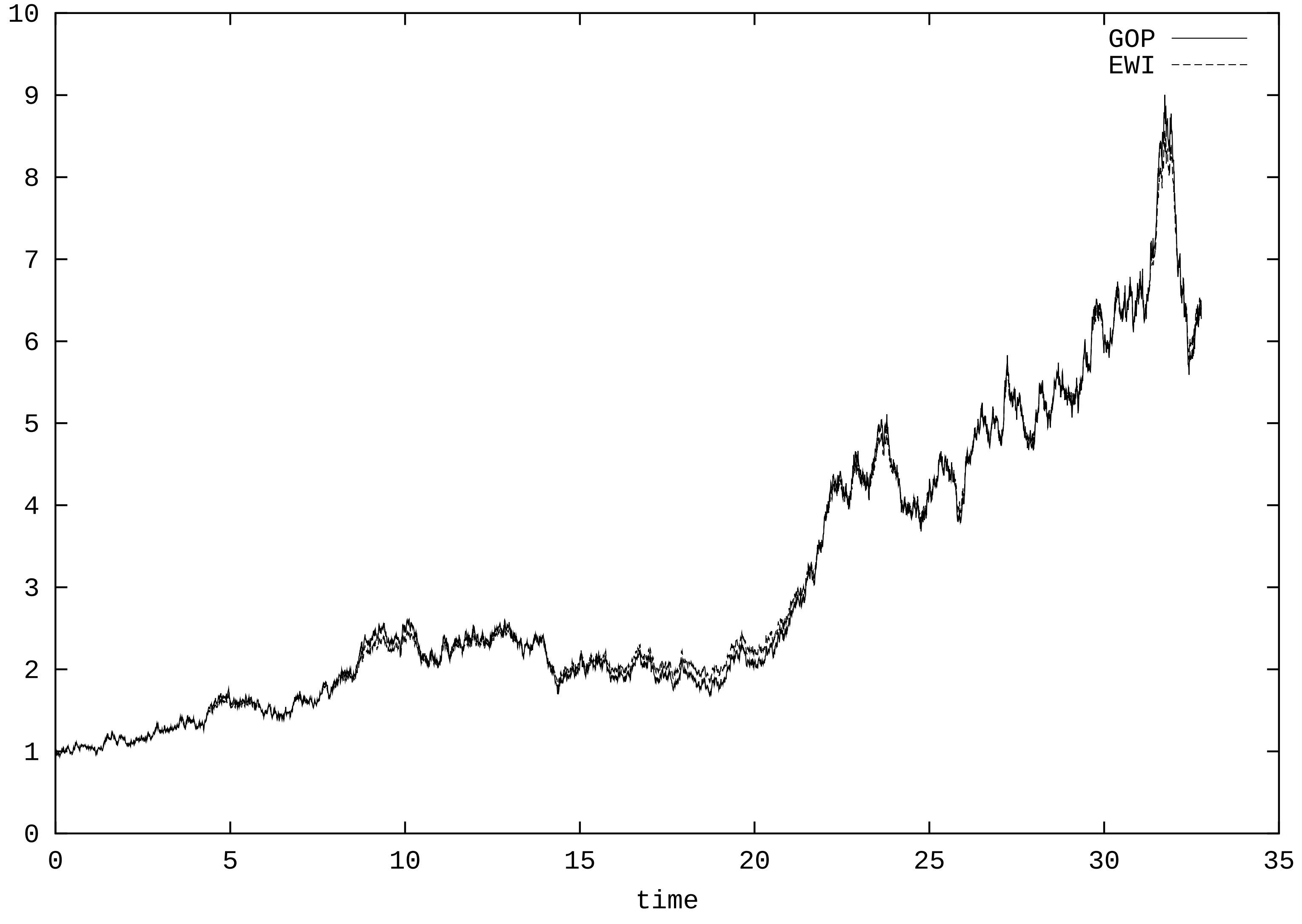

Figure 2.3: Simulated NP and EWI.

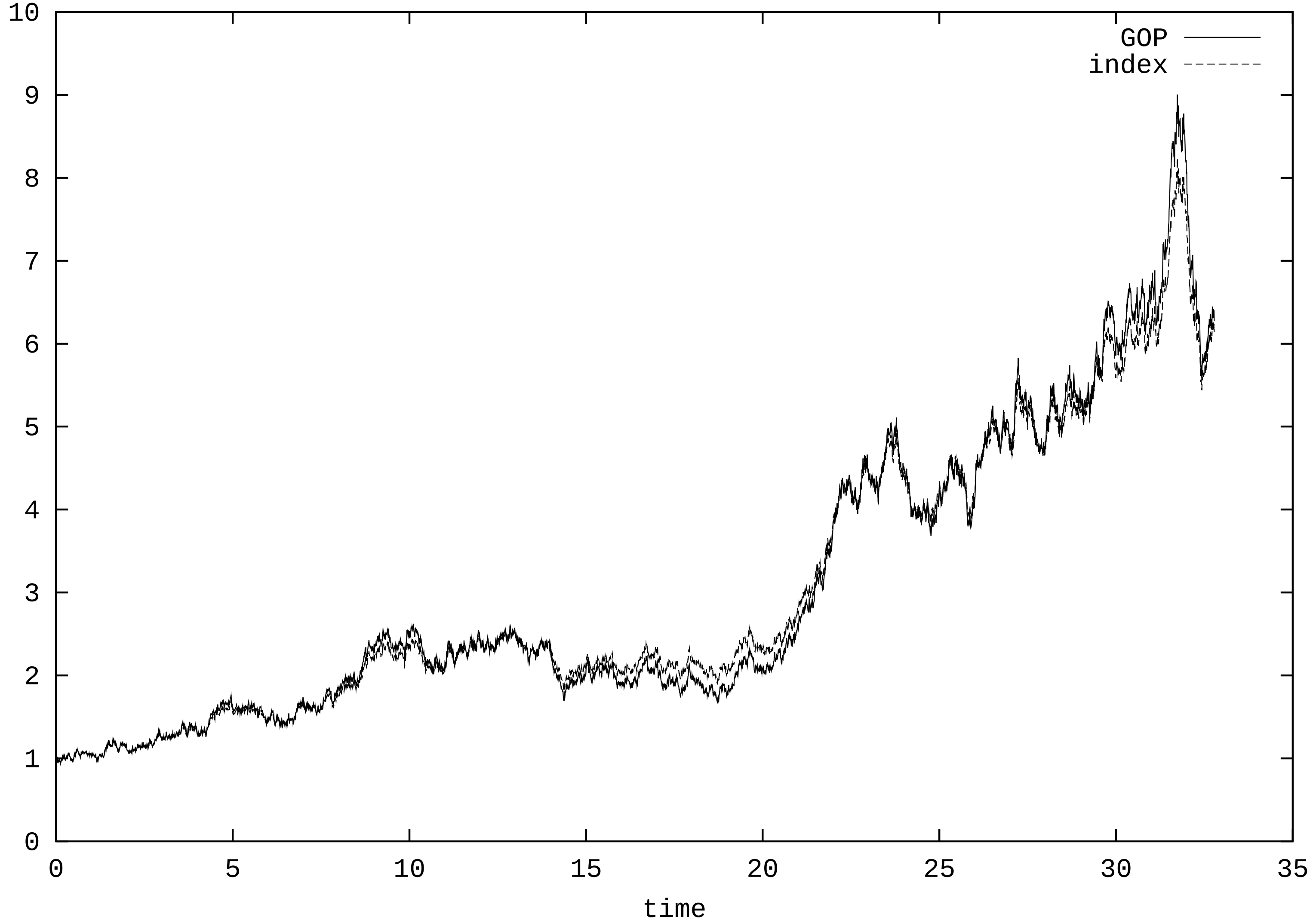

Figure 2.4: Simulated diversified accumulation index and NP.

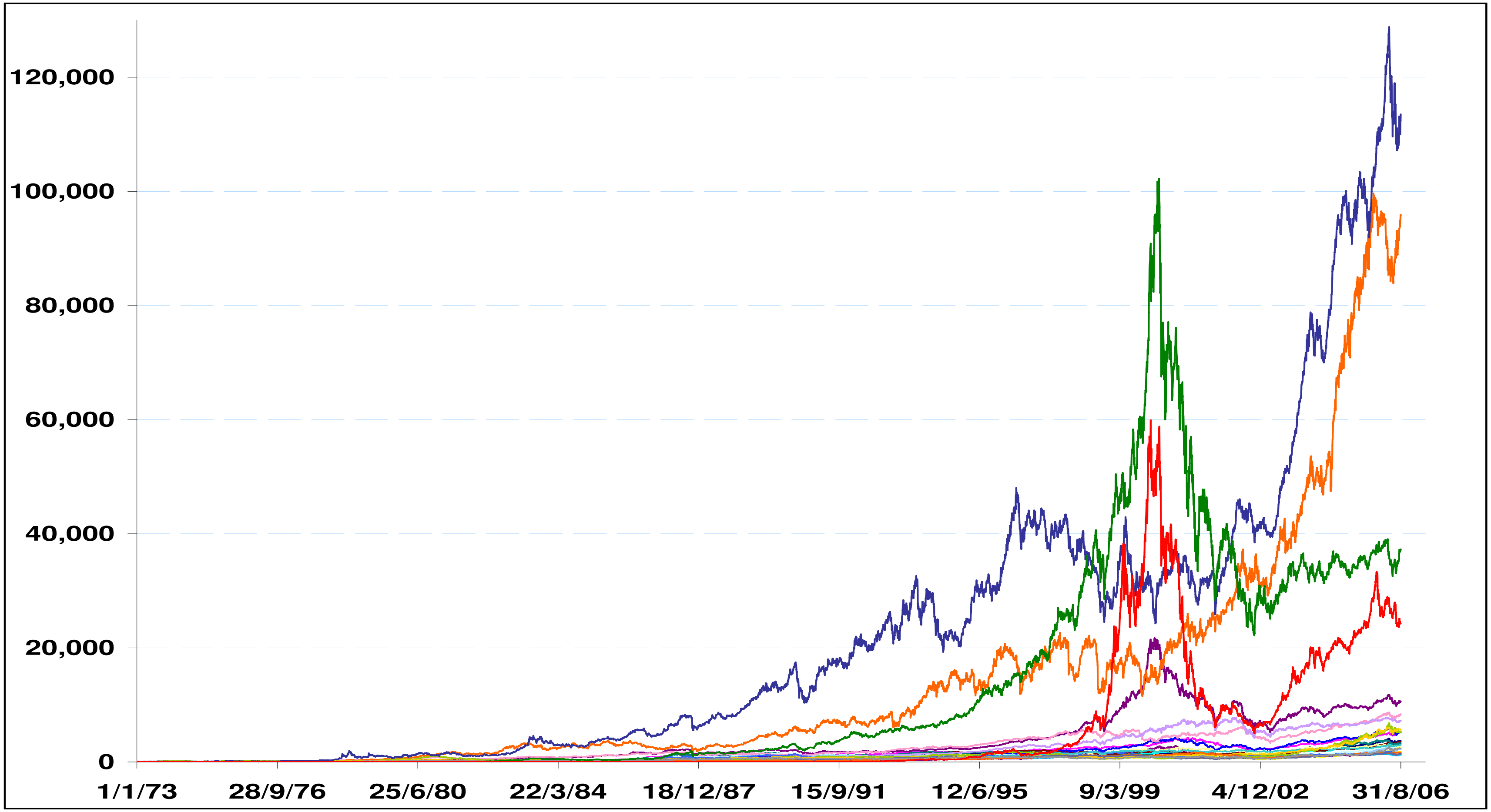

Figure 2.5: World industry sector stock market indices as primary security accounts.

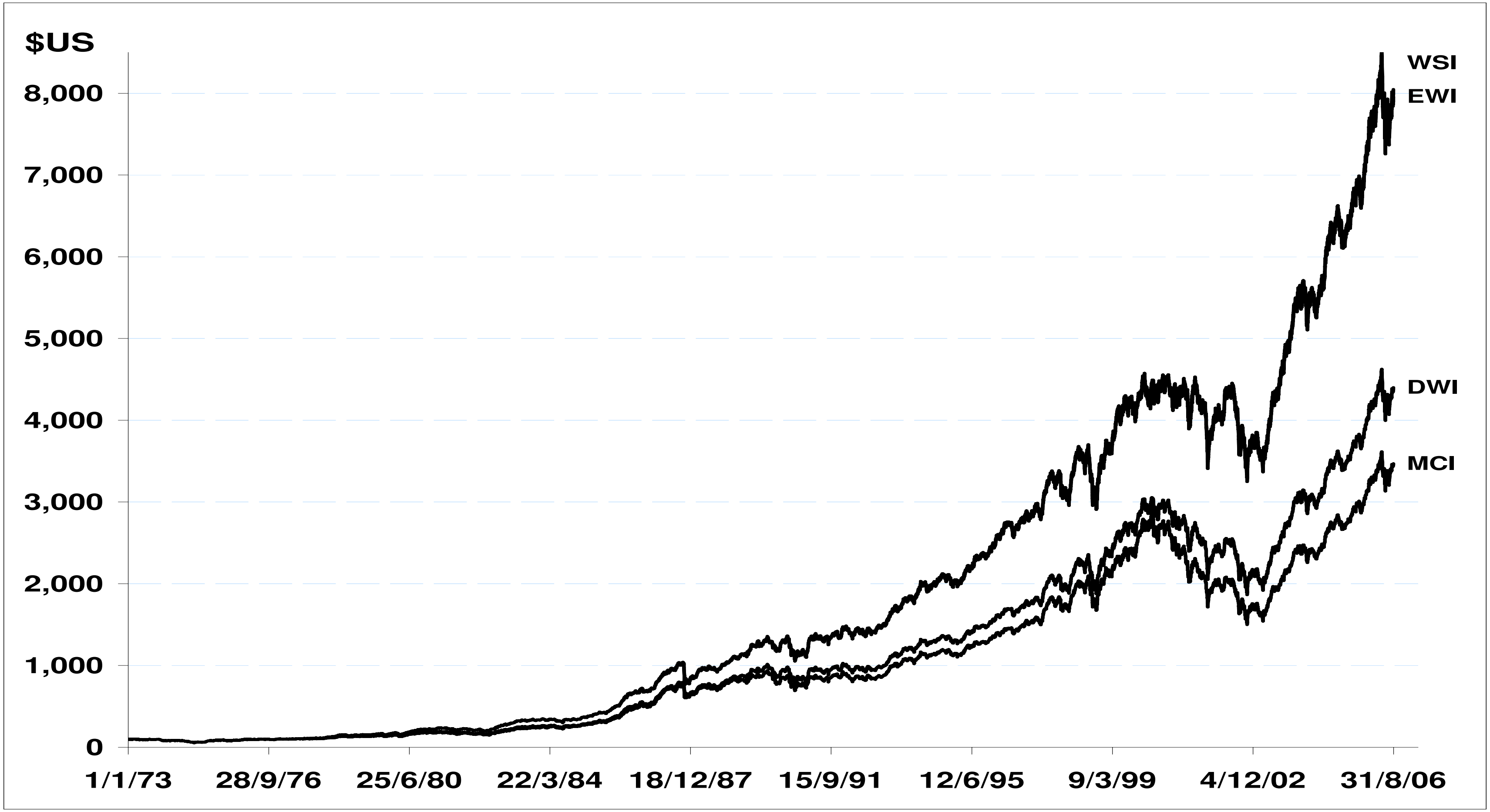

Figure 2.6: Constructed MCI, DWI, EWI and WSI. In the long term the WSI and EWI outperform all other indices, Le & Platen (2006).

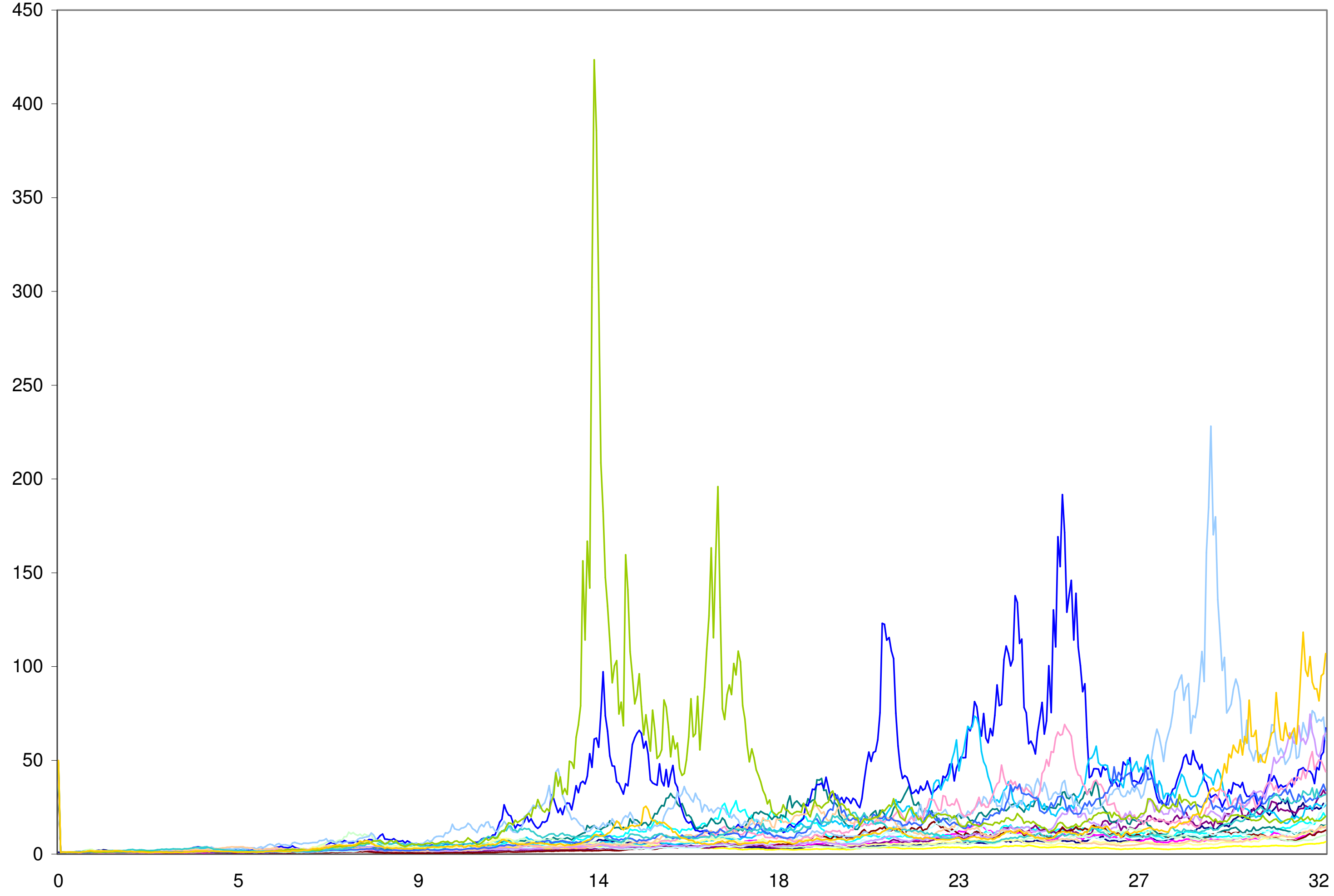

Figure 2.7: Primary security accounts under the MMM.

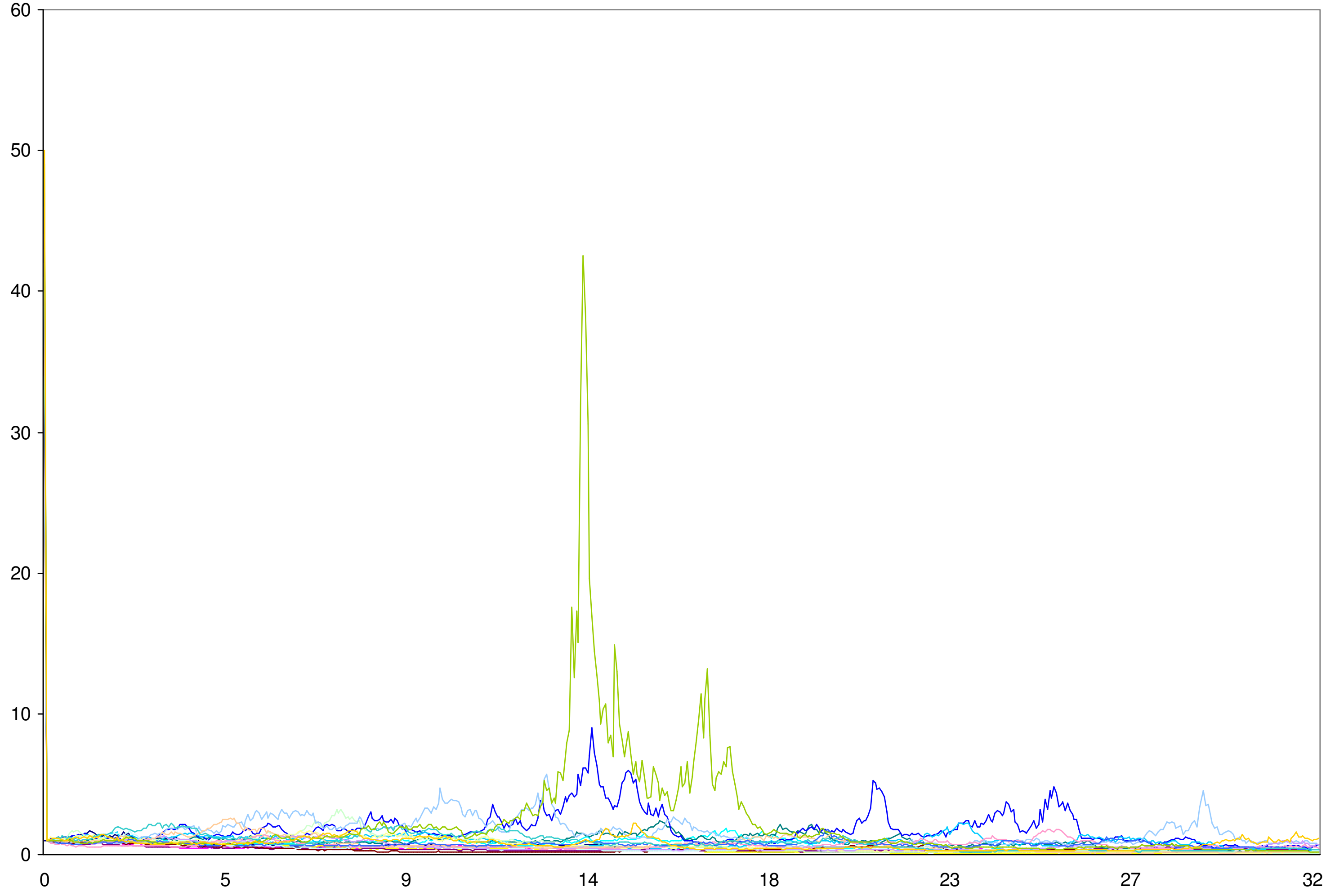

Figure 2.8: Benchmarked primary security accounts.

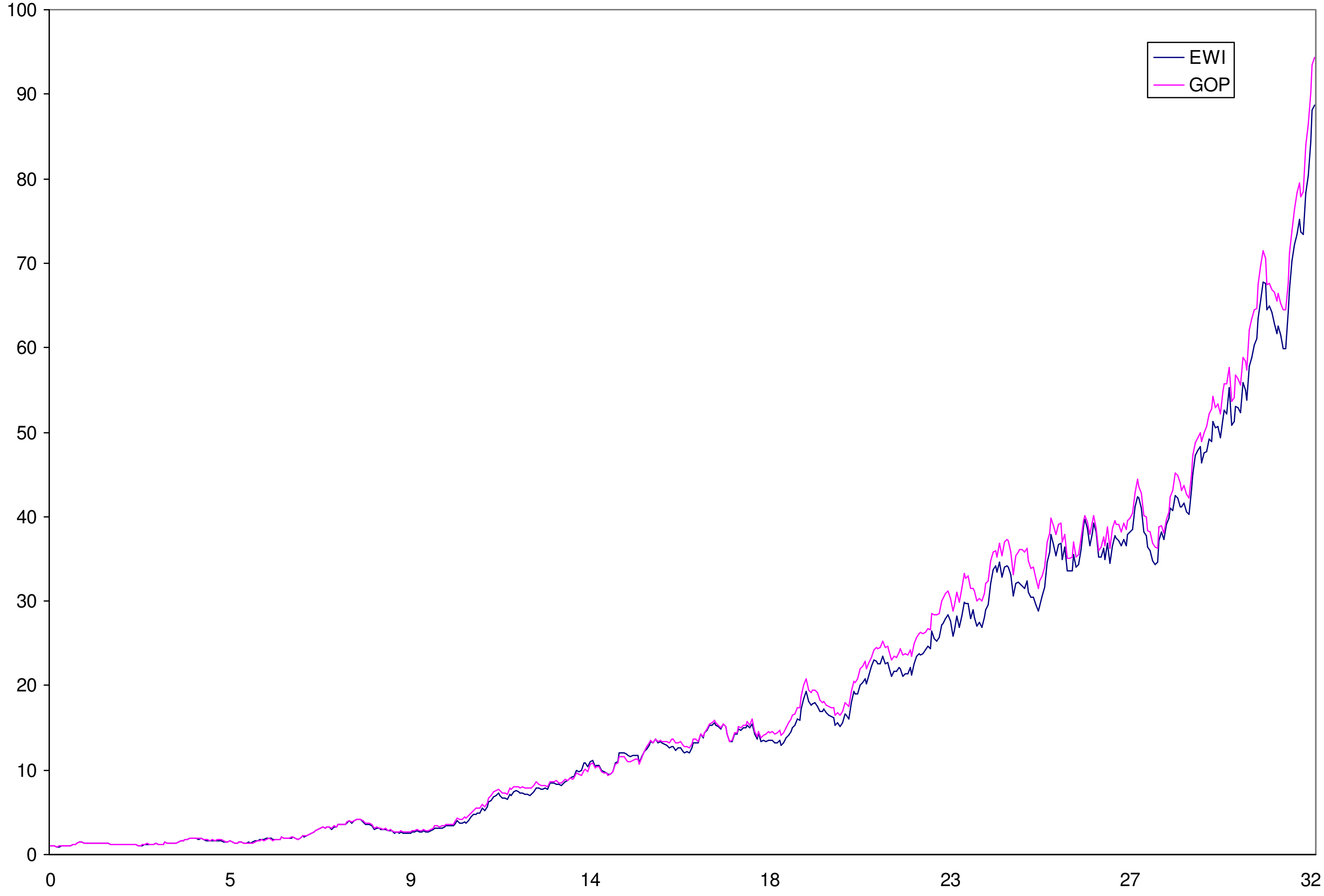

Figure 2.9: NP and EWI.

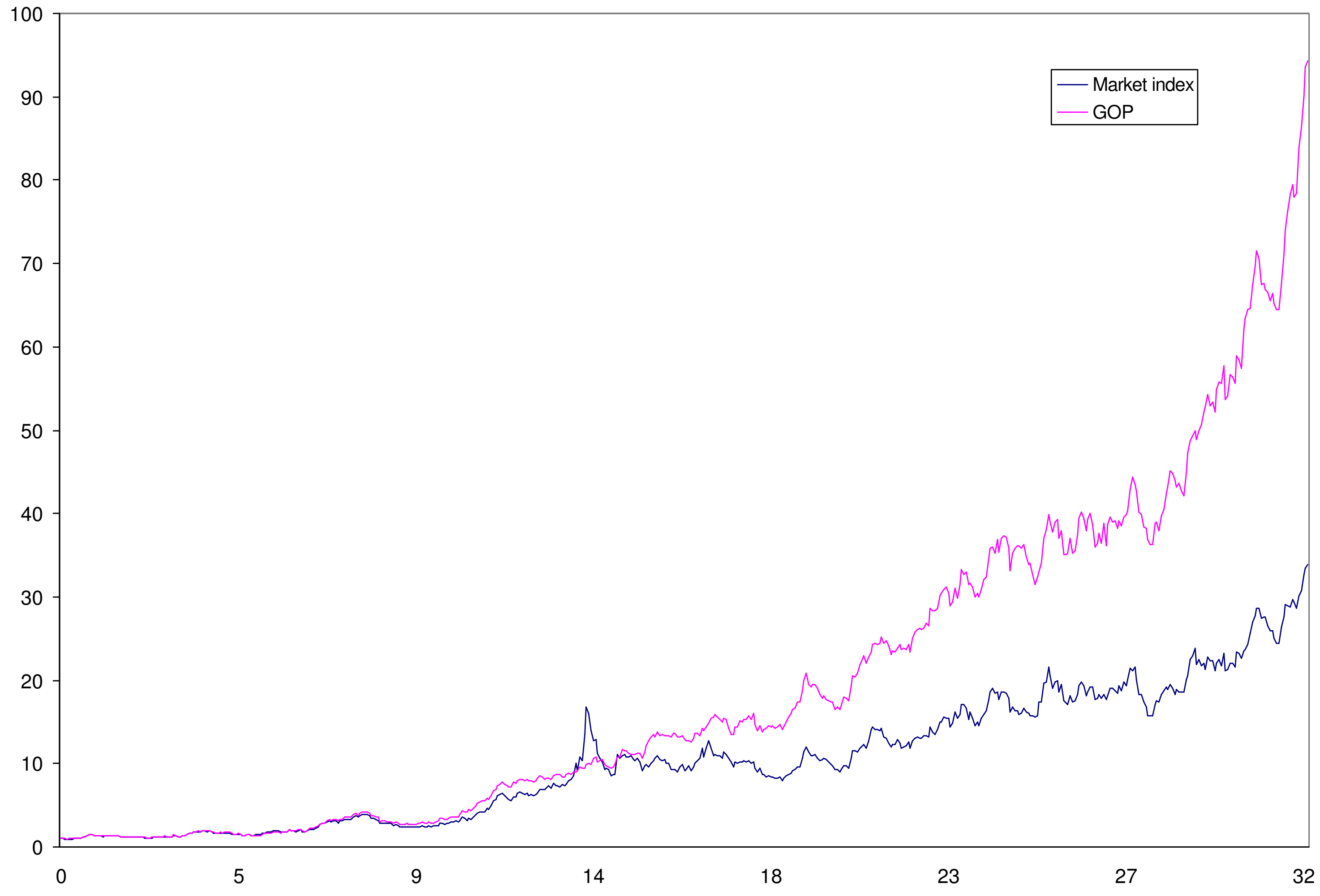

Figure 2.10: NP and market index.

3 Portfolio Optimization

- **NP best long term investment**

  Kelly (1956), Latané (1959), Breiman (1961), Hakansson (1971b), Thorp (1972)

- **optimal portfolio separated into two funds**

  Tobin (1958), Sharpe (1964)

- **mean-variance efficient portfolio**

  Markowitz (1959)

- **intertemporal capital asset pricing model**

  Merton (1973a)

Expected Utility Maximization

- **utility function** $U(\cdot),\ U'(\cdot) > 0 \qquad U''(\cdot) < 0$

- **examples**

  power utility

$$U(x) = \frac{1}{\gamma}\, x^{\gamma}$$

  for $\gamma \neq 0$ and $\gamma < 1$

  log-utility

$$U(x) = \ln(x)$$

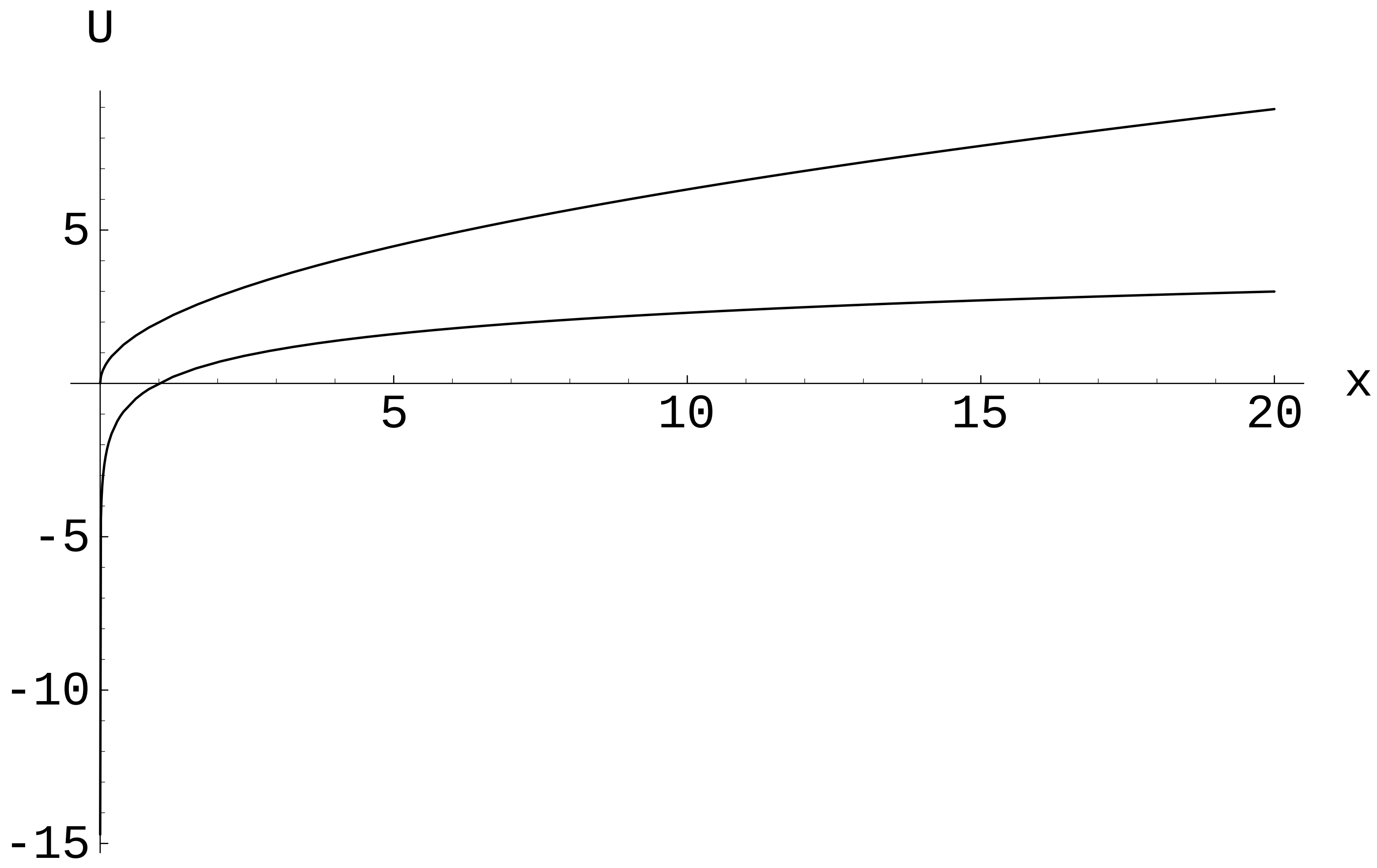

Figure 3.1: Examples for power utility (upper graph) and log-utility (lower graph).

- assume $\hat{S}^0$ is **scalar diffusion**

- **maximize expected utility**

$$v^{\tilde{\delta}} = \max_{\bar{S}^\delta \in \bar{\mathcal{V}}^+_{S_0}} E_0\left(U\left(\bar{S}^\delta_T\right)\right)$$

$\bar{\mathcal{V}}^+_{S_0}$ strictly positive, discounted, fair portfolios

$$\bar{S}^\delta_0 = S_0 > 0$$

Theorem 3.1

Benchmarked, expected utility maximizing portfolio:

$$\hat{S}^{\tilde{\delta}}_t = \hat{u}(t, \hat{S}^0_t) = E_t\left(U'^{-1}\left(\lambda\, \hat{S}^0_T\right) \hat{S}^0_T\right),$$

$$\lambda \quad s.t. \qquad S_0 = \hat{S}^{\tilde{\delta}}_0\, S^{\delta_*}_0 = \hat{u}(0, \hat{S}_0)\, S^{\delta_*}_0$$

two fund separation *with risk aversion coefficient*

$$J^{\tilde{\delta}}_t = \frac{1}{1 - \frac{\hat{S}^0_t}{\hat{u}(t,\hat{S}^0_t)} \frac{\partial \hat{u}(t,\hat{S}^0_t)}{\partial \hat{S}^0_t}}$$

$\Longrightarrow$ *invest* $\frac{1}{J^{\tilde{\delta}}_t}$ *of wealth in NP and remainder in savings account*

- **log-utility function** $U(x) = \ln(x)$

$$U'(x) = \frac{1}{x}$$

$$U'^{-1}(y) = \frac{1}{y}$$

$$U''(x) = -\frac{1}{x^2}$$

utility concave

$$\hat{u}(t, \hat{S}_t^0) = E_t\left(U'^{-1}\left(\lambda\, \hat{S}_T^0\right) \hat{S}_T^0\right) = E_t\left(\frac{1}{\lambda\, \hat{S}_T^0}\, \hat{S}_T^0\right) = \frac{1}{\lambda}$$

Lagrange multiplier

$$\lambda = \frac{S_0^{\delta_*}}{S_0}$$

risk aversion coefficient

$$J_t^{\tilde{\delta}} = 1$$

- expected log-utility

$$\begin{aligned} v^{\tilde{\delta}} &= E_0\left(\ln\left(\bar{S}_T^{\delta_*}\right)\right) \\ &= \ln(\lambda) + \ln(S_0) + \frac{1}{2}\int_0^T E_0\left(\left(\theta(s, \bar{S}_s^{\delta_*})\right)^2\right) ds \end{aligned}$$

if local martingale part forms a martingale

- **power utility**

$$U(x) = \frac{1}{\gamma}\, x^{\gamma}$$

for $\gamma < 1, \gamma \neq 0$

$$U'(x) = x^{\gamma-1}$$

$$U'^{-1}(y) = y^{\frac{1}{\gamma-1}}$$

$$U''(x) = (\gamma - 1)\, x^{\gamma-2}$$

concave

$$\hat{u}(t, \hat{S}_t^0) = E_t\left(\left(\frac{\lambda}{\bar{S}_T^{\delta_*}}\right)^{\frac{1}{\gamma-1}} \frac{1}{\bar{S}_T^{\delta_*}}\right) = \lambda^{\frac{1}{\gamma-1}}\, E_t\left(\left(\bar{S}_T^{\delta_*}\right)^{\frac{\gamma}{1-\gamma}}\right)$$

$\bar{S}^{\delta_*}$ geometric Brownian motion

$$\hat{u}(t, \hat{S}_t^0) = \lambda^{\frac{1}{\gamma-1}} \left(\bar{S}_t^{\delta_*}\right)^{\frac{\gamma}{1-\gamma}} \times E_t\left(\exp\left\{\frac{\gamma}{1-\gamma}\left(\frac{\theta^2}{2}(T-t) + \theta\,(W_T - W_t)\right)\right\}\right)$$

$$= \lambda^{\frac{1}{\gamma-1}} \left(\bar{S}_t^{\delta_*}\right)^{\frac{\gamma}{1-\gamma}} \exp\left\{\frac{\theta^2}{2}\,\frac{\gamma}{(1-\gamma)^2}\,(T-t)\right\}$$

Lagrange multiplier

$$\lambda = S_0^{\gamma-1}\,\bar{S}_0^{\delta_*}\ \exp\left\{\frac{\theta^2}{2}\,\frac{\gamma}{1-\gamma}\,T\right\}$$

$\hat{S}_t^0 = (\bar{S}_t^{\delta_*})^{-1}$

$$\frac{\partial \hat{u}(t,\hat{S}_t^0)}{\partial \hat{S}^0} = \frac{\hat{u}(t,\hat{S}_t^0)}{\hat{S}_t^0}\,\frac{\gamma}{\gamma-1}$$

$\Longrightarrow$ risk aversion coefficient

$$J_t^{\tilde{\delta}} = 1-\gamma$$

expected utility

$$v^{\tilde{\delta}} = E_0\left(\frac{1}{\gamma}\left(\bar{S}_T^{\tilde{\delta}}\right)^{\gamma}\right) = \exp\left\{-\frac{\theta^2}{2}\,\frac{\gamma}{1-\gamma}\,T\right\}(S_0)^{\gamma}$$

Utility Indifference Pricing

discounted payoff $\bar{H} = \frac{H}{S_T^0}$ (not perfectly hedgable)

$$v_{\varepsilon,V}^{\tilde{\delta}} = E_0\left(U\left((S_0 - \varepsilon\, V)\,\frac{\bar{S}_T^{\tilde{\delta}}}{S_0} + \varepsilon\,\bar{H}\right)\right)$$

- assume $\hat{S}^0$ is **scalar diffusion**

Definition 3.2 ***utility indifference price*** V ***s.t.***

$$\lim_{\varepsilon \to 0} \frac{v_{\varepsilon,V}^{\tilde{\delta}} - v_{0,V}^{\tilde{\delta}}}{\varepsilon} = 0$$

Davis (1997)

- Taylor expansion

$$v^{\tilde{\delta}}_{\varepsilon,V} \approx E_0\left(U\left(\bar{S}^{\tilde{\delta}}_T\right) + \varepsilon\, U'\left(\bar{S}^{\tilde{\delta}}_T\right)\left(\bar{H} - \frac{V\,\bar{S}^{\tilde{\delta}}_T}{S_0}\right)\right)$$

$$= v^{\tilde{\delta}}_{0,V} + \varepsilon\, E_0\left(U'\left(\bar{S}^{\tilde{\delta}}_T\right)\,\bar{H}\right) - V\,\varepsilon\, E_0\left(U'\left(\bar{S}^{\tilde{\delta}}_T\right)\frac{\bar{S}^{\tilde{\delta}}_T}{S_0}\right)$$

$$\bar{S}^{\tilde{\delta}}_T = \hat{S}^{\tilde{\delta}}_T/\hat{S}^0_T = U'^{-1}\left(\lambda\,\hat{S}^0_T\right) \quad \Longrightarrow$$

$$v^{\tilde{\delta}}_{\varepsilon,V} - v^{\tilde{\delta}}_{0,V} = \varepsilon\left(E_0\left(\lambda\,\hat{S}^0_T\,\bar{H}\right) - V\,E_0\left(\lambda\,\hat{S}^0_T\,\frac{\bar{S}^{\tilde{\delta}}_T}{S_0}\right)\right)$$

$$\Longrightarrow$$

- **utility indifference price**

$$V = \frac{E_0\left(\frac{\lambda}{\bar{S}_T^{\delta_*}}\,\bar{H}\right)}{E_0\left(\frac{\lambda}{\bar{S}_T^{\delta_*}}\,\frac{\bar{S}_T^{\tilde{\delta}}}{S_0}\right)} = \frac{E_0\left(\frac{H}{S_T^{\delta_*}}\right)}{\frac{1}{S_0}\,\frac{S_0}{S_0^{\delta_*}}}$$

independent of U and of given model $\Longrightarrow$

- **real world pricing formula**

$$V = S_0^{\delta_*}\,E_0\left(\frac{H}{S_T^{\delta_*}}\right)$$

Hedging

- benchmarked derivative price

$$\hat{U}_H(t) = E_t\left(\frac{H}{S_T^{\delta_*}}\right)$$

- **real world martingale representation**

$$\frac{H}{S_T^{\delta_*}} = \hat{U}_H(t) + \sum_{k=1}^{d} \int_t^T x_H^k(s)\, dW_s^k + M_H(t)$$

M_H-orthogonal martingale

$$E\left(\left[M_H, W^k\right]_t\right) = 0$$

Föllmer-Schweizer decomposition, Föllmer & Schweizer (1991)

least square projection

Theorem 3.3 *Derivative price*

$$S_t^{\delta_H} = S_t^{\delta_*} \, E_t\left(\frac{H}{S_T^{\delta_*}}\right)$$

fractions

$$\pi_{\delta_H}(t) = \left(b_{\delta_H}(t)^\top \, b_t^{-1}\right)^\top$$

with portfolio volatilities

$$b_{\delta_H}^k(t) = \frac{1}{\hat{U}_H(t)} \sum_{j=0}^{d} \delta_H^j(t) \, \hat{S}_t^j \, b_t^{j,k} = \frac{x_H^k(t)}{\hat{U}_H(t)} + \theta_t^k.$$

4 Modeling Stochastic Volatility

- consider well diversified accumulation index $\approx$ NP

$$dS_t^{\delta_*} = S_t^{\delta_*} \left(r_t \, dt + |\theta_t| \, \left(|\theta_t| \, dt + d\hat{W}_t \right) \right)$$

$$d \ln \left(S_t^{\delta_*} \right) = \left(r_t \, dt + \frac{1}{2} \, |\theta_t|^2 \right) dt + |\theta_t| \, d\hat{W}_t$$

- **volatility**

$$|\theta_t| = \sqrt{\frac{d}{dt} \left[\ln \left(S^{\delta_*} \right) \right]_t}$$

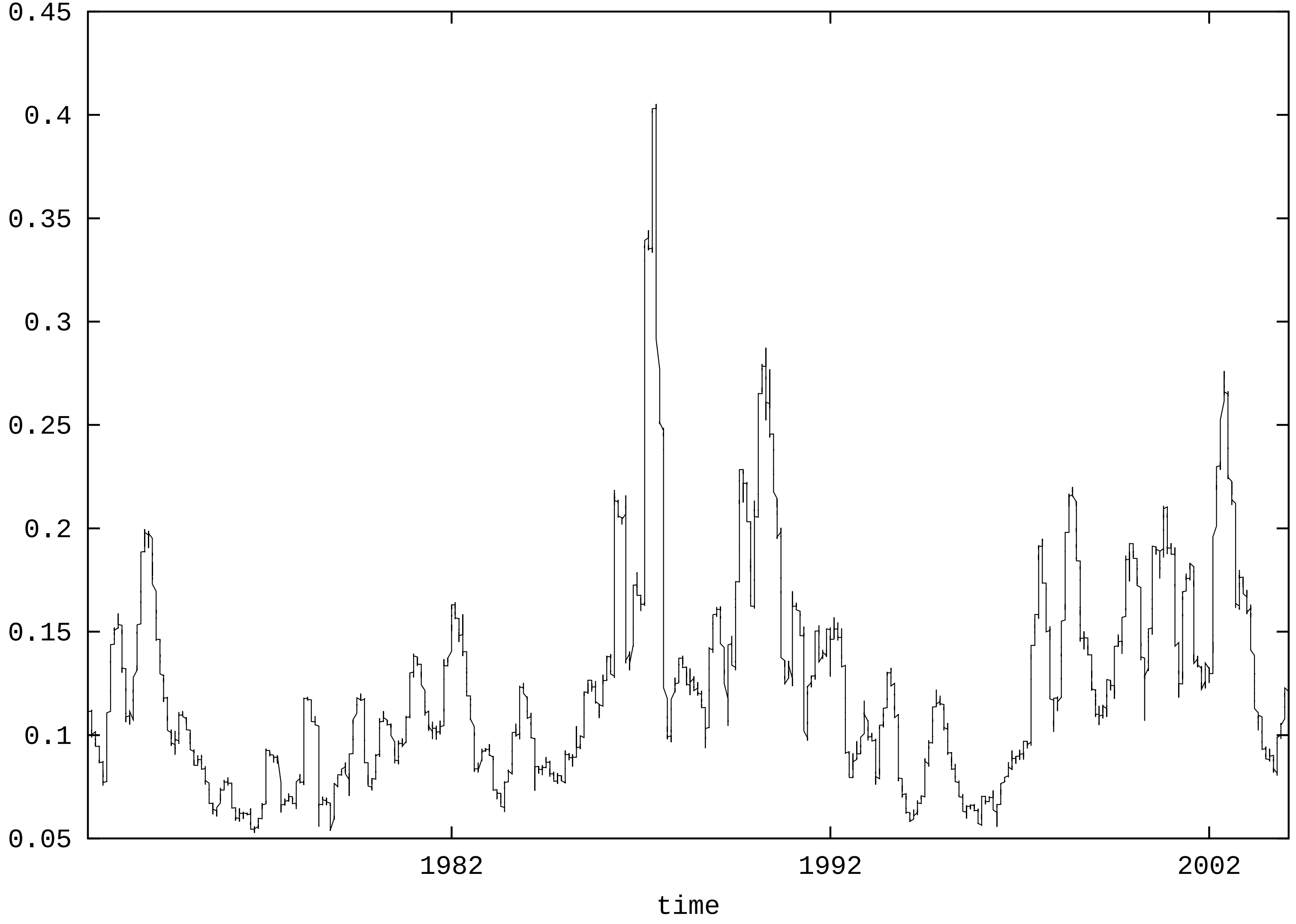

Figure 4.1: Estimated volatility of WSI from 1973–2004.

Log-Returns of a Stock Index

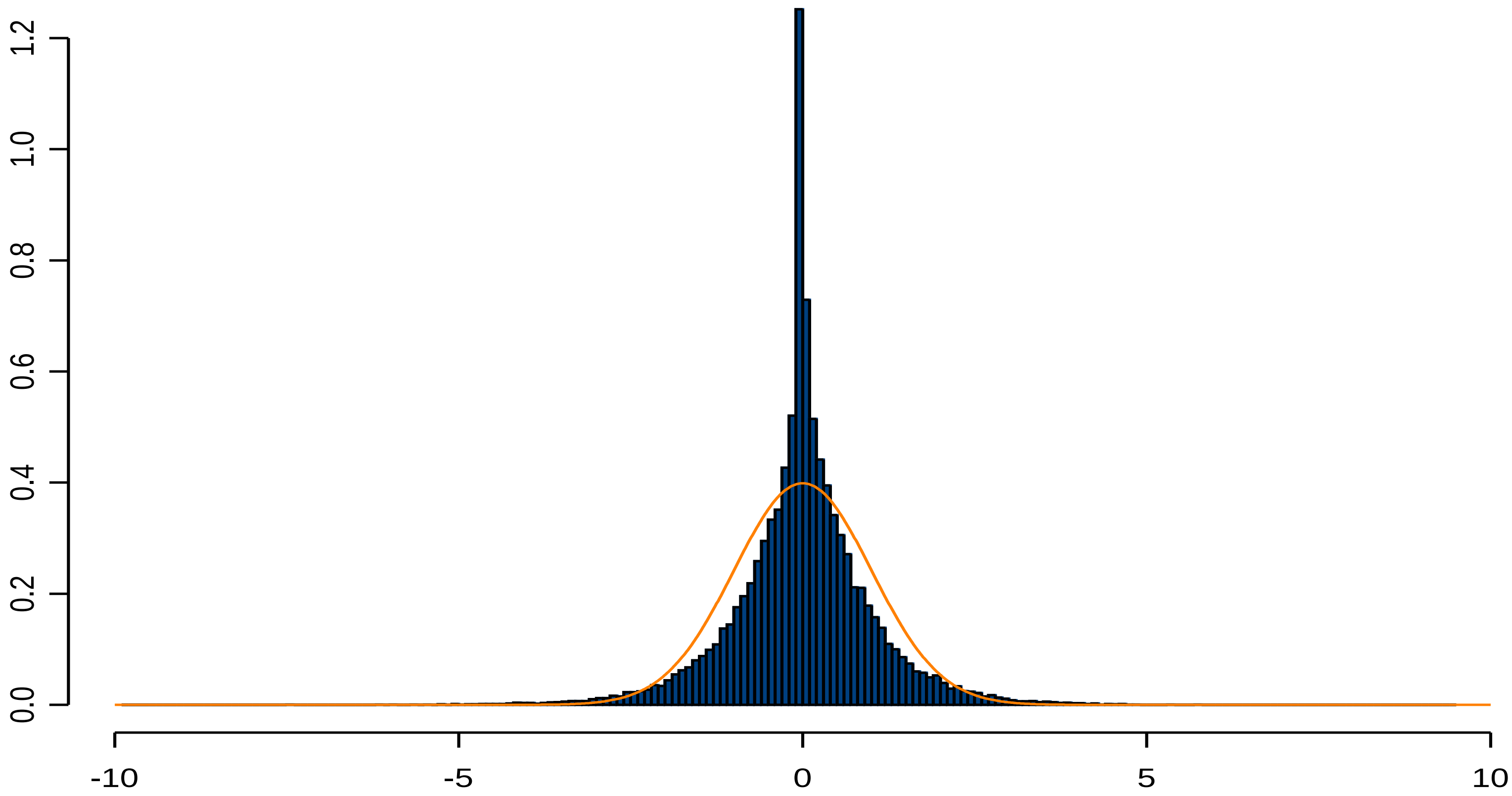

Figure 4.2: Histogram for S&P500 hourly deseasonalized log-returns.

fitted Gaussian density, much fatter tails

Implied Volatilities

- European call for Black-Scholes model

$$c_{\tau,K}(0, S, \sigma)$$

maturity $\tau \in [0, T]$

strike price $K > 0$

underlying index $S > 0$

volatility σ

- observed European call option price

$$V_{c,\tau,K}(0, S_0)$$

- **implied volatility**

$$\sigma_{\text{BS}}^{\text{call}}(0, S_0, \tau, K)$$

obtained such that

$$V_{c,\tau,K}(0, S_0) = c_{\tau,K}\left(0, S_0, \sigma_{\text{BS}}^{\text{call}}(0, S_0, \tau, K)\right)$$

by some root finding method

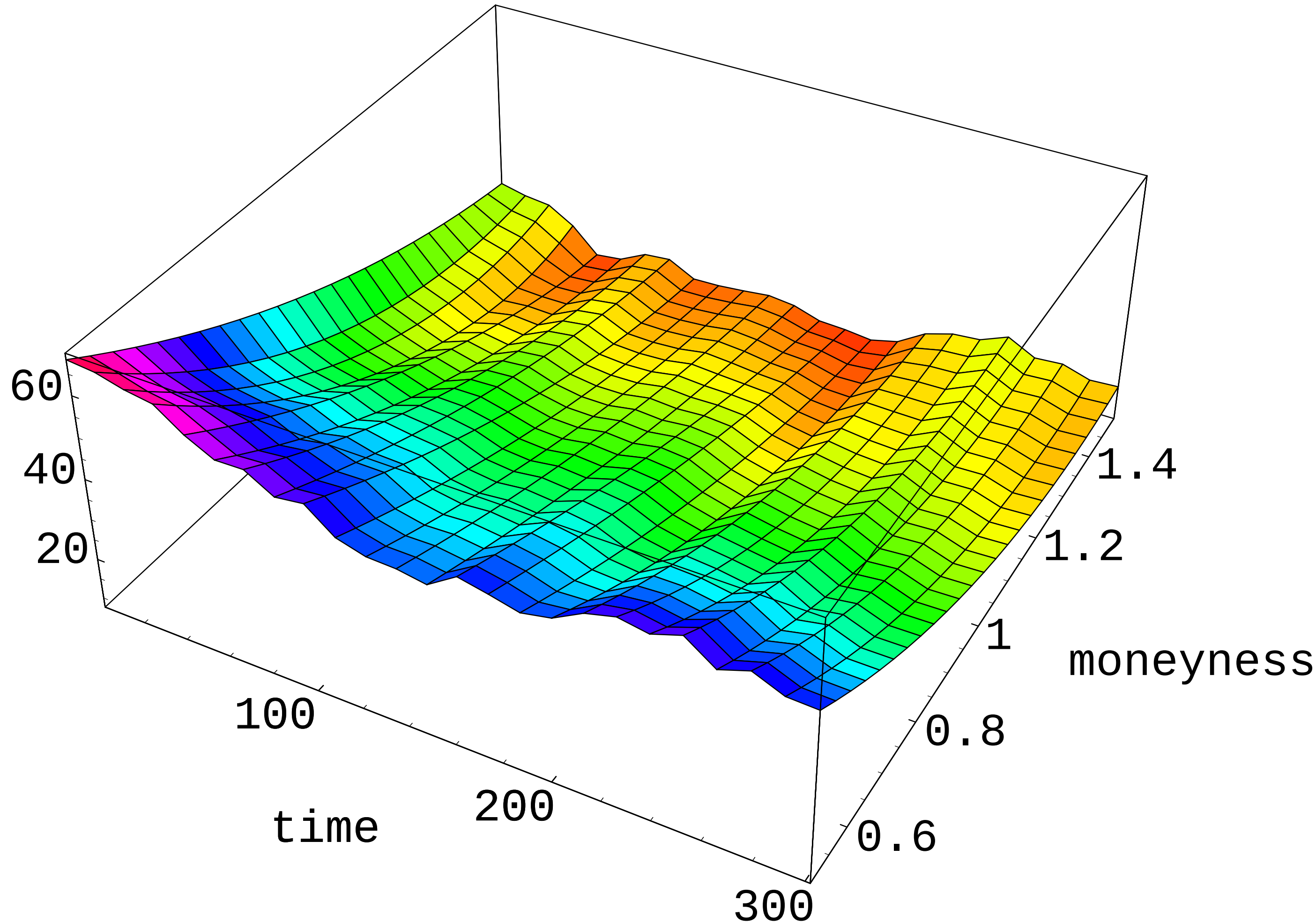

Figure 4.3: Implied volatilities for S&P500 three month options.

implied volatility surface

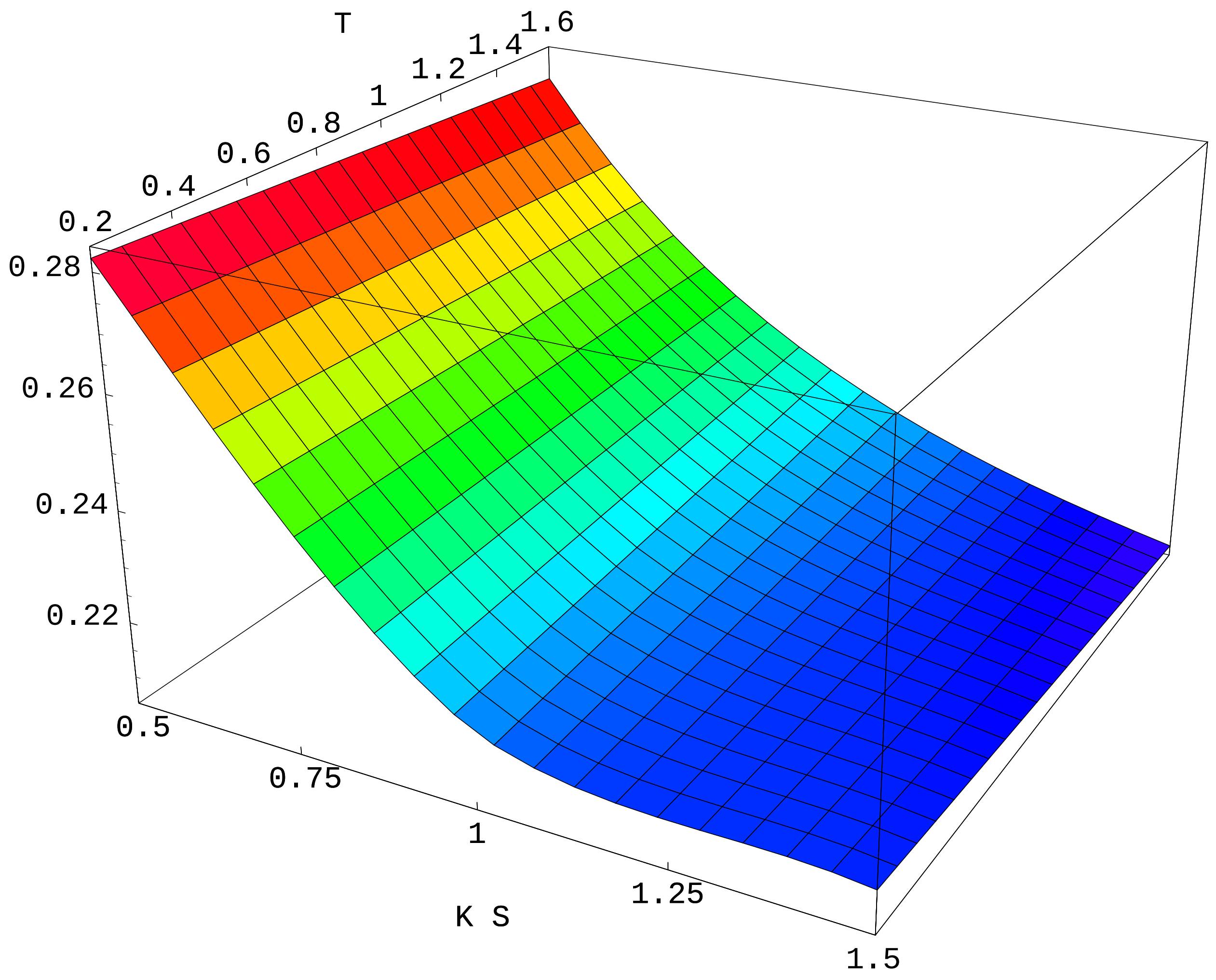

Figure 4.4: Average S&P500 implied volatility surface.

A Modified Constant Elasticity of Variance Model

- **CEV Model**

  Cox (1975), Cox & Ross (1976)

  Andersen & Andreasen (2000)

  Lewis (2000)

  Lo, Yuen & Hui (2000)

  Brigo & Mercurio (2005)

  Heath & Pl. (2005b)

- **classical risk neutral CEV**

  Beckers (1980)

  Schroder (1989)

  Delbaen & Shirakawa (2002)

  certain problems in risk neutral pricing of derivatives

Modified CEV Model

Heath & Pl. (2005b)

- savings account

$$dS_t^0 = r\, S_t^0\, dt$$

total market price for risk process

$$\{\theta_t \geq 0,\ t \in [0,T]\}$$

- drifted Wiener process

$$d\hat{W}_\theta(t) = \theta_t\, dt + d\hat{W}_t$$

- NP

$$dS_t^{\delta_*} = S_t^{\delta_*} \left(r_t \, dt + \theta_t \, d\hat{W}_\theta(t) \right)$$

- NP volatility

$$\theta_t = (S_t^{\delta_*})^{a-1} \, \psi$$

exponent $a \in (-\infty, \infty)$

scaling parameter $\psi > 0$

stochastic for $a \neq 1$

- **hypothetical risk neutral CEV dynamics**

$$dS_t^{\delta_*} = S_t^{\delta_*}\, r_t\, dt + (S_t^{\delta_*})^a\, \psi\, d\hat{W}_\theta(t)$$

$\hat{W}_\theta$ - Wiener process under a hypothetical risk neutral measure P_θ

- If $a = 1$ $\Longrightarrow$ BS model

$\Longrightarrow$ equivalent risk neutral probability measure P_θ

with market price of risk $\theta_t = \psi$

- **real world CEV dynamics**

$$dS_t^{\delta_*} = \left(S_t^{\delta_*}\, r_t + \left(S_t^{\delta_*}\right)^{2a-1}\, \psi^2\right) dt + \left(S_t^{\delta_*}\right)^{a}\, \psi\, d\hat{W}_t$$

modified CEV model remains for $a < 1$ strictly positive

this is **not** the case for its “risk neutral” dynamics

- **squared Bessel process**

$$X_t = \left(S_t^{\delta_*}\right)^{2\,(1-a)}$$

for $a \neq 1$ with SDE

$$dX_t = \Big(2(1-a)\,r_t\,X_t + \psi^2\,(1-a)\,(3-2a)\Big)\,dt + 2\,\psi\,(1-a)\,\sqrt{X_t}\,d\hat{W}_t$$

and **dimension**

$$\delta = \frac{3-2\,a}{1-a}$$

- **Radon-Nikodym derivative process**

$$\Lambda_\theta = \{\Lambda_\theta(t),\, t \in [0,T]\}$$

(state price density)

- **hypothetical risk neutral measure P_θ**

$$\frac{dP_\theta}{dP} = \Lambda_\theta(T)$$

$$\Lambda_\theta(t) \;=\; \frac{S_0^{\delta_*}}{S_t^{\delta_*}}\, S_t^0 = \left(\frac{X_0}{X_t}\right)^q S_t^0$$

$$q = \frac{1}{2\,(1-a)}$$

- **squared Bessel process**

$$dX_t \;=\; \big(2\,(1-a)\,r_t\,X_t + \psi^2\,(1-a)(1-2\,a)\big)\;dt$$

$$+\,2\,\psi\,(1-a)\,\sqrt{X_t}\,d\hat{W}_\theta(t)$$

dimension

$$\delta_\theta = \frac{1-2\,a}{1-a}$$

- dimension

  risk neutral

$$\delta_\theta = \frac{1 - 2\,a}{1 - a}$$

  real world

$$\delta = \frac{3 - 2\,a}{1 - a}$$

- if $a < 1 \quad \Longrightarrow \quad \delta > 2$ and $\delta_\theta < 2$

  if $a > 1 \quad \Longrightarrow \quad \delta < 2$ and $\delta_\theta > 2$

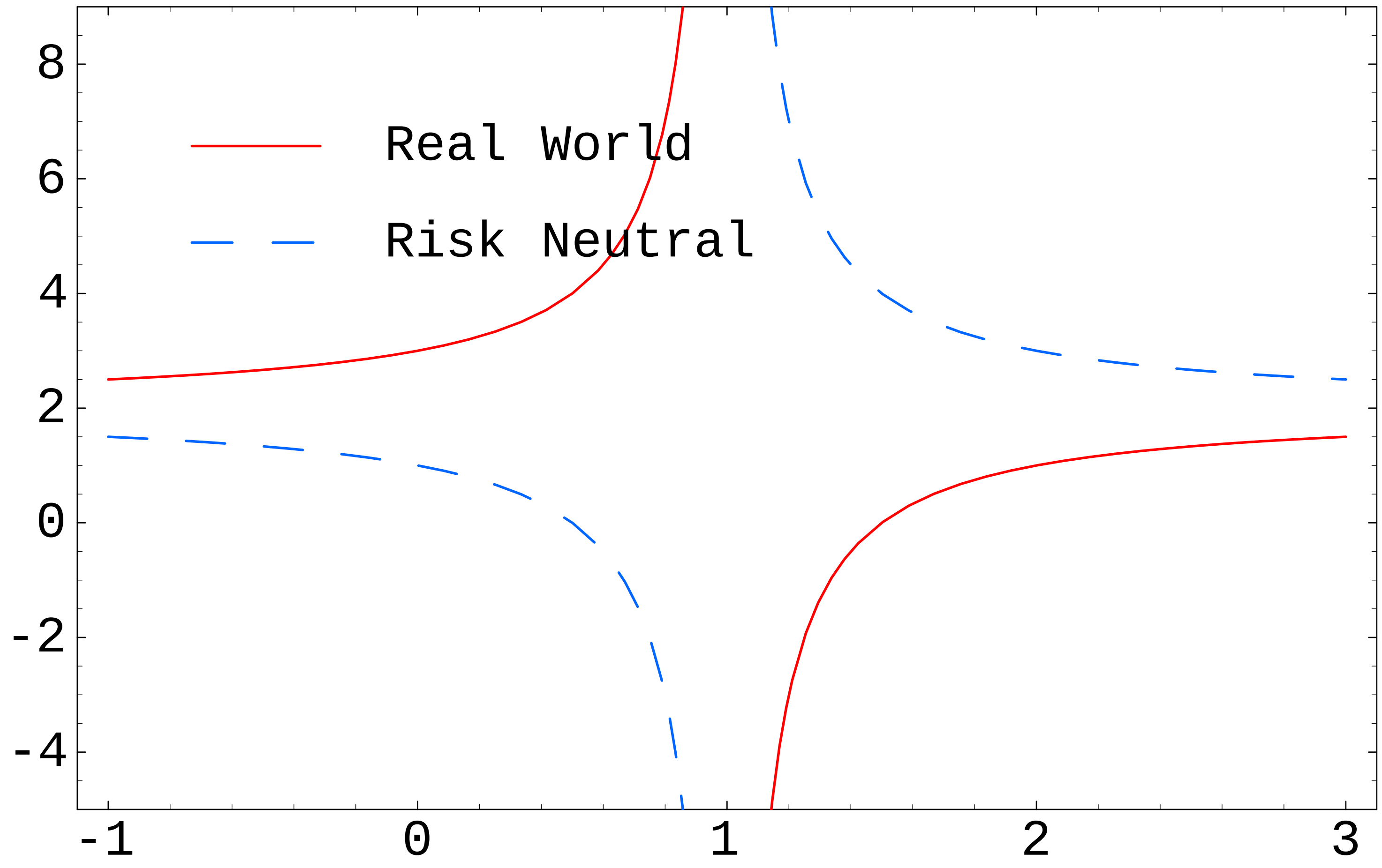

Figure 4.5: Dimensions δ and δ_θ as a function of the exponent a.

- **hypothetical risk neutral probability**

$$
\begin{aligned}
P_\theta(A) &= \int_A dP_\theta(\omega) = \int_A \frac{dP_\theta(\omega)}{dP(\omega)}\, dP(w) \\
&= \int_A \Lambda_\theta(T)\, dP(\omega)
\end{aligned}
$$

$\Longrightarrow$

$$
P_\theta(\Omega) = \int_\Omega \Lambda_\theta(T)\, dP(\omega) = E_0\left(\Lambda_\theta(T)\right)
$$

- for $a < 1 \quad \Longrightarrow \quad \delta_\theta < 2$

$\Longrightarrow \quad \Lambda_\theta$ is strict supermartingale $\quad \Longrightarrow$

$P_\theta(\Omega) < \Lambda_\theta(0) = 1$

$\Longrightarrow \quad P_\theta$ is not a probability measure

- for $a \neq 1$ P and P_θ are not equivalent

  $\Longrightarrow$ modified CEV model does *not* have *equivalent risk neutral probability measure*

- consider only $a < 1$ since for $a > 1$ NP explodes

Real World Pricing

- **contingent claim**

$$H_\tau = H_\tau(S_\tau^{\delta_*}), \quad E\left(\frac{H_\tau}{S_\tau^{\delta_*}}\right) < \infty$$

- **fair price**

$$u_{H_\tau}(t, S_t^{\delta_*}) = S_t^{\delta_*}\, \hat{u}_{H_\tau}(t, S_t^{\delta_*})$$

benchmarked price

$$\hat{u}_{H_\tau}(t, S_t^{\delta_*}) = E_t\left(\frac{H_\tau(S_\tau^{\delta_*})}{S_\tau^{\delta_*}}\right)$$

Martingale Representation

$$\frac{H_\tau}{S_\tau^{\delta_*}} = \hat{u}_{H_\tau}(\tau, S_\tau^{\delta_*})$$

$$= \hat{u}_{H_\tau}(t, S_t^{\delta_*}) + \int_t^\tau \left(S_s^{\delta_*}\right)^a \psi \frac{\partial \hat{u}_{H_\tau}(s, S_s^{\delta_*})}{\partial S^{\delta_*}} \, d\hat{W}_s$$

Stochastic Volatility Models

- volatility function of underlying diffusion
  $\Longrightarrow$ LV model

- volatility as separate, possibly correlated process

  Hull & White (1987, 1988)
  Johnson & Shanno (1987)
  Scott (1987)
  Wiggins (1987)
  Chesney & Scott (1989)
  Melino & Turnbull (1990)
  Stein & Stein (1991)
  Hofmann, Pl. & Schweizer (1992)
  Heston (1993), . . .

Empirical Evidence on Log-Returns

- Markowitz & Usmen (1996)
  daily S&P500 $\longrightarrow$ Student $\boldsymbol{t(4.5)}$

- Hurst & Platen (1997)
  daily stock market indices $\longrightarrow$ Student $\boldsymbol{t(4)}$

- Fergusson & Pl. (2006), Pl. & Rendek (2008)
  daily WSI, EWI104s with high accuracy $\longrightarrow$ Student $\boldsymbol{t(4)}$

- Breymann, Dias & Embrechts (2003)
  copula of joint distribution of log-returns of exchange rates $\longrightarrow$
  Student $\boldsymbol{t}$ copula with roughly four degrees of freedom

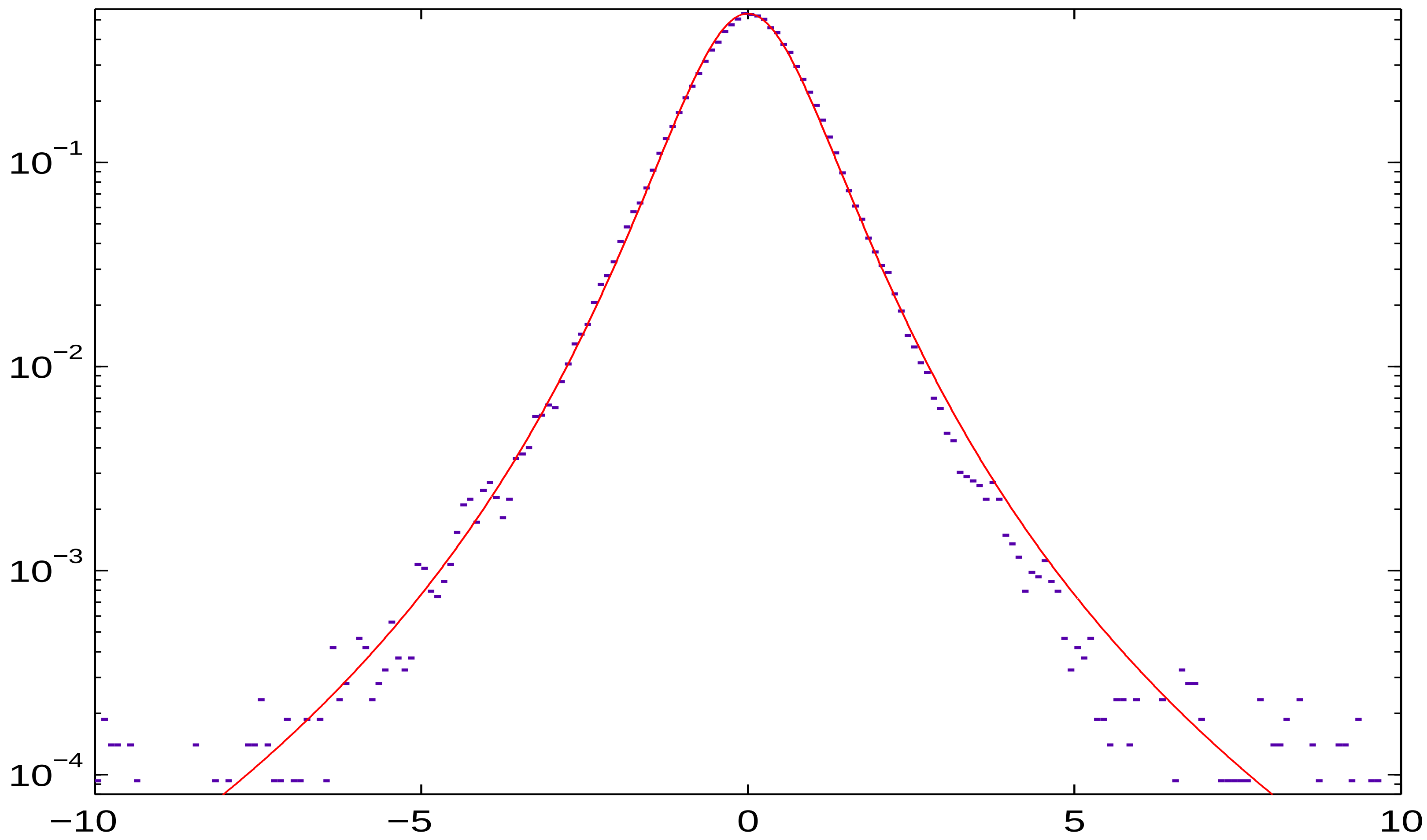

Figure 4.6: Log-histogram of the EWI104s log-returns and Student t density with four degrees of freedom.

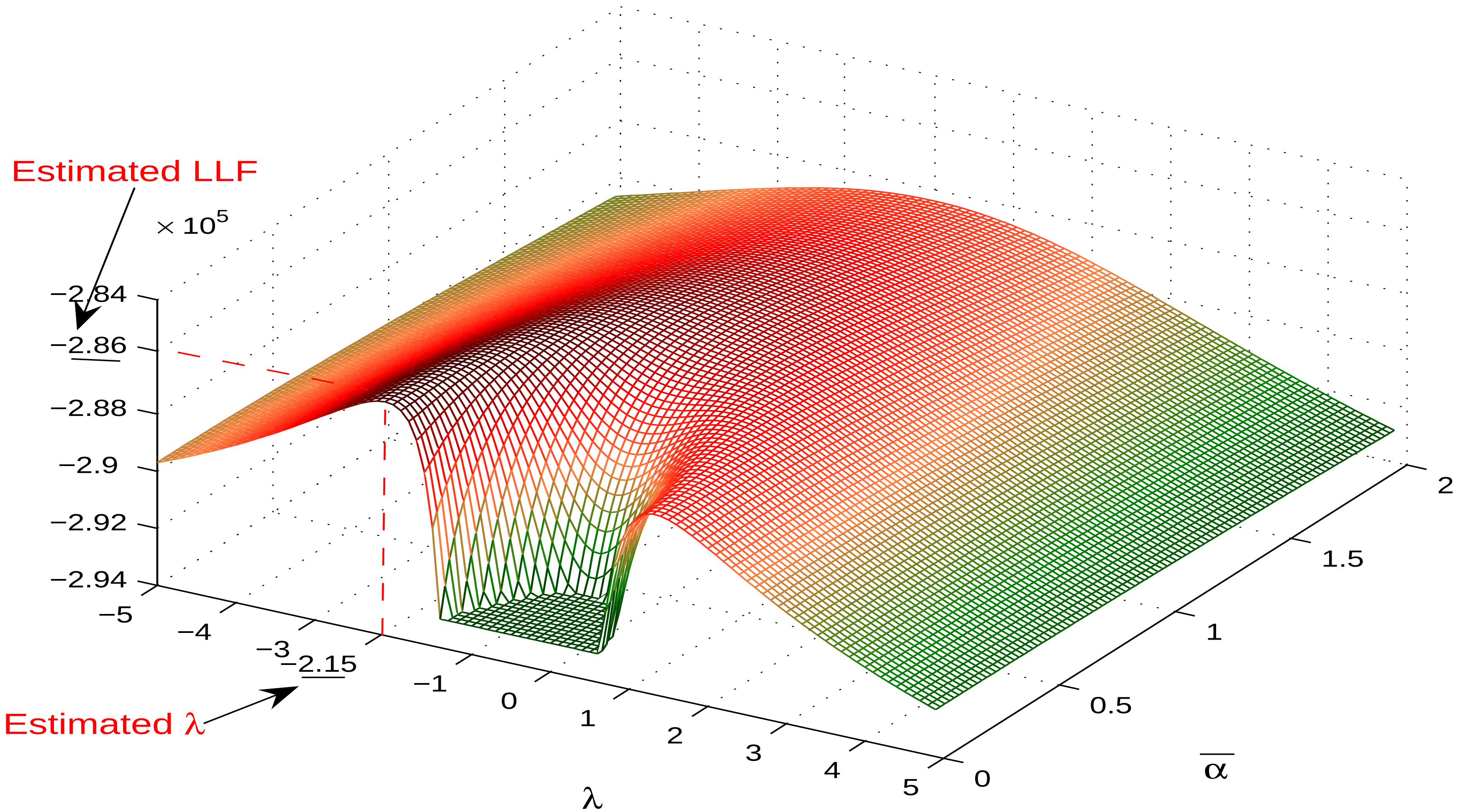

Figure 4.7: Log-likelihood function based on the EWI104s.

Country	Student-t	NIG	Hyperbolic	VG	ν
Australia	0.000000	76.770817	150.202282	181.632971	4.281222
Austria	0.000000	39.289103	77.505683	102.979330	4.725907
Belgium	0.000000	31.581622	60.867570	83.648470	4.989912
Brazil	2.617693	5.687078	63.800349	60.078395	2.713036
Canada	0.000000	47.506215	79.917741	104.297607	5.316154
Denmark	0.000000	41.509921	87.199686	114.853658	4.512101
Finland	0.000000	28.852844	68.677271	88.553080	4.305638
France	0.000000	26.303544	57.639325	80.567283	4.722787
Germany	0.000000	27.290205	52.667918	71.120798	5.005856
Greece	0.000000	60.432172	104.789463	125.601499	4.674626
Hong.Kong	0.000000	42.066531	100.834255	122.965326	3.930473
India	0.000000	74.773701	163.594078	198.002956	3.998713
Ireland	0.000000	77.727856	136.505582	170.013644	4.761519
Italy	0.000000	25.196598	55.185625	75.481897	4.668983
Japan	0.000000	37.630363	77.163656	102.967380	4.649745
Korea.S.	0.000000	120.904983	304.829431	329.854620	3.289204

Malaysia	0.000000	79.714054	186.013963	221.061290	3.785195
Netherlands	0.000000	26.832761	51.625813	71.541627	5.084056
Norway	0.000000	42.243851	89.012090	115.059003	4.472349
Portugal	0.000000	61.177624	137.681039	165.689683	3.984860
Singapore	0.000000	36.379685	77.600590	98.124375	4.251472
Spain	0.000000	56.694545	109.533768	138.259224	4.517153
Sweden	0.000000	77.618384	143.420049	178.983373	4.546640
Taiwan	0.000000	41.162560	96.283628	115.186585	3.914719
Thailand	0.000000	78.250621	254.590254	267.508143	3.032038
UK	0.000000	26.693076	55.937248	80.678494	4.952843
USA	0.000000	40.678242	79.617362	100.901197	4.636661

Table 1: L_n test statistic of the EWI104s for different currency denominations

Index Log-returns

- **Student t distributed with about 4 degrees of freedom**

- **clustering**

  Find corresponding stationary volatility processes !

  Kessler (1997), Prakasa Rao (1999)

- **normal mixing**

  log-return conditionally Gaussian distributed
  with stochastic variance process

  for small time step size $h > 0$

$$\Delta \ln(S_t^{\delta_*}) = \ln\left(\frac{S_{t+h}^{\delta_*}}{S_t^{\delta_*}}\right)$$

$$\sim \mathcal{N}\left(\frac{\theta_t^2}{2}\,h, \theta_t^2\,h\right)$$

  when θ_t^2 has inverse gamma stationary density

  $\Longrightarrow$ estimated log-return appears to be Student t distributed

A Class of Continuous Stochastic Volatility Models

- **discounted NP**

$$\bar{S}_t^{\delta_*} = \frac{S_t^{\delta_*}}{S_t^0}$$

$$d\bar{S}_t^{\delta_*} = \bar{S}_t^{\delta_*} \left(\theta_t^2 \, dt + \theta_t \, d\hat{W}_t\right)$$

Particular Stochastic Volatility Models

- **Hull and White'87 model**

  Hull & White (1987)

$$d\theta_t^2 = \mu\,\theta_t^2\,dt + \xi\,\theta_t^2\,d\bar{W}_t$$

squared volatility does not have a stationary dynamics

- **Heston model**

  Hull & White (1988), Heston (1993)

  mean reverting dynamics

  k, $\bar{\theta}$ and γ are positive constants

  $\Longrightarrow$

$$d\theta_t^2 = k\,(\bar{\theta}^2 - \theta_t^2)\,dt + \gamma\,|\theta_t|\,d\bar{W}_t$$

  $\theta_0^2 > 0,\ \frac{\kappa}{\gamma^2} \geq \frac{1}{2}$

  $\Longrightarrow$ has as stationary density gamma density

- **Scott model**

  Scott (1987)
  Stein & Stein (1991)

  Ornstein-Uhlenbeck process

  $k, \bar{\theta}, \gamma$ positive constants

$$d\theta_t = k\,(\bar{\theta} - \theta_t)\,dt + \gamma\,d\bar{W}_t$$

  $\theta_0 \in \Re$

  $\Longrightarrow$ θ_t^2 has as stationary density a gamma density

- **Wiggins model**

  Wiggins (1987)
  Chesney & Scott (1989)
  Melino & Turnbull (1990)

$$d\ln(\theta_t) = k\left(\ln(\bar{\theta}) - \ln(\theta_t)\right)dt + \gamma\, d\bar{W}_t$$

$\theta_0 > 0$

θ_t^2 has as stationary density a log-normal density

- **inverted gamma distribution**

$$\bar{p}_{\theta^2}(y) = \frac{(\frac{1}{2}\,\nu)^{\frac{1}{2}\,\nu}}{\varepsilon^2\,\Gamma(\frac{1}{2}\,\nu)}\,\left(\frac{y}{\varepsilon^2}\right)^{-\frac{1}{2}\,\nu-1}\,\exp\left\{-\frac{\frac{1}{2}\,\nu\,\varepsilon^2}{y}\right\}$$

$y > 0$, degrees of freedom $\nu > 0$, scaling parameter ε

- **SDE for squared volatility**

$$d\theta_t^2 = k\,\theta_t^{4(\xi-1)}\,(\bar{\theta}^2 - \theta_t^2)\,dt + \gamma\,\theta_t^{2\,\xi}\,d\bar{W}_t$$

degree of freedom

$$\nu = \frac{2\,(2\,\xi - 1)}{1 - \frac{\varepsilon^2}{\bar{\theta}^2}}$$

$\Longrightarrow$ stationary density inverted gamma

- $\frac{3}{2}$ **volatility model**

  $\xi = \frac{3}{2}$

  Pl. (1997), Lewis (2000), Pl. (2001)

$$d\theta_t^2 = k\,\theta_t^2\,(\bar{\theta}^2 - \theta_t^2)\,dt + \gamma\,\theta_t^3\,d\bar{W}_t$$

- **ARCH diffusion**

  $\xi = 1$

  Engle (1982), Nelson (1990) and Frey (1997)

$$d\theta_t^2 = k\,(\bar{\theta}^2 - \theta_t^2)\,dt + \gamma\,\theta_t^2\,d\bar{W}_t$$

  continuous time limit of innovation process

  GARCH(1,1) and NGARCH(1,1)

Implied Volatilities for the ARCH Diffusion Model

Hurst (1997), Lewis (2000), Heath, Hurst & Pl. (2001)

GARCH(1,1) of Bollerslev (1986) when $\hat{W}$ and $\bar{W}$ are independent

continuous time limit of NGARCH($\mathbf{1}, \mathbf{1}$) model of Engle & Ng (1993)

two-factor model

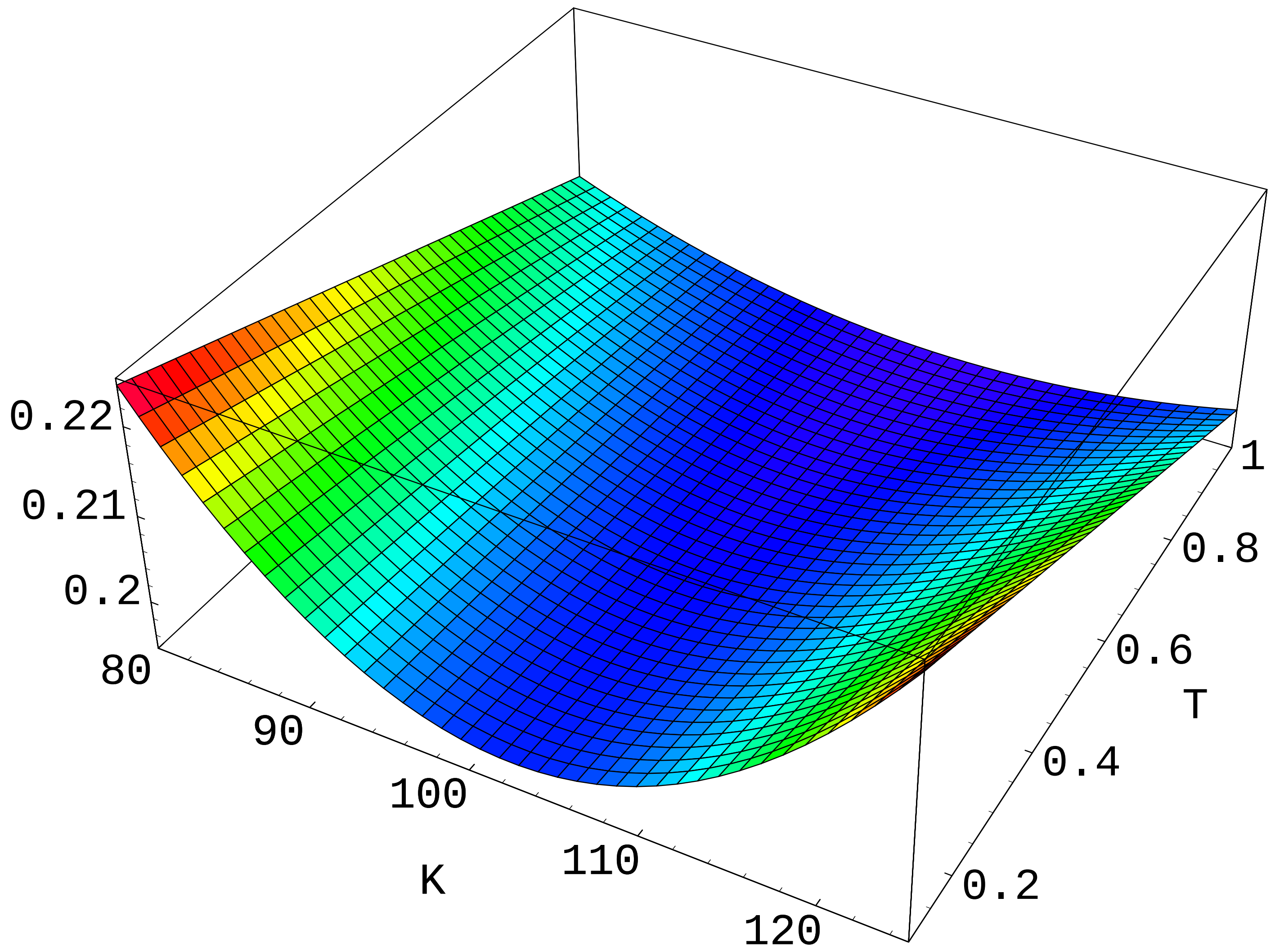

Figure 4.8: Implied volatility surface for zero correlation.

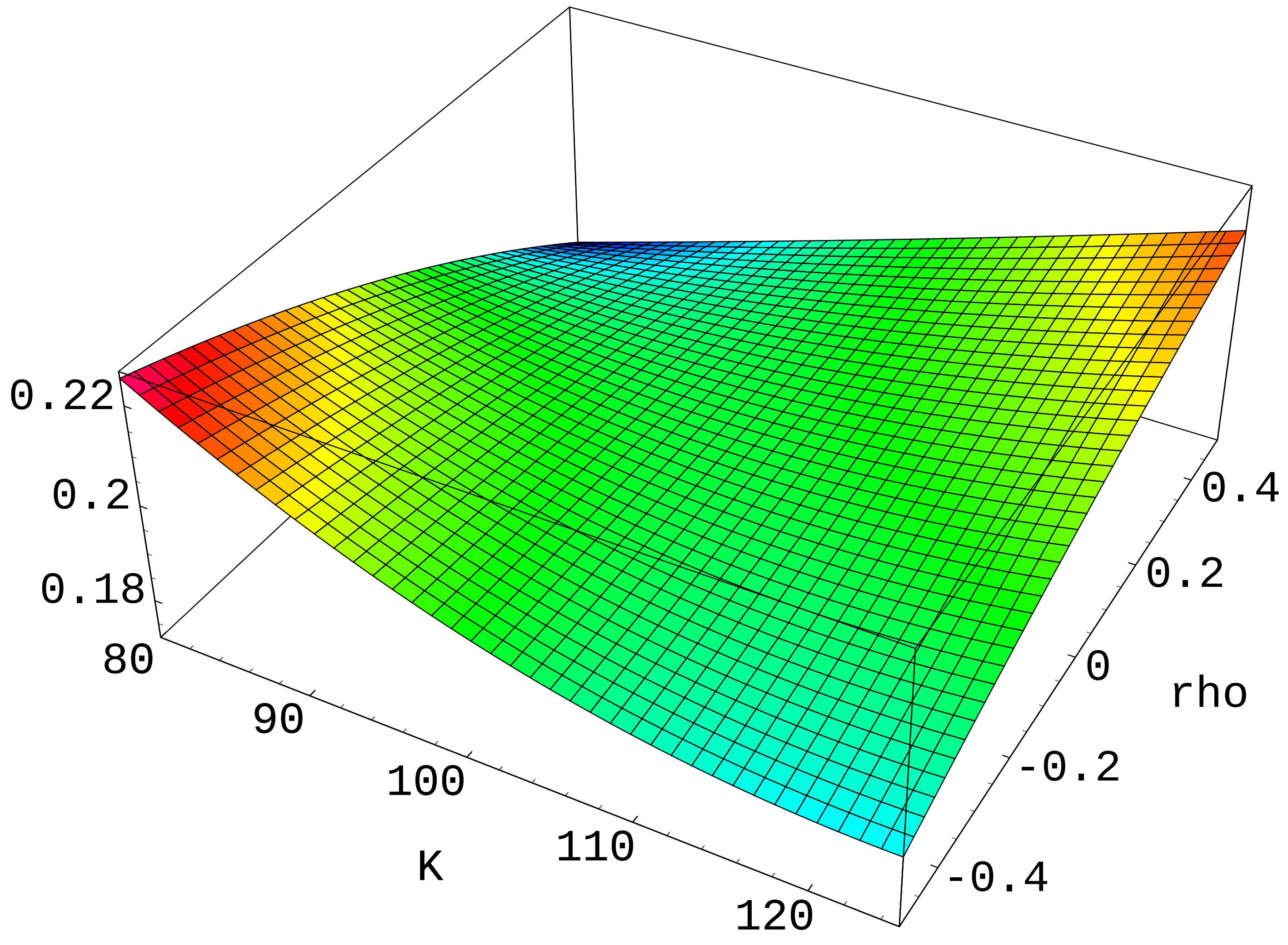

Figure 4.9: Effect of changing correlation on implied volatilities.

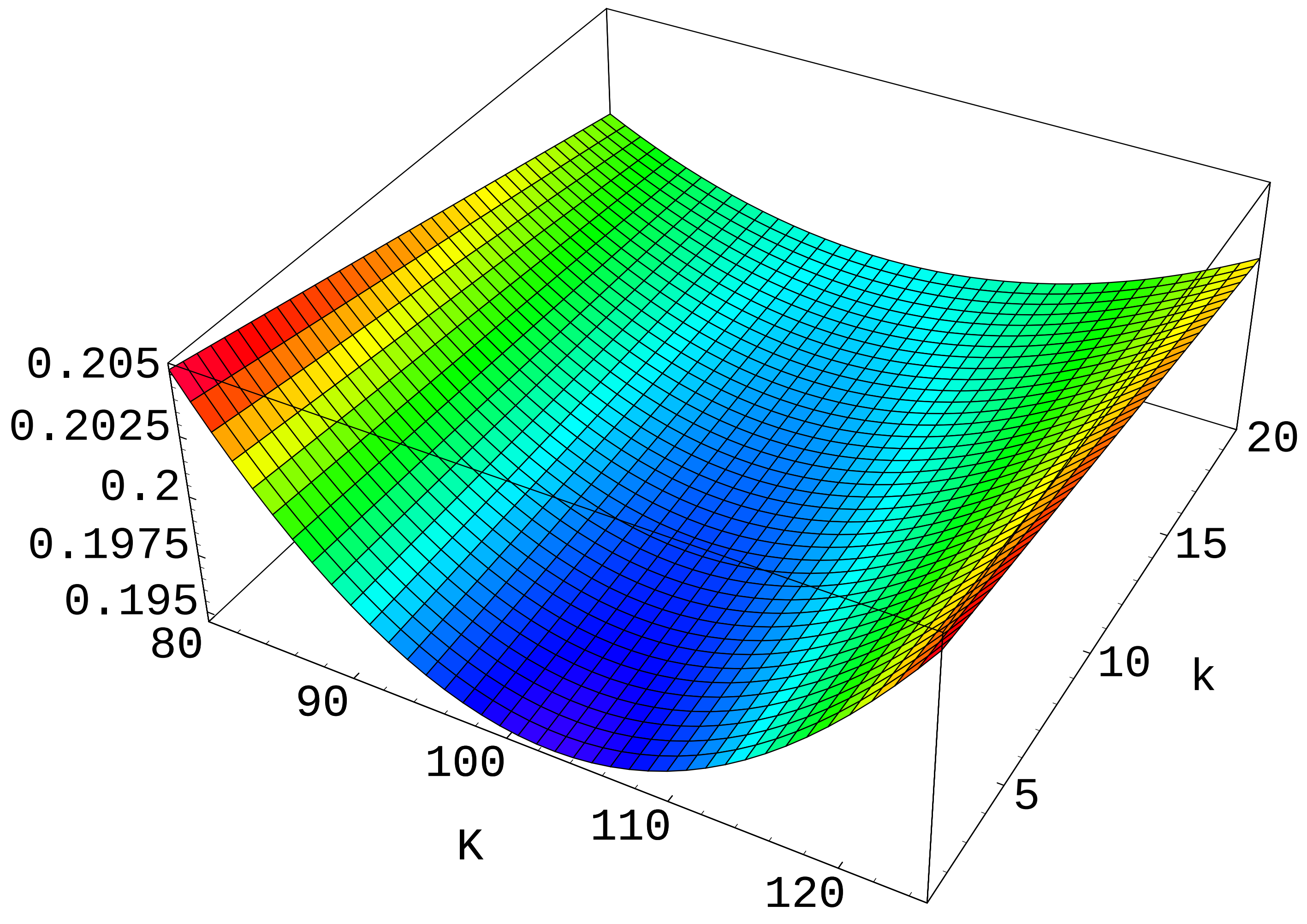

Figure 4.10: Effect of changing speed of adjustment on implied volatilities.

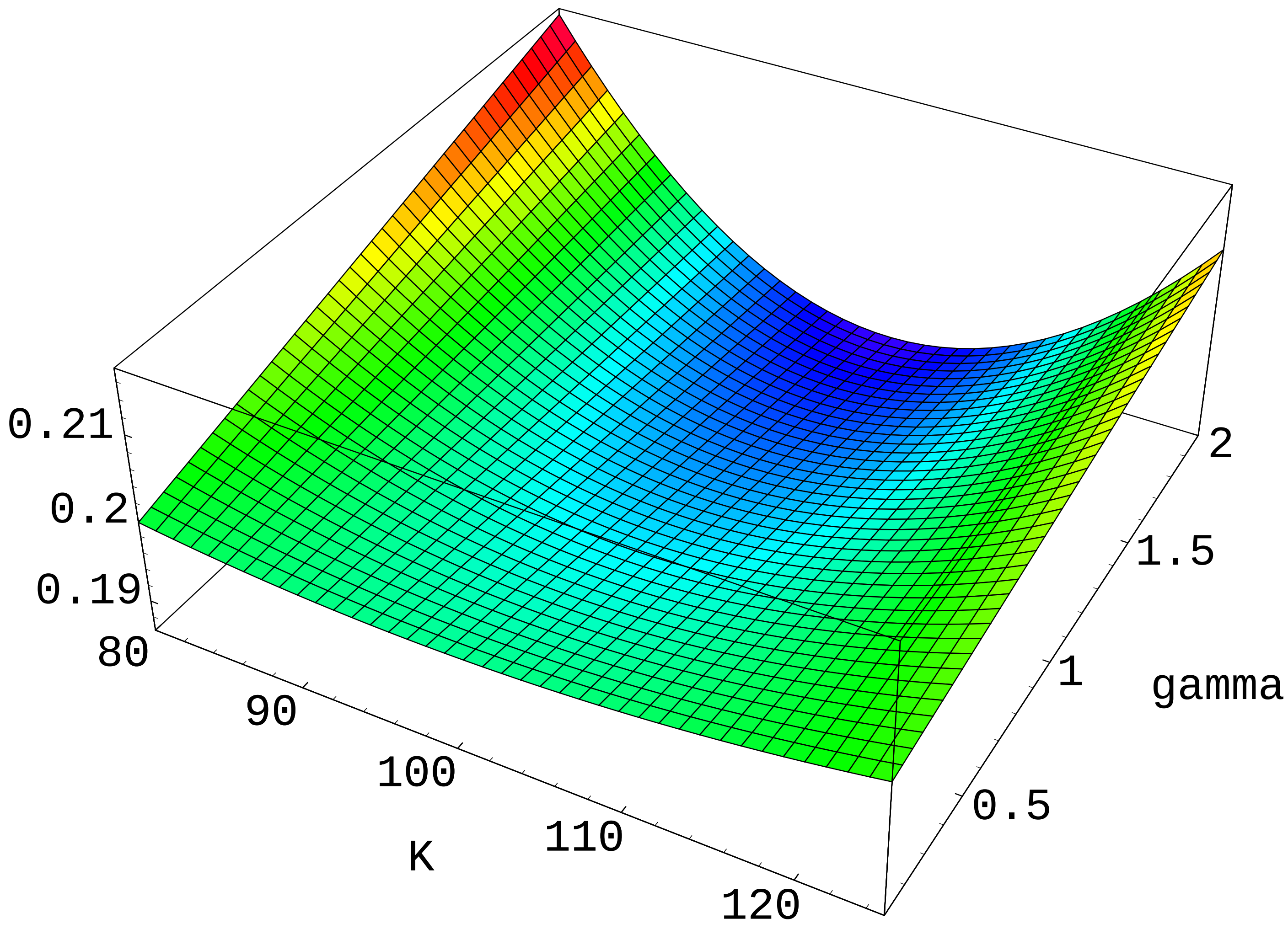

Figure 4.11: Effect of changing volatility of the squared volatility on implied volatilities.

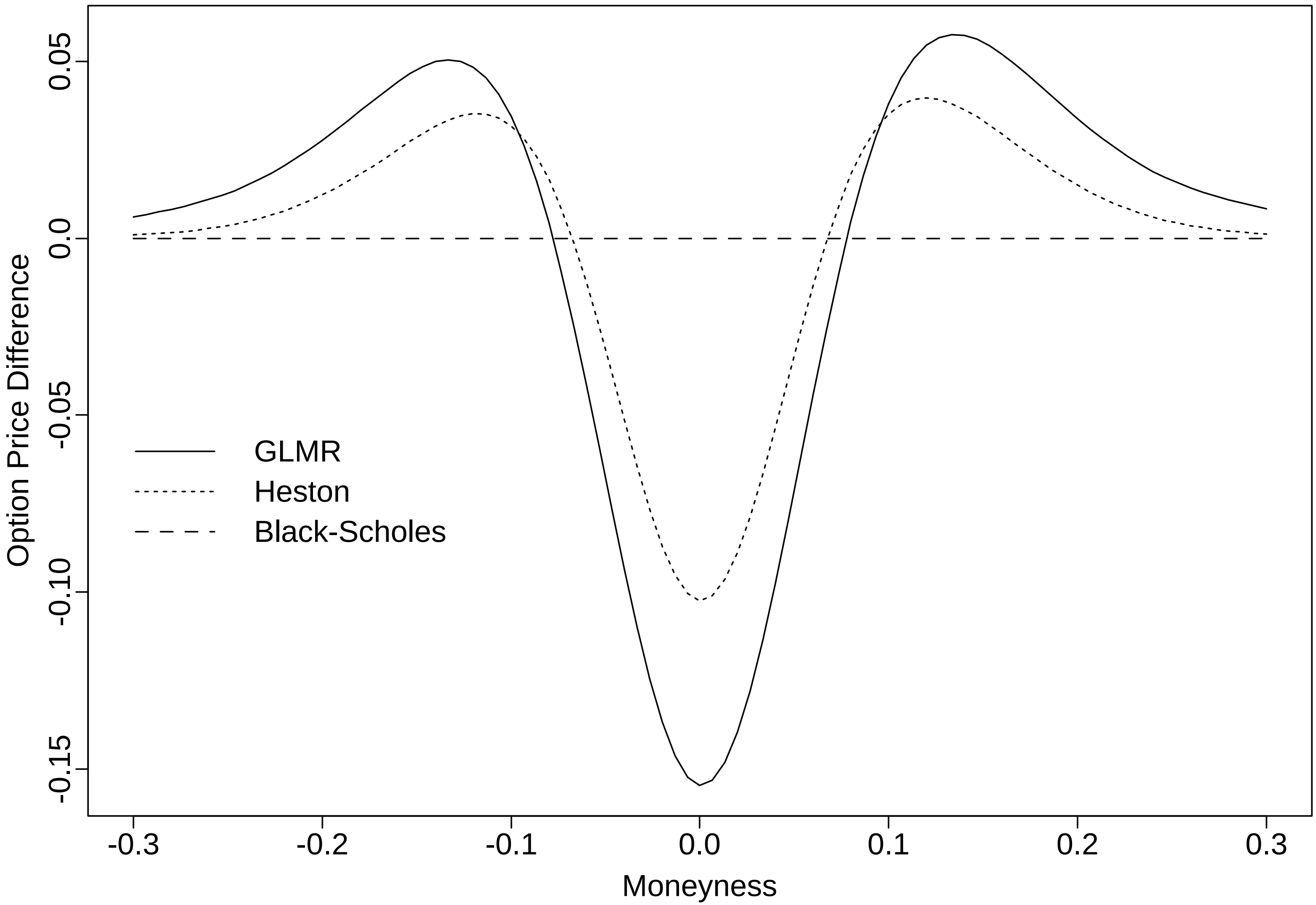

Figure 4.12: Option price differences between the ARCH diffusion (GLMR), Heston and Black-Scholes models, with time to maturity of three months.

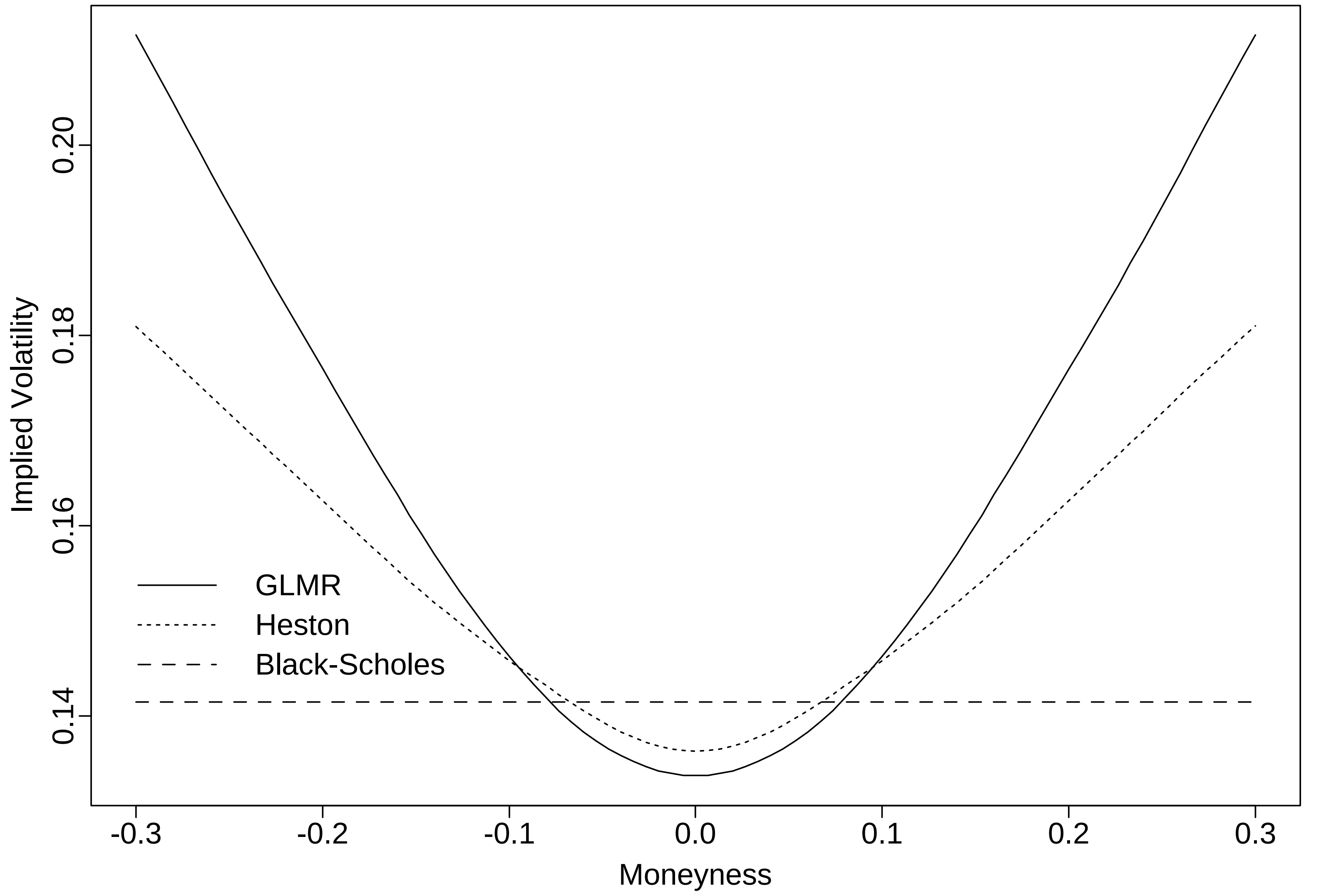

Figure 4.13: Implied volatilities of the ARCH diffusion (GLMR), Heston and Black-Scholes models, with time to maturity of three months.

5 Minimal Market Model

- **search for parsimonious one-factor model**

- **Diversification Theorem** $\Longrightarrow$

  well diversified stock market index approximates NP

- **discounted NP**

$$d\bar{S}_t^{\delta_*} = \bar{S}_t^{\delta_*} \, |\theta_t| \, (|\theta_t| \, dt + dW_t),$$

where

$$dW_t = \frac{1}{|\theta_t|} \sum_{k=1}^{d} \theta_t^k \, dW_t^k$$

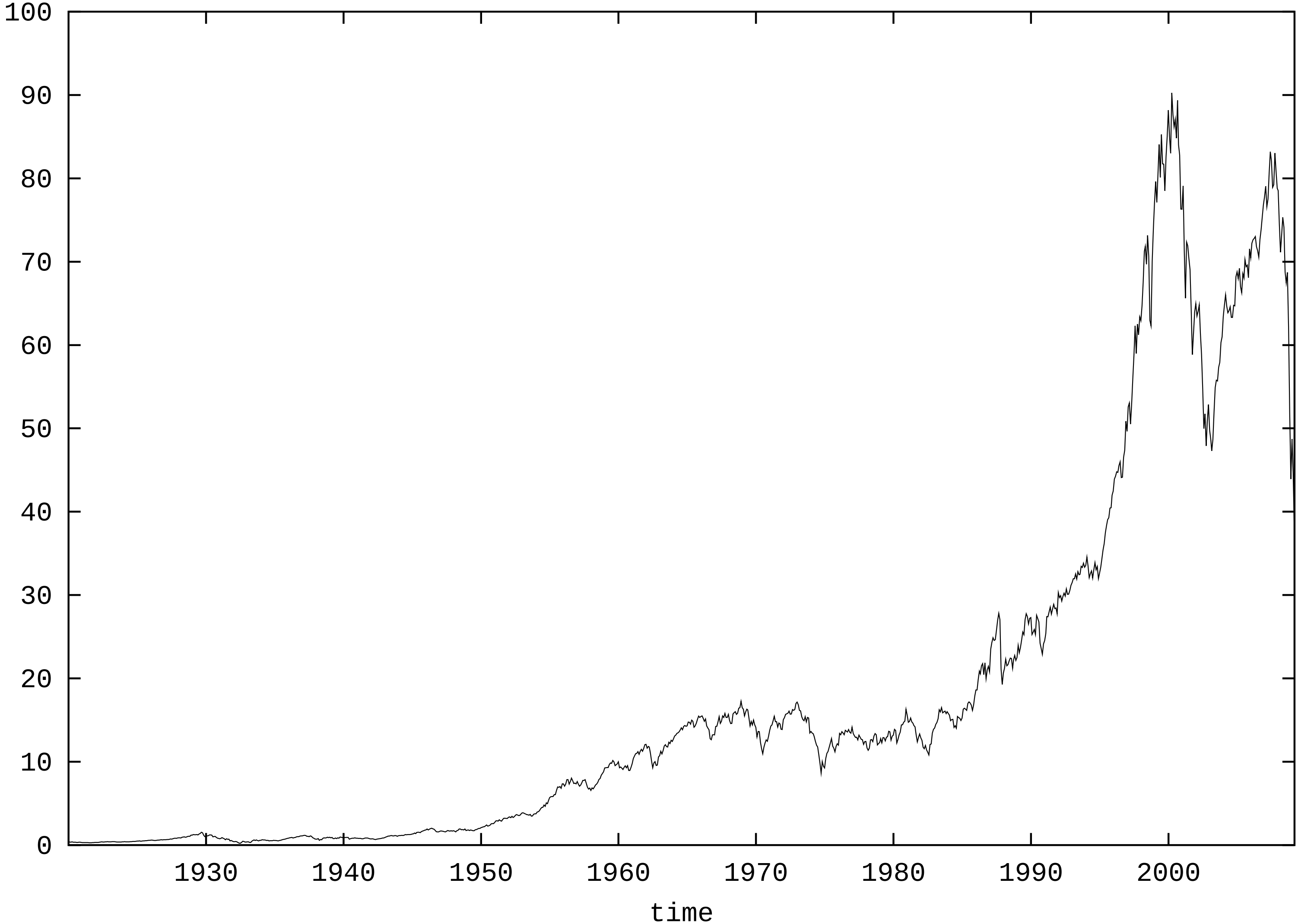

Figure 5.1: Discounted S&P500 total return index.

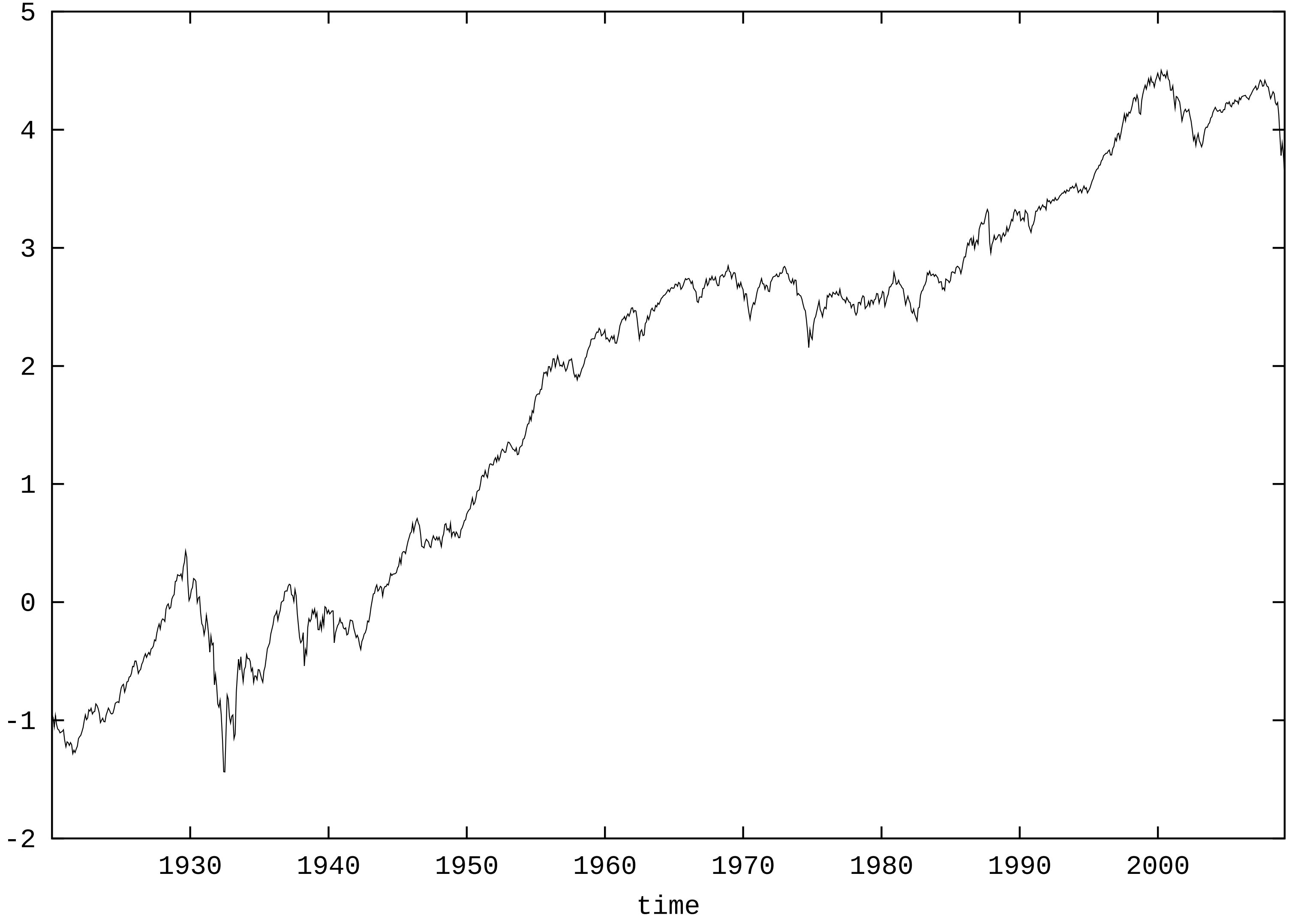

Figure 5.2: Logarithm of discounted S&P500.

- **logarithm of discounted NP**

- is **not** a transformed Wiener process

- is **not** a process with independent increments

- has some **feedback effect**

Drift Parametrization

- **discounted NP drift**

$$\alpha_t = \bar{S}_t^{\delta_*} \, |\theta_t|^2$$

strictly positive, predictable

$\Longrightarrow$

- **volatility**

$$|\theta_t| = \sqrt{\frac{\alpha_t}{\bar{S}_t^{\delta_*}}}$$

leverage effect creates natural **feedback**
under independent drift

- **discounted NP**

$$d\bar{S}_t^{\delta_*} = \alpha_t \, dt + \sqrt{\bar{S}_t^{\delta_*} \, \alpha_t} \, dW_t$$

drift has economic meaning,

fundamental value

- **transformed time**

$$\varphi_t = \frac{1}{4} \int_0^t \alpha_s \, ds$$

- **squared Bessel process of dimension four**

$$X_{\varphi_t} = \bar{S}_t^{\delta_*}$$

$$dW(\varphi_t) = \sqrt{\alpha_t}\, dW_t$$

$$dX_\varphi = 4\, d\varphi + 2\, \sqrt{X_\varphi}\, dW(\varphi)$$

Revuz & Yor (1999)

economically founded dynamics in φ-time

still no specific dynamics in t-time

Time Transformed Bessel Process

$$d\sqrt{\bar{S}_t^{\delta_*}} = \frac{3\,\alpha_t}{8\,\sqrt{\bar{S}_t^{\delta_*}}}\,dt + \frac{1}{2}\,\sqrt{\alpha_t}\,dW_t$$

- **quadratic variation**

$$\left[\sqrt{\bar{S}^{\delta_*}}\right]_t = \frac{1}{4}\int_0^t \alpha_s\,ds$$

- **transformed time**

$$\varphi_t = \left[\sqrt{\bar{S}^{\delta_*}}\right]_t$$

observable

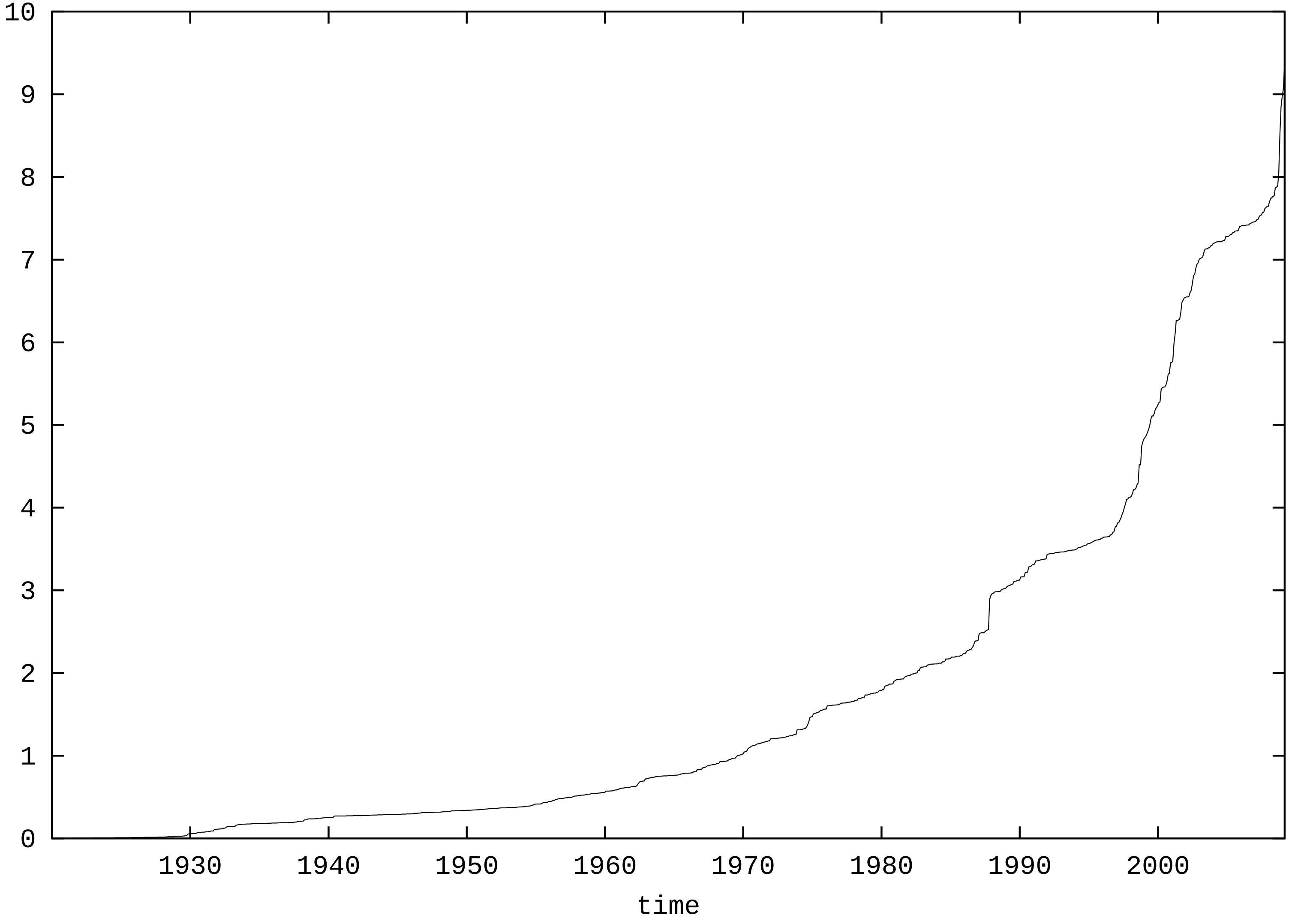

Figure 5.3: Empirical quadratic variation of the square root of the discounted NP.

Stylized Minimal Market Model

Pl. (2001, 2002, 2006)

- assume **discounted NP drift** as

$$\alpha_t = \alpha \exp\{\eta t\}$$

- initial parameter $\alpha > 0$

- net growth rate η

- **transformed time**

$$\varphi_t = \frac{\alpha}{4} \int_0^t \exp\{\eta\, z\}\, dz$$

$$= \frac{\alpha}{4\,\eta} \left(\exp\{\eta\, t\} - 1\right)$$

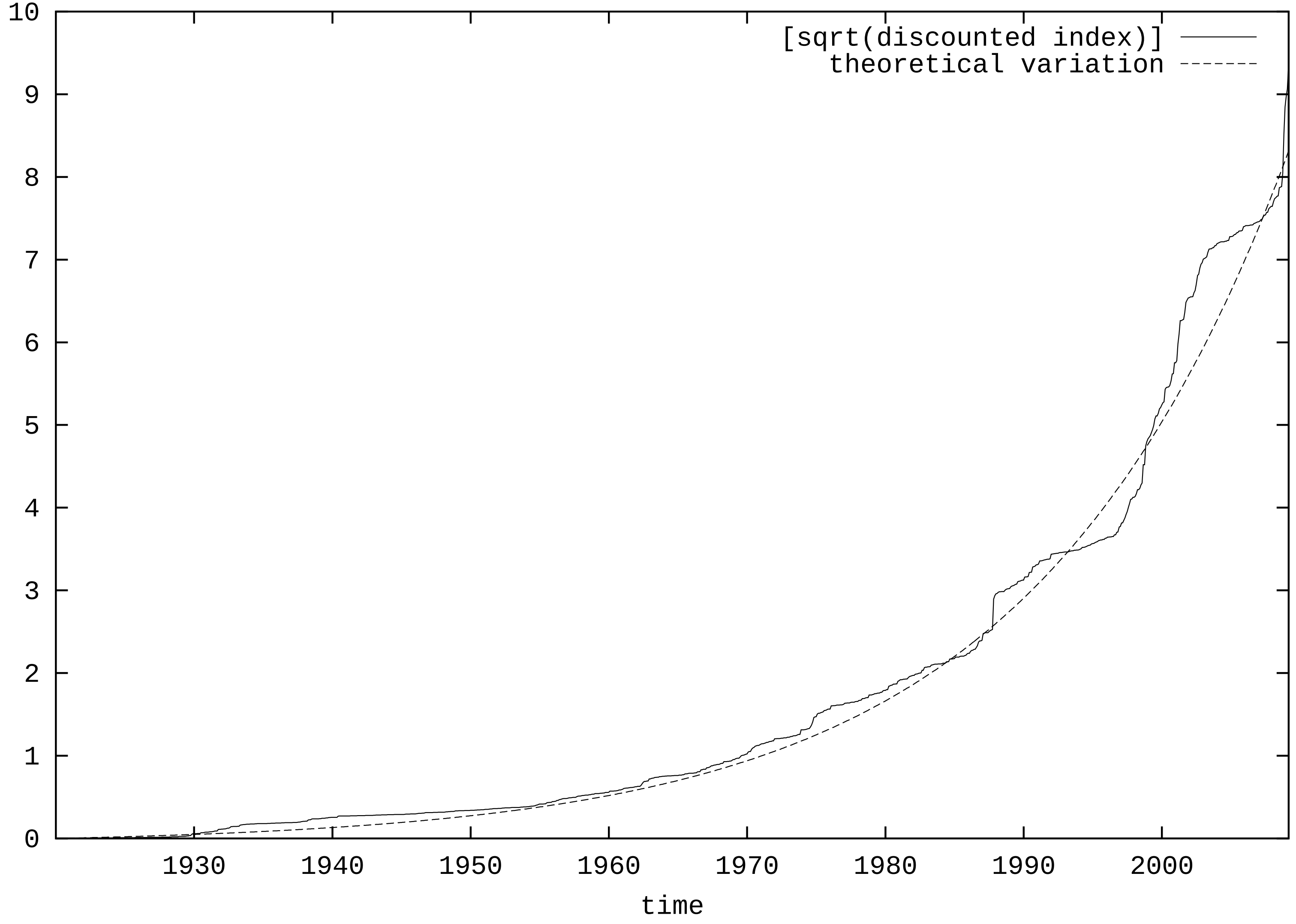

Figure 5.4: Fitted and observed transformed time.

- **normalized NP**

$$Y_t = \frac{\bar{S}_t^{\delta_*}}{\alpha_t}$$

$$dY_t = (1 - \eta\, Y_t)\, dt + \sqrt{Y_t}\, dW_t$$

square root process of dimension four

$\Longrightarrow$ parsimonious model, long term viability

$\Longrightarrow$

- **discounted NP**

$$\bar{S}_t^{\delta_*} = Y_t \, \alpha_t$$

- **NP**

$$S_t^{\delta_*} = S_t^0 \, \bar{S}_t^{\delta_*} = S_t^0 \, Y_t \, \alpha_t$$

- scaling parameter $\alpha_0 = 0.01468$
- net growth rate $\eta = 0.054$
- reference level $\frac{1}{\eta} = 18.5$
- speed of adjustment η
- half life time of major displacement $\frac{\ln(2)}{\eta} \approx 13$ years

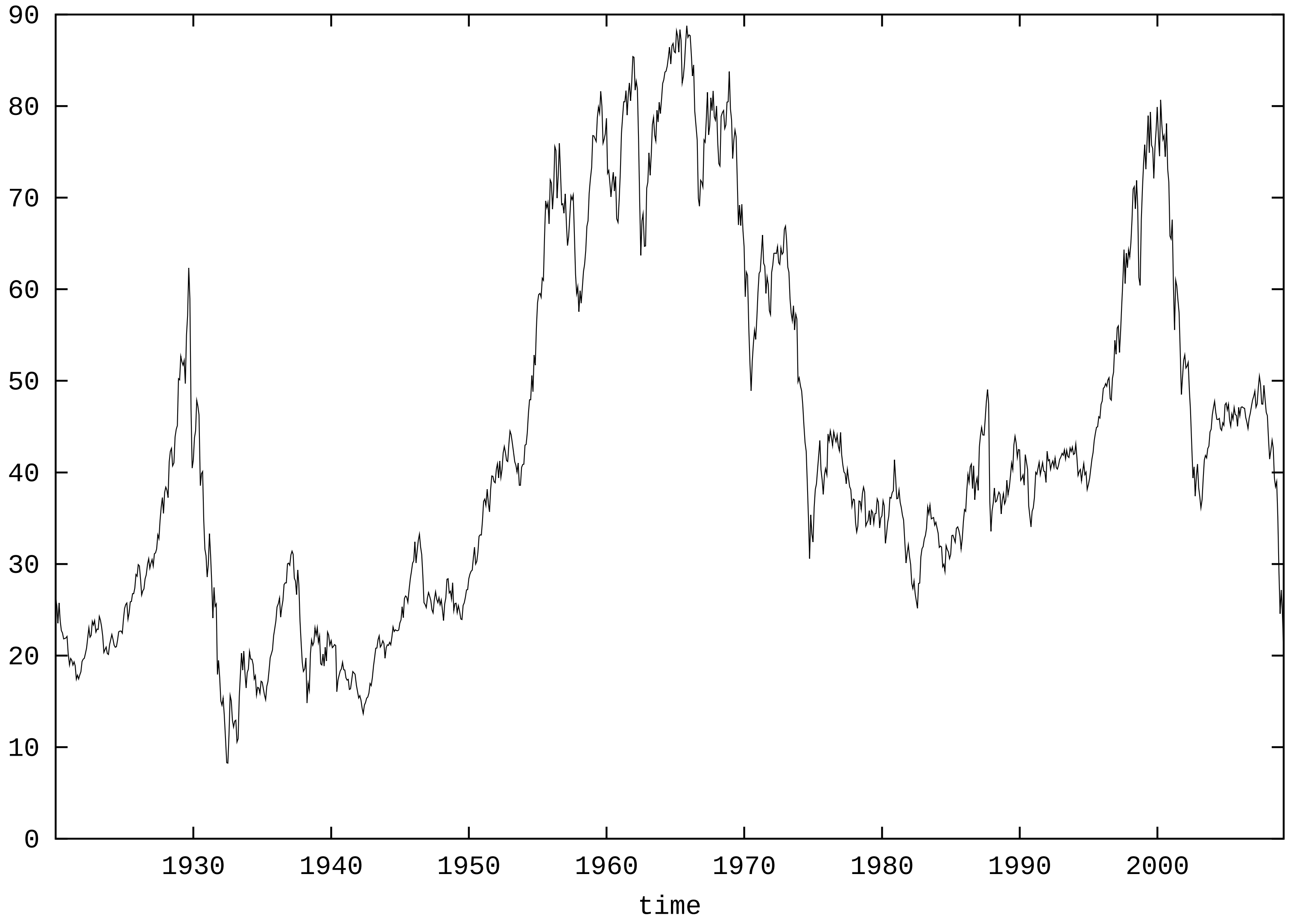

Figure 5.5: Normalized NP.

- **logarithm of discounted NP**

$$\ln(\bar{S}_t^{\delta_*}) = \ln(Y_t) + \ln(\alpha_0) + \eta\, t$$

- no need for extra volatility process in MMM

- realistic long term dynamics

Volatility of NP under the stylized MMM

$$|\theta_t| = \frac{1}{\sqrt{Y_t}}$$

- **squared volatility**

$$d|\theta_t|^2 = d\left(\frac{1}{Y_t}\right) = |\theta_t|^2\,\eta\,dt - \left(|\theta_t|^2\right)^{\frac{3}{2}}\;dW_t$$

$3/2$ volatility model

Platen (1997), Lewis (2000)

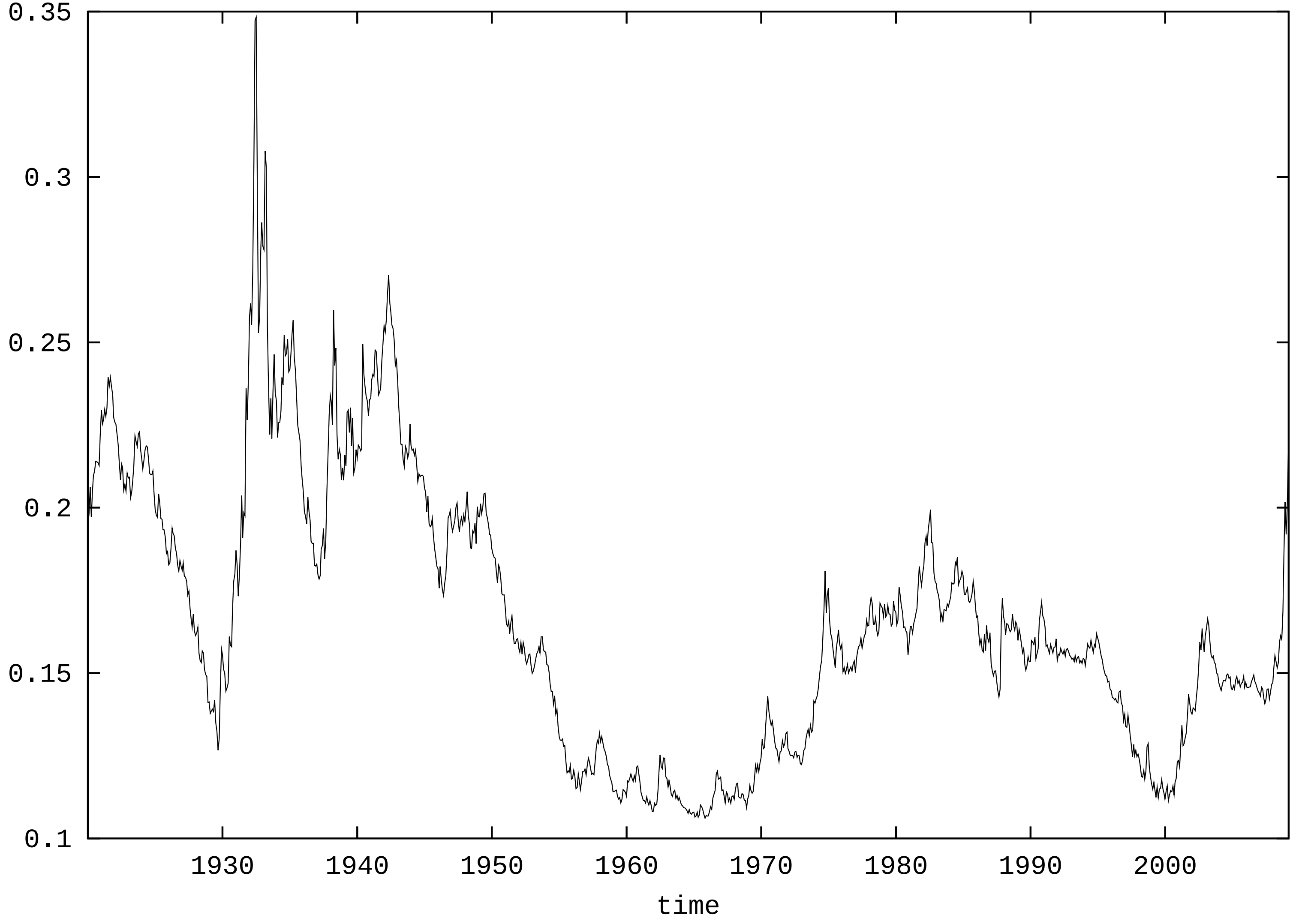

Figure 5.6: Volatility of the NP under the stylized MMM.

Transition Density of Stylized MMM

- transition density of **discounted NP** $\bar{S}^{\delta_*}$

$$p(s,x;t,y) = \frac{1}{2\,(\varphi_t - \varphi_s)} \left(\frac{y}{x}\right)^{\frac{1}{2}} \exp\left\{-\frac{x+y}{2\,(\varphi_t - \varphi_s)}\right\}$$

$$\times\, I_1\left(\frac{\sqrt{x\,y}}{\varphi_t - \varphi_s}\right)$$

$$\varphi_t = \frac{\alpha}{4\,\eta}\,(\exp\{\eta\,t\} - 1)$$

$I_1(\cdot)$ modified Bessel function of the first kind

- **non-central chi-square** distributed random variable:

$$\frac{y}{\varphi_t - \varphi_s} = \frac{\bar{S}_t^{\delta_*}}{\varphi_t - \varphi_s}$$

with $\delta = 4$ degrees of freedom and **non-centrality parameter:**

$$\frac{x}{\varphi_t - \varphi_s} = \frac{\bar{S}_s^{\delta_*}}{\varphi_t - \varphi_s}$$

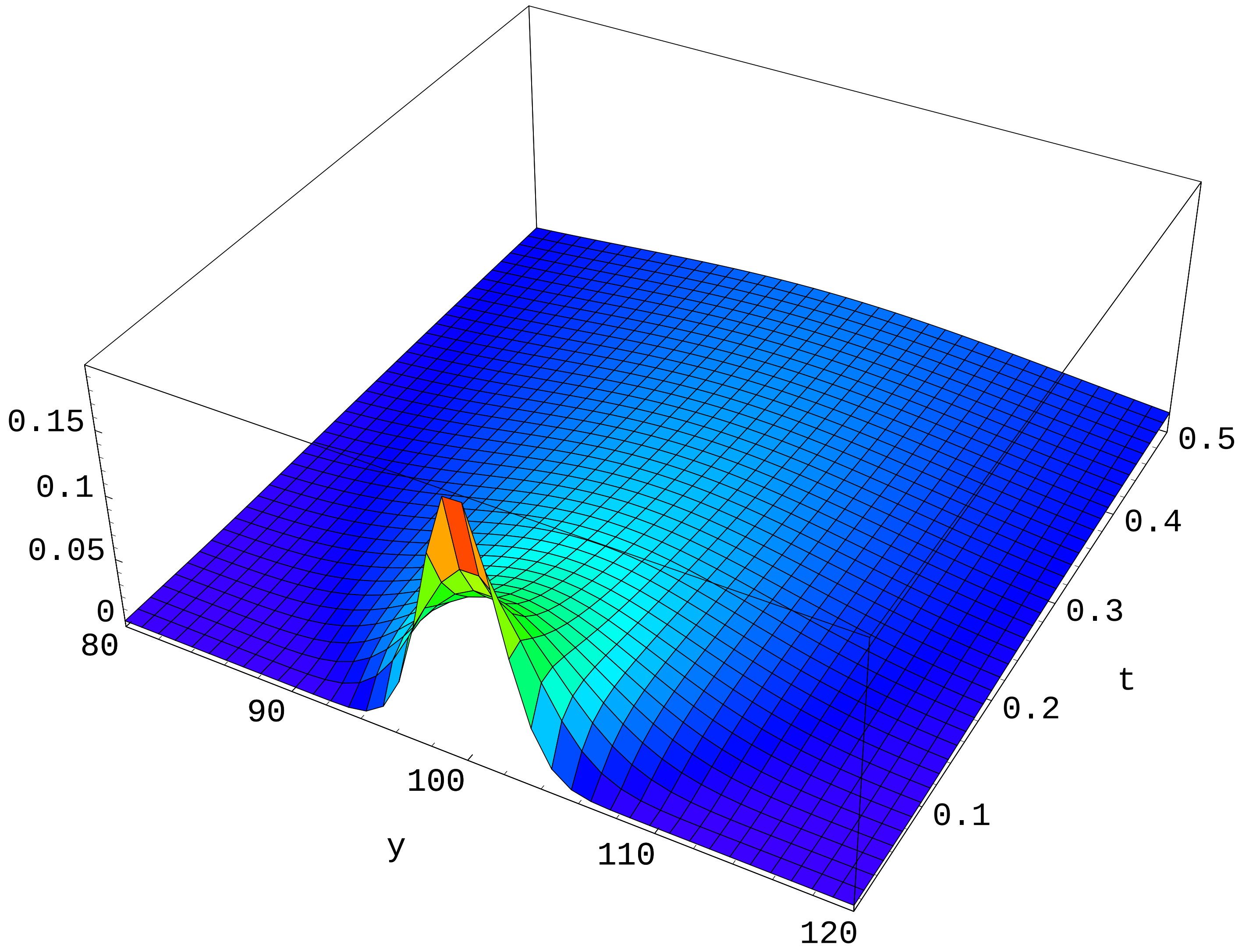

Figure 5.7: Transition density of squared Bessel process for $\delta = 4$.

Zero Coupon Bond under the stylized MMM

- **zero coupon bond**

$$P(t,T) = S_t^{\delta_*}\, E_t\left(\frac{1}{S_T^{\delta_*}}\right) = P_T^*(t)\, E_t\left(\frac{\bar{S}_t^{\delta_*}}{\bar{S}_T^{\delta_*}}\right)$$

with **savings bond**

$$P_T^*(t) = \exp\left\{-\int_t^T r_s\, ds\right\}$$

$\delta = 4 \quad \Longrightarrow$

$$P(t,T) = P^*_T(t)\left(1 - \exp\left\{-\frac{\bar{S}^{\delta_*}_t}{2\,(\varphi_T - \varphi_t)}\right\}\right) < P^*_T(t)$$

for $t \in [0, T)$, Platen (2002)

with time transform

$$\varphi_t = \frac{\alpha}{4\,\eta}\,(\exp\{\eta\,t\} - 1)$$

$P(0,T) = 0.00077$

$P^*_T(0) = 0.0335$

$\frac{P(0,T)}{P^*_T(0)} \approx 0.023$

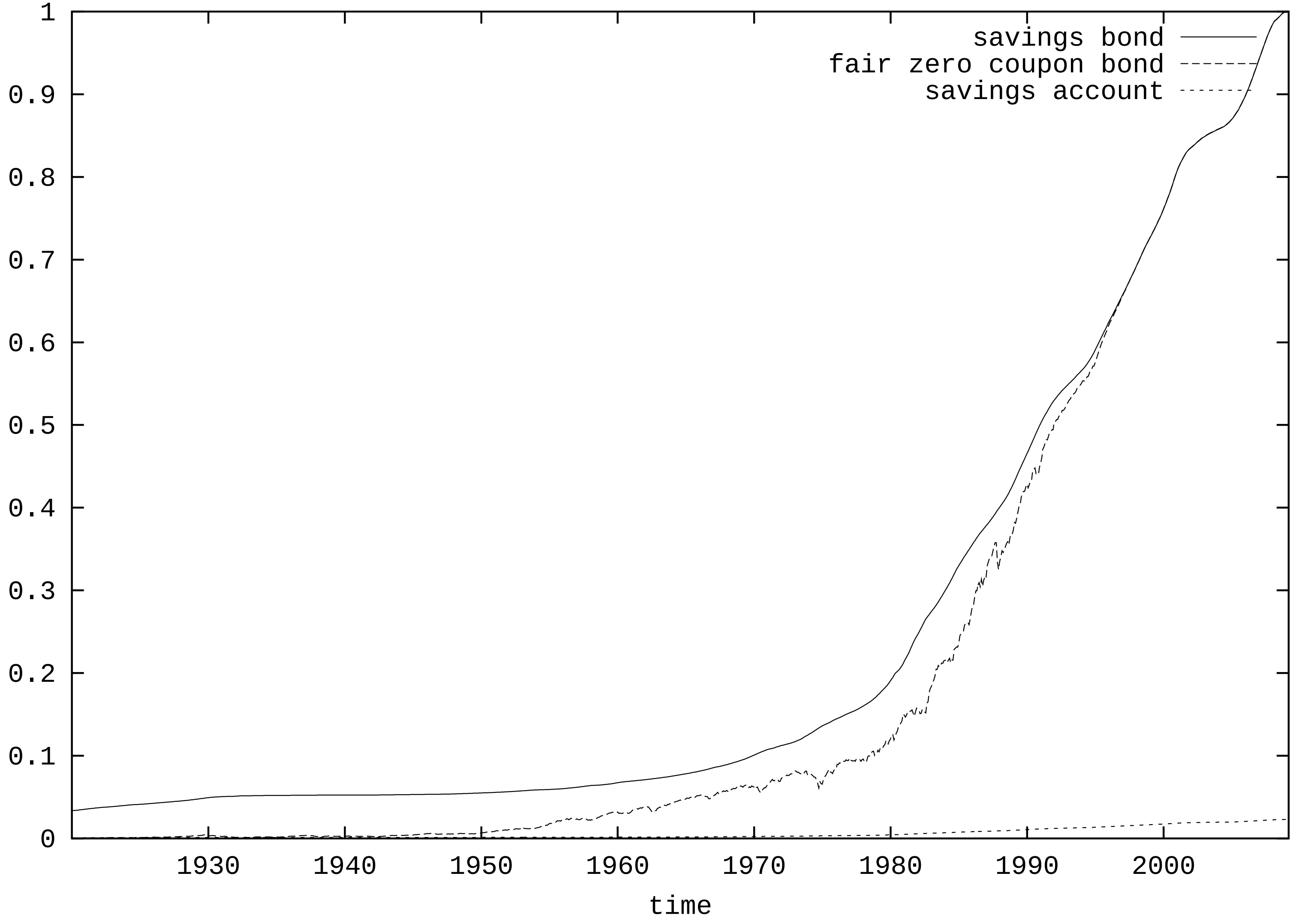

Figure 5.8: Savings bond, fair zero coupon bond and savings account.

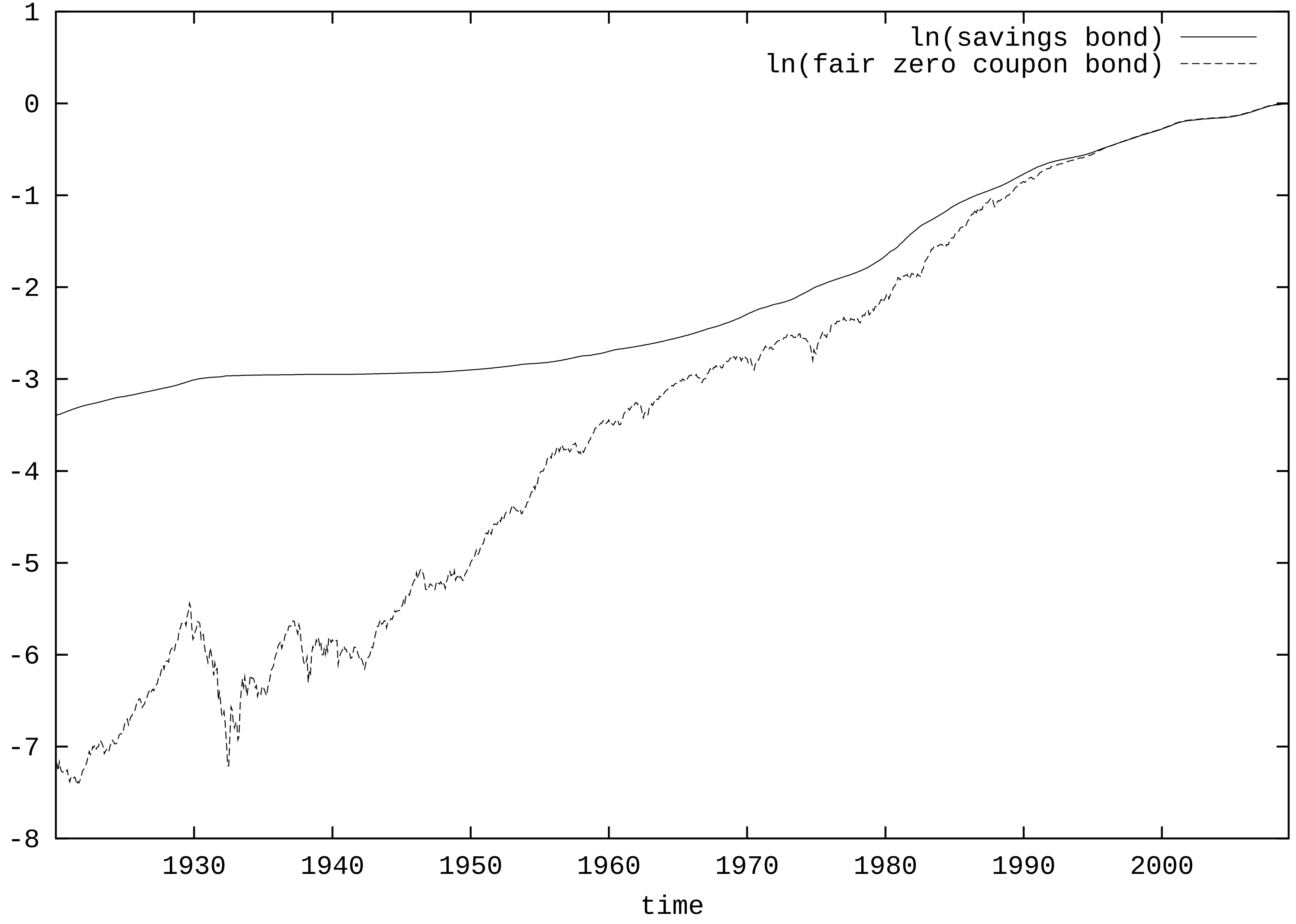

Figure 5.9: Logarithms of savings bond and fair zero coupon bond.

Forward Rates under the MMM

- **forward rate**

$$\begin{aligned} f(t,T) &= -\frac{\partial}{\partial T} \ln(P(t,T)) \\ &= r_T + n(t,T) \end{aligned}$$

- **market price of risk contribution**

$$n(t,T) = -\frac{\partial}{\partial T} \ln\left(1 - \exp\left\{-\frac{\bar{S}_t^{\delta_*}}{2(\varphi_T - \varphi_t)}\right\}\right)$$

$$= \frac{1}{\left(\exp\left\{\frac{\bar{S}_t^{\delta_*}}{2(\varphi_T - \varphi_t)}\right\} - 1\right)} \frac{\bar{S}_t^{\delta_*}}{(\varphi_T - \varphi_t)^2} \frac{\alpha_T}{8}$$

$$\lim_{T\to\infty} n(t,T) = \eta$$

Pl. (2005a), Miller & Pl. (2005)

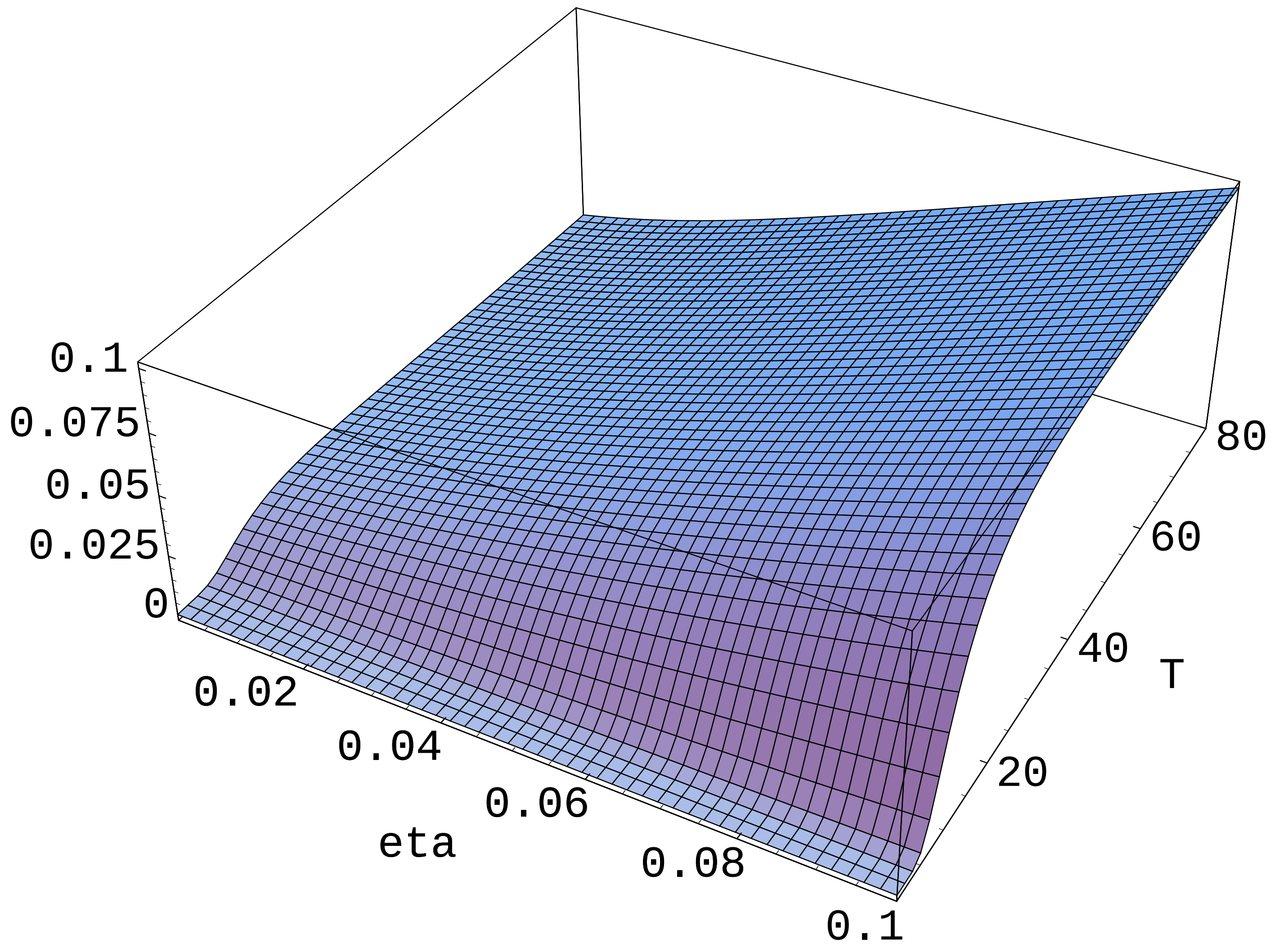

Figure 5.10: Market price of risk contribution in dependence on η and T.

Free Snack from Savings Bond

- potential existence of weak form of arbitrage?

  borrow amount $P(0,T)$ from savings account

$$S_t^{\delta} = P(t,T) - P(0,T)\,\exp\{r\,t\}$$

  such that $S_0^{\delta} = 0$

$$S_T^{\delta} = 1 - P(0,T)\,\exp\{r\,T\} > 0$$

  lower bound

$$S_t^{\delta} \geq -P(0,T)\,\exp\{r\,t\}$$

- since S_t^δ may become negative

  allowed by strong arbitrage concept

- **cheap thrills** may exist

  Loewenstein & Willard (2000)

  MMM does **not** admit an equivalent risk neutral probability measure

$\Longrightarrow$

there is a **free lunch with vanishing risk**

Delbaen & Schachermayer (2006)

Absence of an Equivalent Risk Neutral Probability Measure

- fair zero coupon bond has a lower price than the savings bond

$$P(t,T) < P^*_T(t) = \exp\left\{-\int_t^T r_s\,ds\right\} = \frac{S^0_t}{S^0_T}$$

- **Radon-Nikodym derivative**

$$\Lambda_t = \frac{\hat{S}^0_t}{\hat{S}^0_0} = \frac{\bar{S}^{\delta_*}_0}{\bar{S}^{\delta_*}_t}$$

strict $(\underline{\mathcal{A}}, P)$-supermartingale under MMM

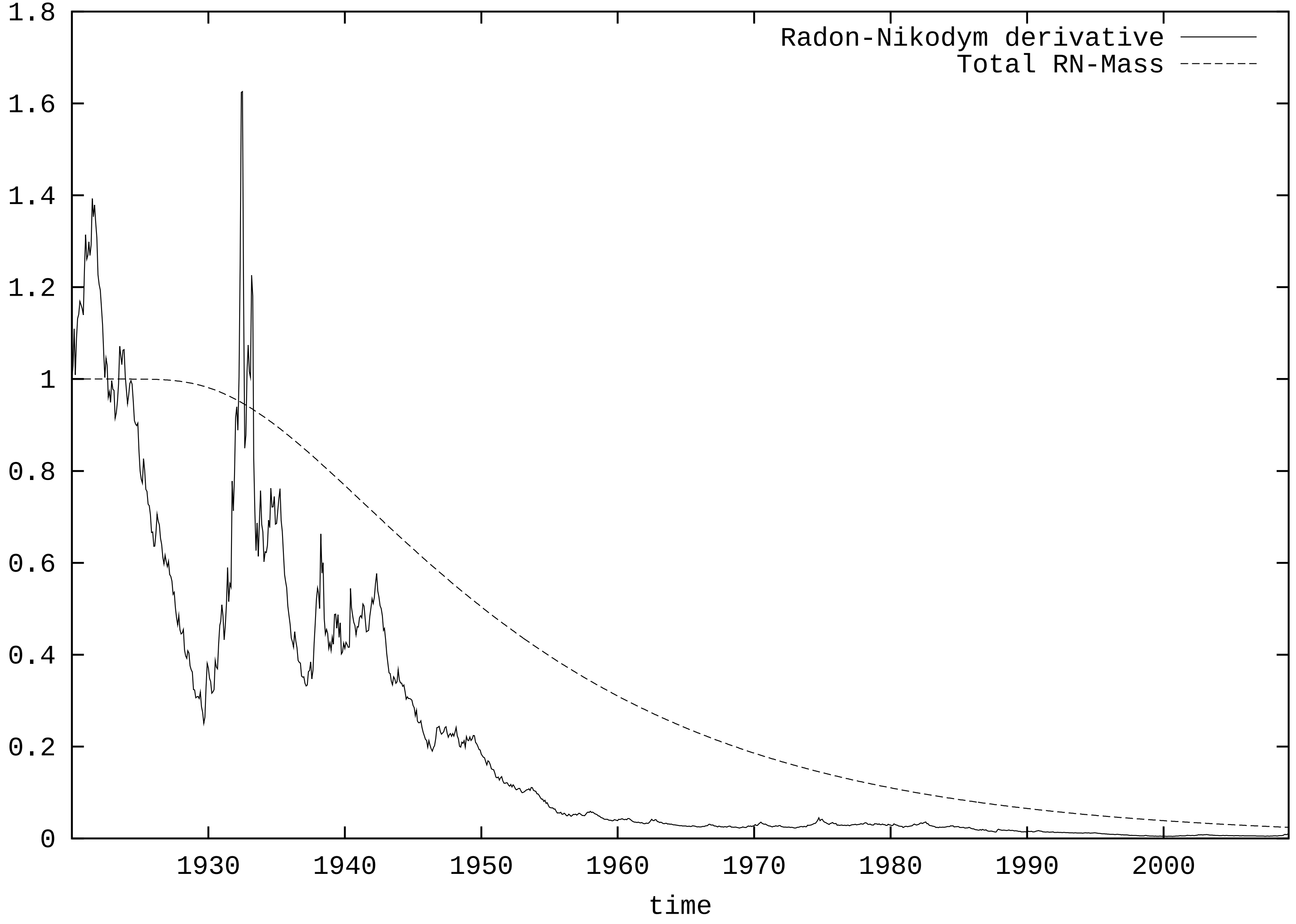

Figure 5.11: Radon-Nikodym derivative and total mass of putative risk neutral measure.

- **hypothetical risk neutral measure**

  total mass:

$$P_{\theta,T}(\Omega) = E_0\left(\Lambda_T\right) = 1 - \exp\left\{-\frac{\bar{S}_0^{\delta_*}}{2\,\varphi_T}\right\} < \Lambda_0 = 1$$

P_θ is **not** a probability measure

European Call Options under the MMM

real world pricing formula $\Longrightarrow$

$$
\begin{aligned}
c_{T,K}(t, S_t^{\delta_*}) &= S_t^{\delta_*}\, E_t\left(\frac{(S_T^{\delta_*} - K)^+}{S_T^{\delta_*}}\right) \\
&= E_t\left(\left(S_t^{\delta_*} - \frac{K\, S_t^{\delta_*}}{S_T^{\delta_*}}\right)^+\right) \\
&= S_t^{\delta_*}\,\left(1 - \chi^2(d_1; 4, \ell_2)\right) \\
&\quad - K\, \exp\{-r\,(T-t)\}\,(1 - \chi^2(d_1; 0, \ell_2))
\end{aligned}
$$

$\chi^2(\cdot;\cdot,\cdot)$ non-central chi-square distribution with

$$d_1 = \frac{4\,\eta\,K\;\exp\{-r\,(T-t)\}}{S_t^0\,\alpha_t\,(\exp\{\eta\,(T-t)\}-1)}$$

and

$$\ell_2 = \frac{2\,\eta\,S_t^{\delta_*}}{S_t^0\,\alpha_t\,(\exp\{\eta\,(T-t)\}-1)}$$

Hulley, Miller & Pl. (2005)

Hulley & Pl. (2008)

Miller & Pl. (2008)

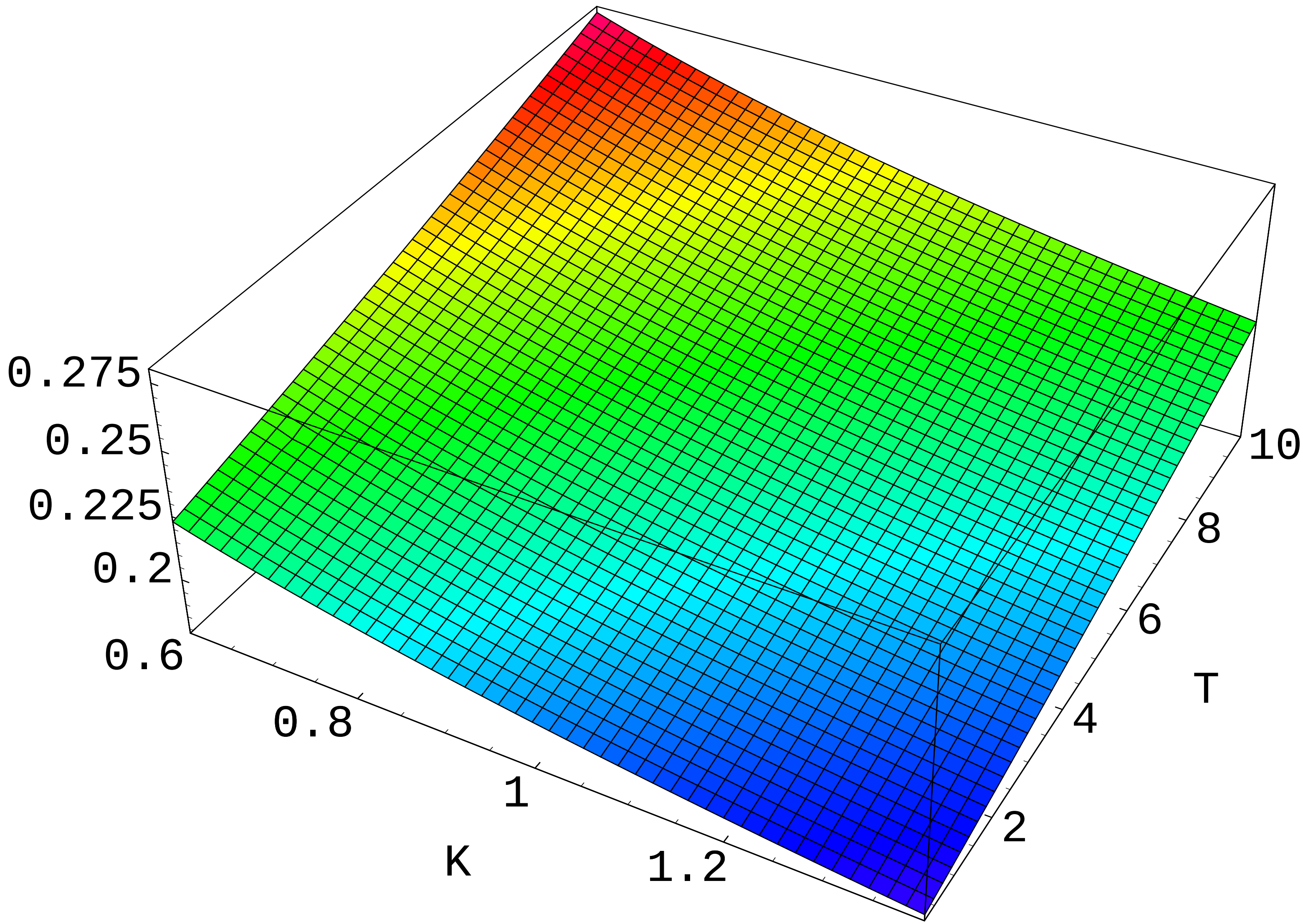

Figure 5.12: Implied volatility surface for the stylized MMM.

- **implied volatility**

  in BS-formula adjust short rate to

$$\hat{r} = -\frac{1}{T-t} \ln(P(t,T))$$

  otherwise put and call implied volatilities do not match

European Put Options under the MMM

- **fair put-call parity relation**

$$p_{T,K}(t,\bar{S}_t^{\delta_*}) = c_{T,K}(t,\bar{S}_t^{\delta_*}) - S_t^{\delta_*} + K\,P(t,T)$$

- **European put option formula**

$$\begin{aligned} p_{T,K}(t,S_t^{\delta_*}) &= -S_t^{\delta_*}\left(\chi^2(d_1;4,\ell_2)\right) \\ &\quad + K\exp\{-r\,(T-t)\}\left(\chi^2(d_1;0,\ell_2) - \exp\{-\ell_2\}\right) \end{aligned}$$

Hulley, Miller & Pl. (2005)

- put-call parity **breaks down** if one uses the savings bond $P^*_T(t)$

$$p_{T,K}(t, \bar{S}^{\delta_*}_t) < c_{T,K}(t, \bar{S}^{\delta_*}_t) - S^{\delta_*}_t + K \; \exp\{-r\,(T-t)\}$$

for $t \in [0, T)$

- **excellent hedge performance for extreme maturities**

  Hulley & Pl. (2008)

  European calls

  European puts

  barrier options

Comparison to Hypothetical Risk Neutral Prices

- **hypothetical risk neutral price** $c^*_{T,K}(t, S^{\delta_*}_t)$

  of a European call option on the NP

- benchmarked hypothetical risk neutral call price

$$\hat{c}^*_{T,K}(t, S^{\delta_*}_t) = \frac{c^*_{T,K}(t, S^{\delta_*}_t)}{S^{\delta_*}_t}$$

  local martingale

  $\hat{c}^*_{T,K}(\cdot, \cdot)$ uniformly bounded

$\Longrightarrow \quad \hat{c}^*_{T,K}$ martingale $\quad \Longrightarrow$

$$\hat{c}^*_{T,K}(t, S^{\delta_*}_t) = \hat{c}_{T,K}(t, S^{\delta_*}_t)$$

$\Longrightarrow$

$$c^*_{T,K}(t, S^{\delta_*}_t) = c_{T,K}(t, S^{\delta_*}_t)$$

- **hypothetical risk neutral put-call parity**

$$p^*_{T,K}(t, S^{\delta_*}_t) = c^*_{T,K}(t, S^{\delta_*}_t) - S^{\delta_*}_t + K\, P^*_T(t)$$

since $P^*_T(t) > P(t,T) \quad \Longrightarrow$

$$p_{T,K}(t, S^{\delta_*}_t) < p^*_{T,K}(t, S^{\delta_*}_t)$$

$t \in [0, T)$

Difference in Asymptotic Put Prices

- hypothetical risk neutral prices can become extreme if NP value tends towards zero

- **asymptotic fair zero coupon bond**

$$\lim_{\bar{S}_t^{\delta_*} \to 0} P_T(t, T) \overset{\text{a.s.}}{=} 0$$

for $t \in [0, T)$

- **asymptotic fair European call**

$$\lim_{\bar{S}_t^{\delta_*} \to 0} c_{T,K}(t, S_t^{\delta_*}) \overset{\text{a.s.}}{=} \lim_{\bar{S}_t^{\delta_*} \to 0} S_t^{\delta_*} \, \hat{c}_{T,K}(t, S_t^{\delta_*}) = 0$$

- **asymptotic fair put**

  fair put-call parity $\Longrightarrow$

$$\lim_{\bar{S}_t^{\delta_*} \to 0} p_{T,K}(t, S_t^{\delta_*}) \stackrel{\text{a.s.}}{=} 0$$

- **asymptotic hypothetical risk neutral put**

  hypothetical risk neutral put-call parity $\Longrightarrow$

$$\lim_{\bar{S}_t^{\delta_*} \to 0} p^*_{T,K}(t, S_t^{\delta_*})$$

$$\stackrel{\text{a.s.}}{=} \lim_{\bar{S}_t^{\delta_*} \to 0} \left(p_{T,K}(t, S_t^{\delta_*}) + K \frac{S_t^0}{S_T^0} \exp\left\{ -\frac{\bar{S}_t^{\delta_*}}{2(\varphi_T - \varphi_t)} \right\} \right)$$

$$\stackrel{\text{a.s.}}{=} K \frac{S_t^0}{S_T^0} > 0$$

dramatic differences can arise

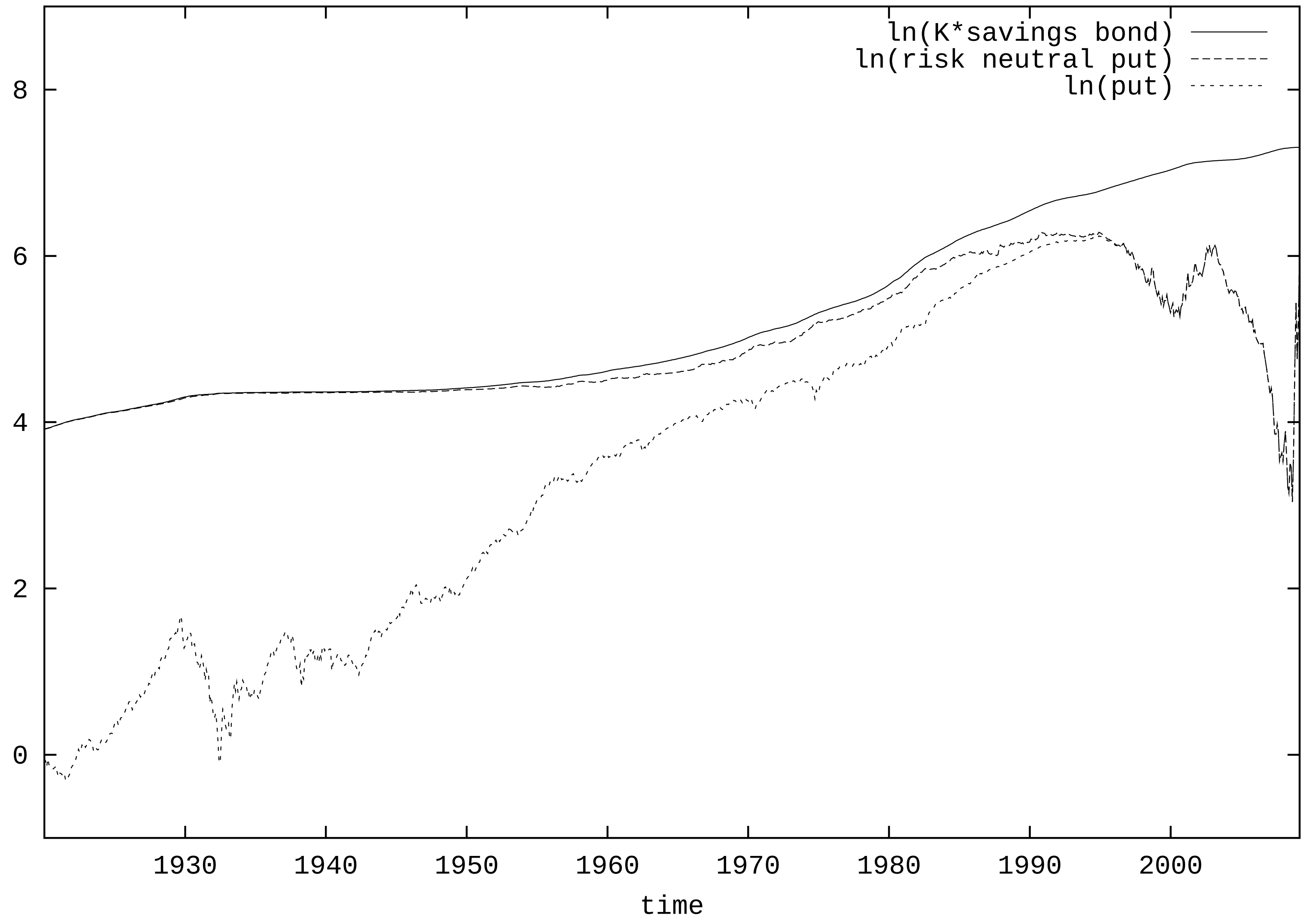

Figure 5.13: Logarithms of savings bond times $\boldsymbol{K}$, risk neutral put and fair put.

MMM with Random Market Activity

- **squared Bessel process with dimension** $\delta = 4$

$$\bar{S}^{\delta_*} = \{\bar{S}^{\delta_*}_t, \, t \in [0, \infty)\}$$

$$d\bar{S}^{\delta_*}_t = \alpha_t \, dt + \sqrt{\alpha_t \, \bar{S}^{\delta_*}_t} \, dW_t$$

- **NP**

$$S_t^{\delta_*} = S_t^0 \, \bar{S}_t^{\delta_*}$$

- **volatility**

$$|\theta_t| = \sqrt{\frac{\alpha_t}{\bar{S}_t^{\delta_*}}}$$

- **discounted NP drift**

α_t to be modeled

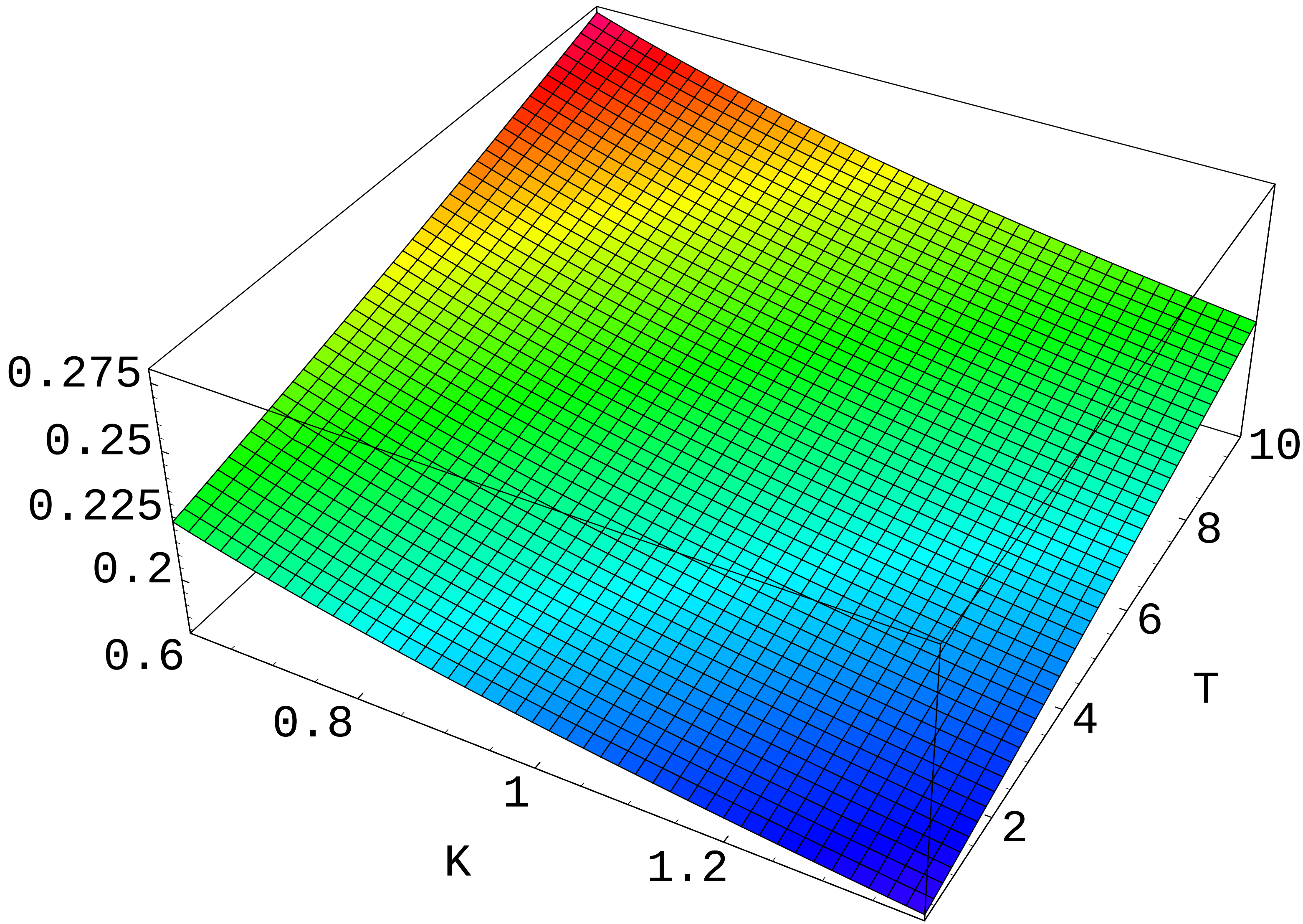

Figure 5.14: Implied volatility surface for the stylized MMM.

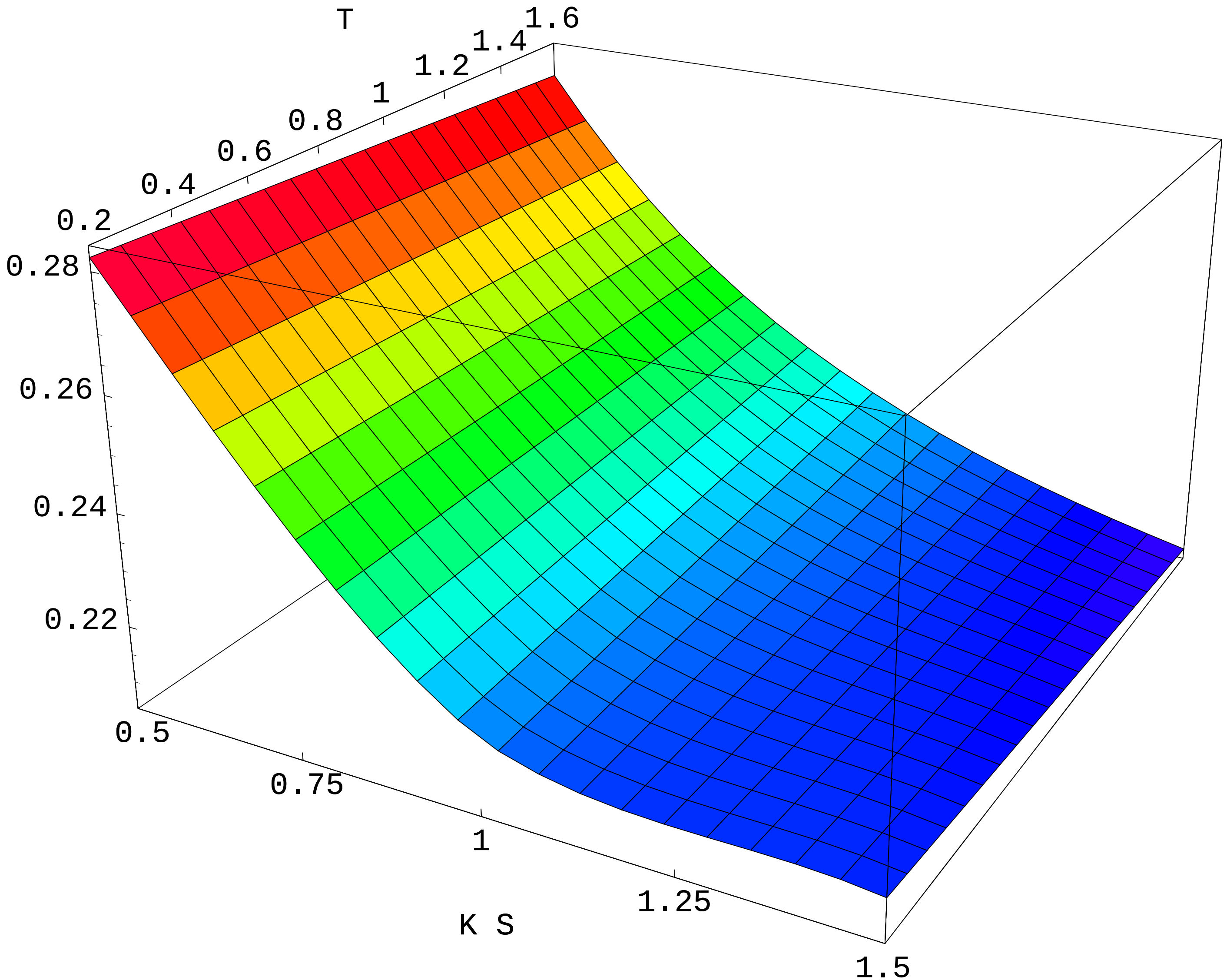

Figure 5.15: Average S&P500 implied volatility surface.

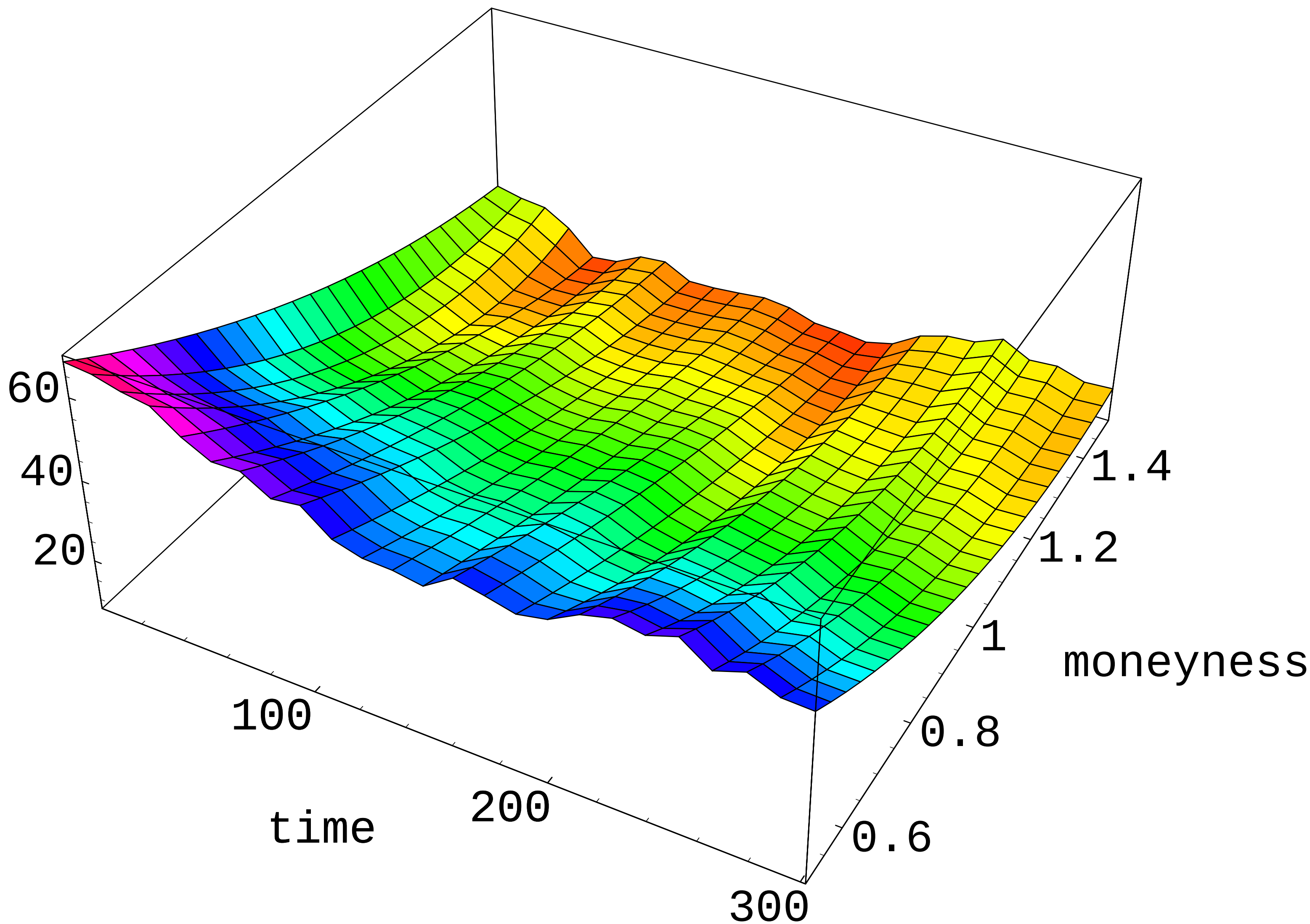

Figure 5.16: Implied volatilities for S&P500 three month options.

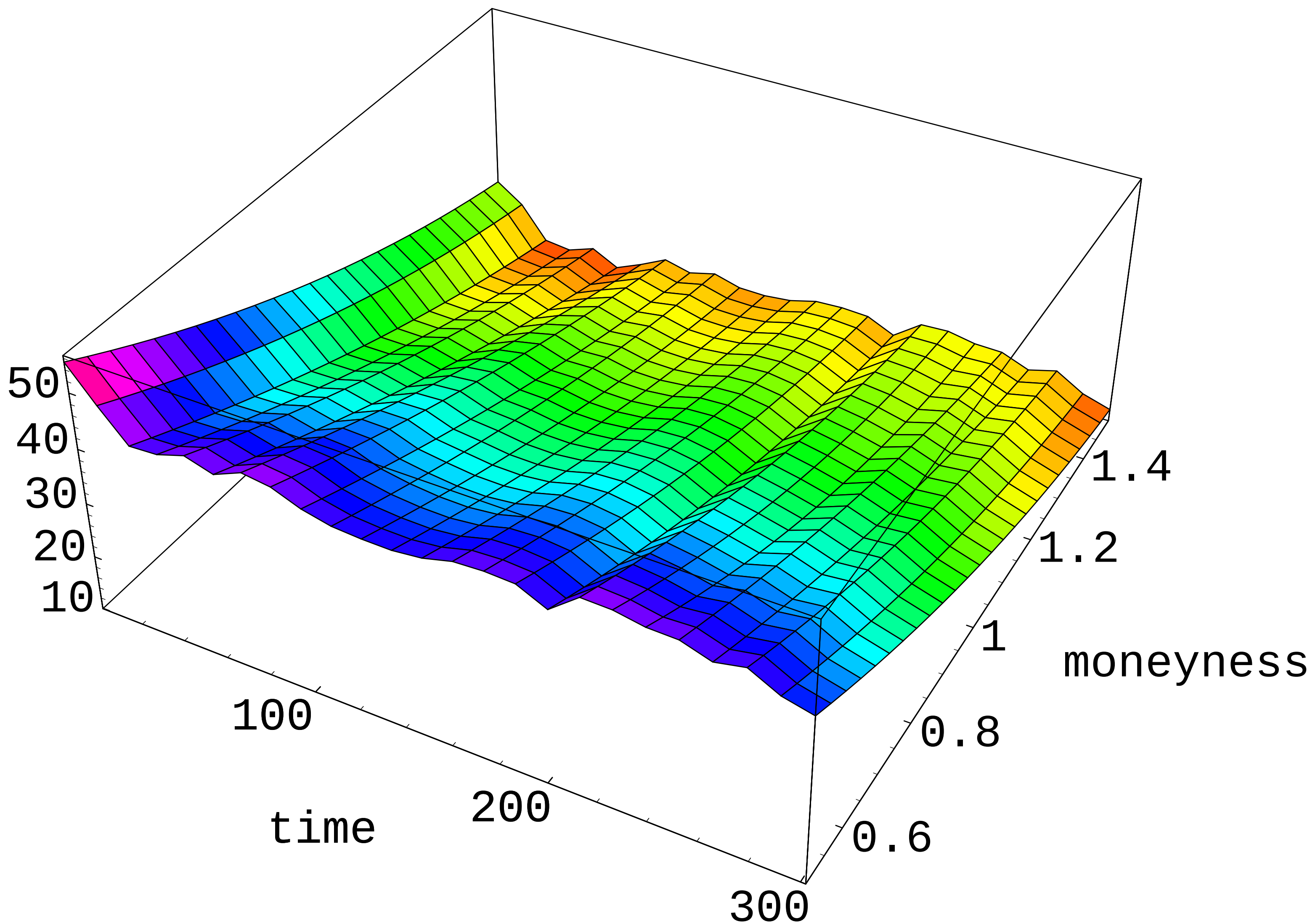

Figure 5.17: Implied volatilities for S&P500 one year options.

- **market activity** m_t

Breymann, Kelly & Pl. (2006)
Pl. & Rendek (2008)

$$\alpha_t = \xi_t \, m_t$$

with

$$\xi_t = \xi_0 \, \exp\{\eta \, t\}$$

$$dm_t = k(m_t) \, \beta^2 \, dt + \beta \, m_t \left(\varrho \, dW_t + \sqrt{1 - \varrho^2} \, d\tilde{W}_t \right)$$

ϱ - correlation parameter

activity volatility $\beta > 0$

$\tilde{W}$ - independent Wiener process

Example:

- drift

$$k(m_t) = (p - g\, m_t)\, \frac{m_t}{2}$$

- stationary density

$$p_m(y) = \frac{g^{p-1}}{\Gamma(p-1)}\, y^{p-2} \exp\{-g\, y\}$$

gamma density with mean $\frac{p-1}{g}$ and variance $\frac{1}{g}$ for $p > 1$ and $g > 0$

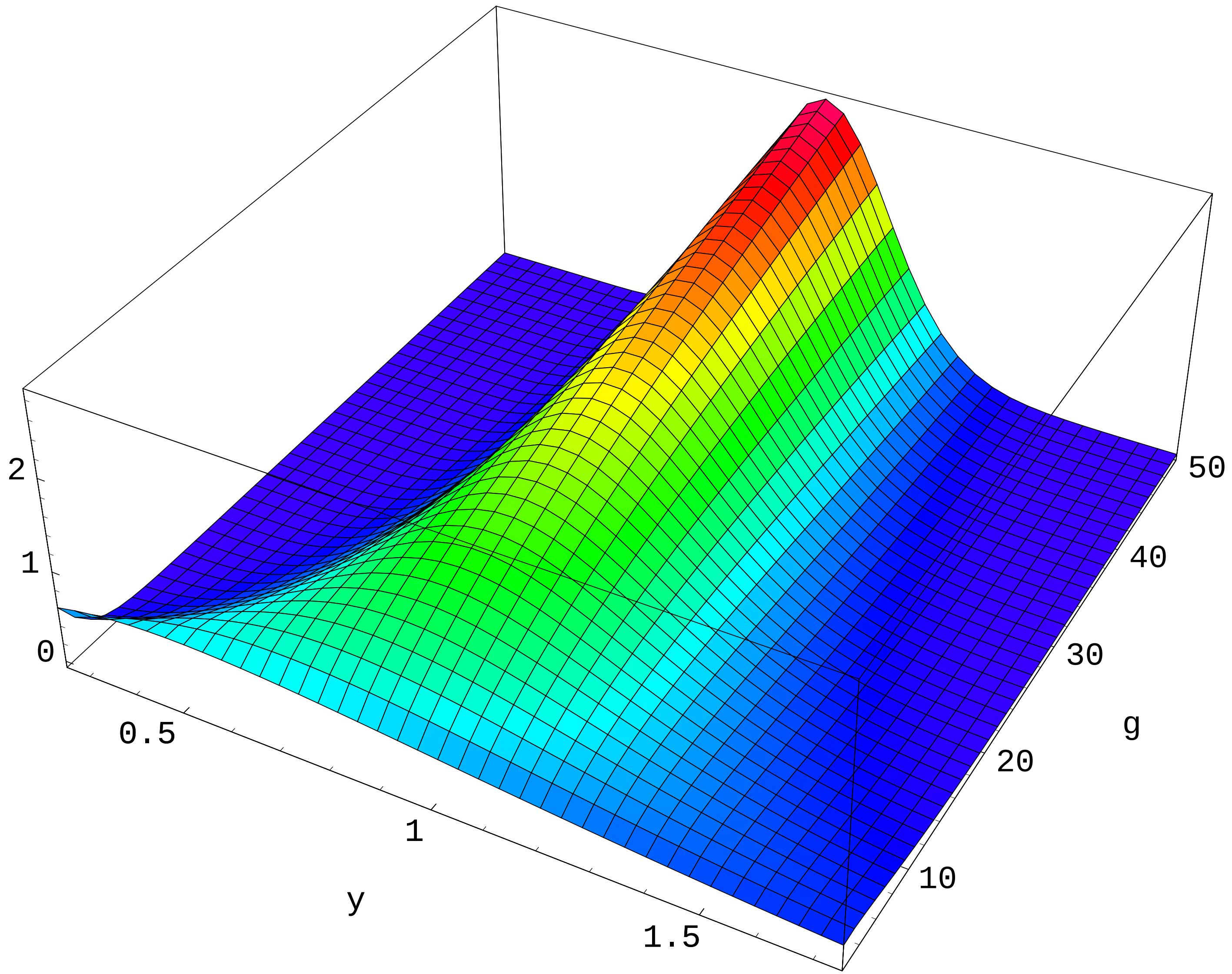

Figure 5.18: Stationary density of market activity $m_t = y$ as function of y and speed of adjustment parameter g.

Zero Coupon Bond

- benchmarked zero coupon bond

$$\hat{P}_T(t, \bar{S}_t^{\delta_*}, m_t) = E_t\left(\frac{1}{S_T^{\delta_*}}\right) = E_t\left(\frac{1}{S_T^0 \, \bar{S}_T^{\delta_*}}\right)$$

- zero coupon bond

$$P_T(t, \bar{S}_t^{\delta_*}, m_t) = S_t^{\delta_*} \, \hat{P}_T(t, \bar{S}_t^{\delta_*}, m_t) = S_t^0 \, \bar{S}_t^{\delta_*} \, \hat{P}_T(t, \bar{S}_t^{\delta_*}, m_t)$$

European Options on a Market Index

- **put option price**

$$p_{T,K}(t, \bar{S}_t^{\delta_*}, m_t) = S_t^0 \, \bar{S}_t^{\delta_*} \, E_t \left(\left(\frac{K}{S_T^0 \, \bar{S}_T^{\delta_*}} - 1 \right)^+ \right)$$

effect of random market time on implied volatilities

curvature for the short dated implied volatility surface

Heath & Pl. (2005a)

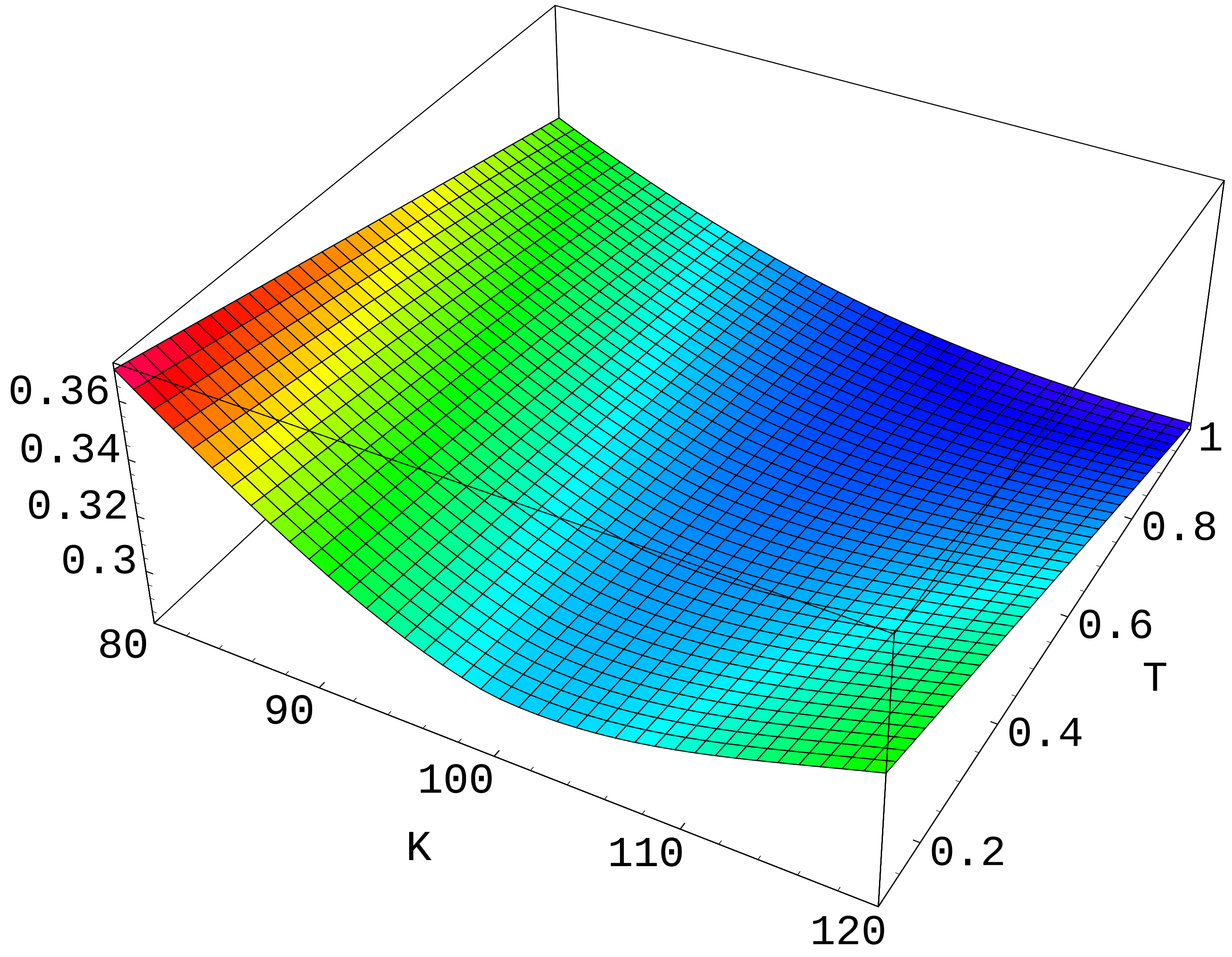

Figure 5.19: Implied volatilities for put options on index as a function of strike K and maturity T.

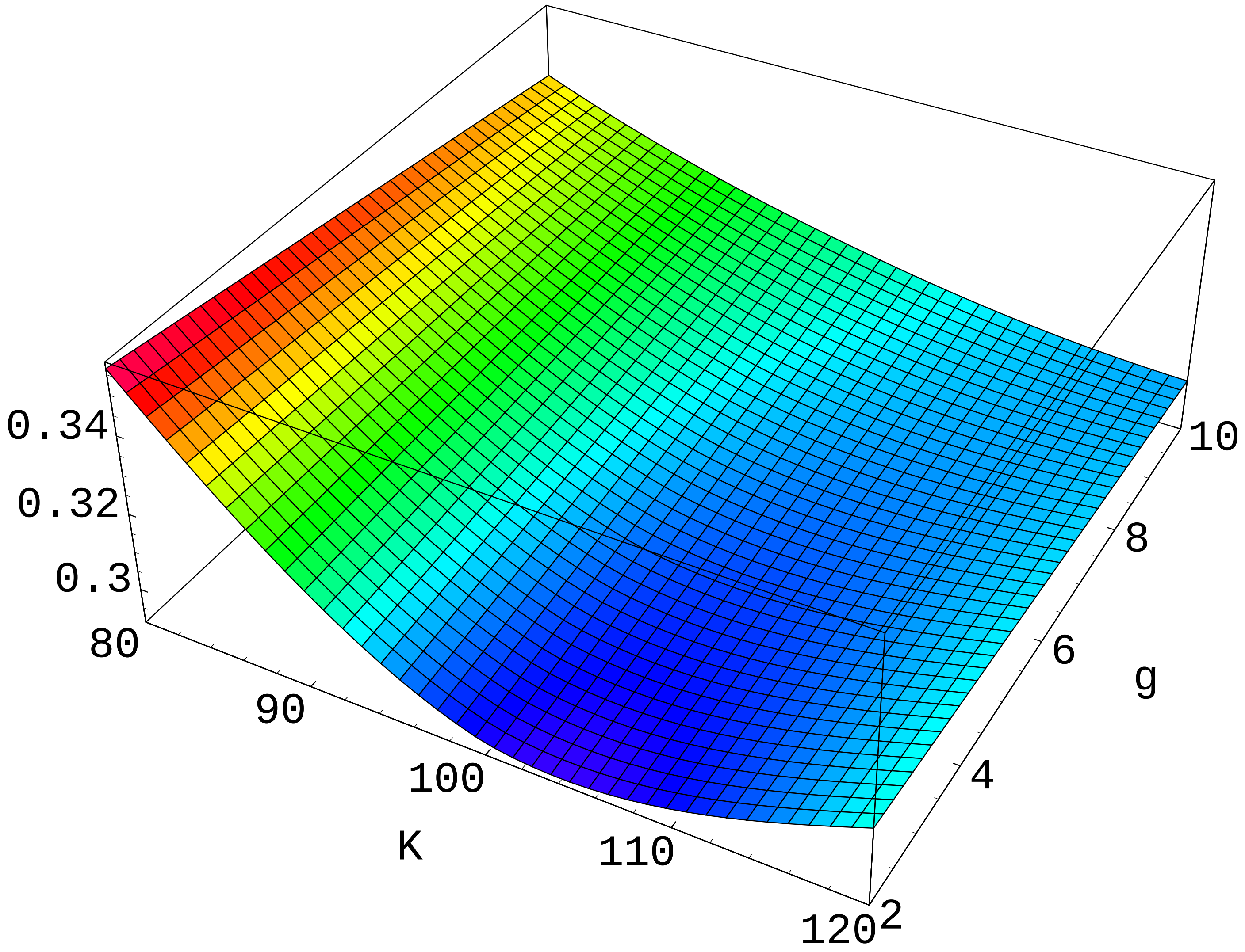

Figure 5.20: Implied volatilities for put options as a function of strike K and speed of adjustment g.

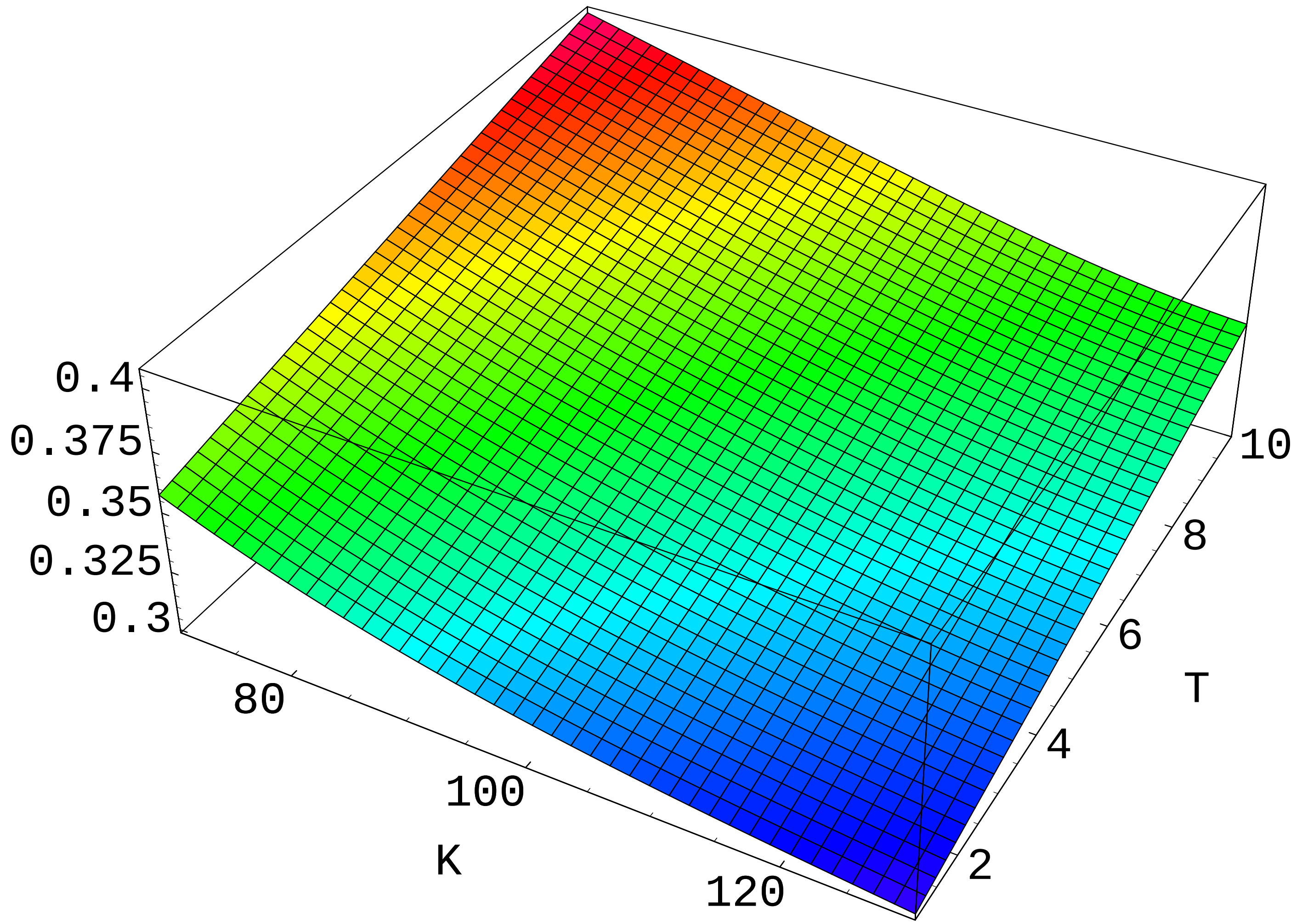

Figure 5.21: Implied volatilities for long dated put options as a function of strike $\boldsymbol{K}$ and maturity $\boldsymbol{T}$.

6 Stylized Multi-Currency MMM

Pl. (2001)

Heath & Pl. (2005a)

$d + 1$ currencies

$S_i^{\delta_*}(t)$ NP in ith currency

r_t^i - ith short rate

$\theta_i^k(t)$ - market price of risk

$$S_i^{\delta_*}(t) = \alpha_t^i \, Y_t^i \, S_i^i(t)$$

$$\alpha_t^i = \alpha_0^i \exp\{\eta^i\, t\}$$

$$S_i^i(t) = \exp\{r^i\, t\}$$

- ith normalized NP

$$dY_t^i = \left(1 - \eta^i\, Y_t^i\right) dt + \sqrt{Y_t^i} \sum_{k=1}^{d+1} q^{i,k}\, dW_t^k$$

scaling levels $q^{i,k}$

$$\sum_{k=1}^{d+1} (q^{i,k})^2 = 1$$

- **(i, j)th exchange rate**

$$X_t^{i,j} = \frac{S_i^{\delta_*}(t)}{S_j^{\delta_*}(t)} = \frac{Y_t^i \, \alpha_t^i \, S_i^i(t)}{Y_t^j \, \alpha_t^j \, S_j^j(t)}$$

$$dX_t^{i,j} = X_t^{i,j} \left((r^i - r^j) \, dt + \sum_{k=1}^{d+1} \left(\frac{q^{i,k}}{\sqrt{Y_t^i}} - \frac{q^{j,k}}{\sqrt{Y_t^j}} \right) \left(\frac{q^{i,k}}{\sqrt{Y_t^i}} \, dt + dW_t^k \right) \right)$$

- jth savings account in ith currency

$$S_i^j(t) = X_t^{i,j}\, S_j^j(t)$$

$$dS_i^j(t) = S_i^j(t)\left(r^i\,dt + \sum_{k=1}^{d+1}\left(\frac{q^{i,k}}{\sqrt{Y_t^i}} - \frac{q^{j,k}}{\sqrt{Y_t^j}}\right)\left(\frac{q^{i,k}}{\sqrt{Y_t^i}}\,dt + dW_t^k\right)\right)$$

- **stochastic market price of risk**

$$\theta_i^k(t) = \frac{q^{i,k}}{\sqrt{Y_t^i}}$$

- (j,k)th volatility

$$b_i^{j,k}(t) = \theta_i^k(t) - \theta_j^k(t)$$

equity prices

commodity prices

7 Valuing Guaranteed Minimum Death Benefits

- **variable annuities**
  fund-linked
  tax-deferred
  guarantees

- **guaranteed minimum death benefits** (GMDBs)

  roll-ups:
  original investment accrued at a pre-defined interest rate

  ratchets:
  death benefit based upon the highest anniversary account value

  embedded options:
  hedging against market downturn occurrence of death

- GMDB put, floating put and/or look-back put option

  long maturities of contracts

  standard option pricing theory ?

  no obvious best choice for price:

  IFRS Phase II
  CFO-Forum (2008)
  CRO-Forum (2008)
  Solvency II
  Verheugen & Hines (2008)

- **pricing and hedging of GMDBs**

  benchmark approach in

  Pl. (2002, 2004b), Pl. & Heath (2006)

  Bühlmann & Pl. (2003)

  best performing portfolio is benchmark

  does **not** require
  existence of an equivalent risk neutral probability measure

  unique **fair** price is the **minimal price**

  may provide significantly lower prices

Financial Model

- **underlying risky security** (unit)

$$dS_t = (\mu_t - \gamma) S_t dt + \sigma_t S_t \, dW_t$$

$\gamma \geq 0$ management fee rates

W_t - Brownian motion

- **savings account**

$$dB_t = r_t B_t \, dt$$

- **market price of risk**

$$\theta_t = \frac{\mu_t - \gamma - r_t}{\sigma_t}$$

- **risky asset**

$$\frac{dS_t}{S_t} = r_t\,dt + \sigma_t\theta_t\,dt + \sigma_t\,dW_t$$

- **self-financing portfolio**

$$V_t = \delta_t^0 B_t + \delta_t^1 S_t$$

$$dV_t = \delta_t^0 \, dB_t + \delta_t^1 dS_t$$

- **fractions**

$$\pi_t^0 = \delta_t^0 \frac{B_t}{V_t}, \qquad \pi_t^1 = \delta_t^1 \frac{S_t}{V_t}$$

$$\pi_t^0 + \pi_t^1 = 1$$

- **instantaneous portfolio return**

$$\frac{dV_t}{V_t} = \pi_t^0 \frac{dB_t}{B_t} + \pi_t^1 \frac{dS_t}{S_t}$$

$$= r_t\,dt + \pi_t^1 \sigma_t (\theta_t\,dt + dW_t)$$

- **numeraire portfolio** (NP) as benchmark

  Long (1990)
  V^* is best performing in several ways

- is growth optimal portfolio

  Kelly (1956)

$$V^* = \max E(\log V_T)$$

$$\pi_t^{1*} = \frac{\mu_t - \gamma - r_t}{\sigma_t^2} = \frac{\theta_t}{\sigma_t}$$

$$\frac{dV_t^*}{V_t^*} = r_t\,dt + \theta_t(\theta_t\,dt + dW_t)$$

Merton (1992)

- **NP best performing portfolio**

$$\limsup_{T\to\infty} \frac{1}{T} \log\left(\frac{V_T^*}{V_0^*}\right) \geq \limsup_{T\to\infty} \frac{1}{T} \log\left(\frac{V_T}{V_0}\right)$$

global well diversified index $\approx$ NP

Pl. (2005b)

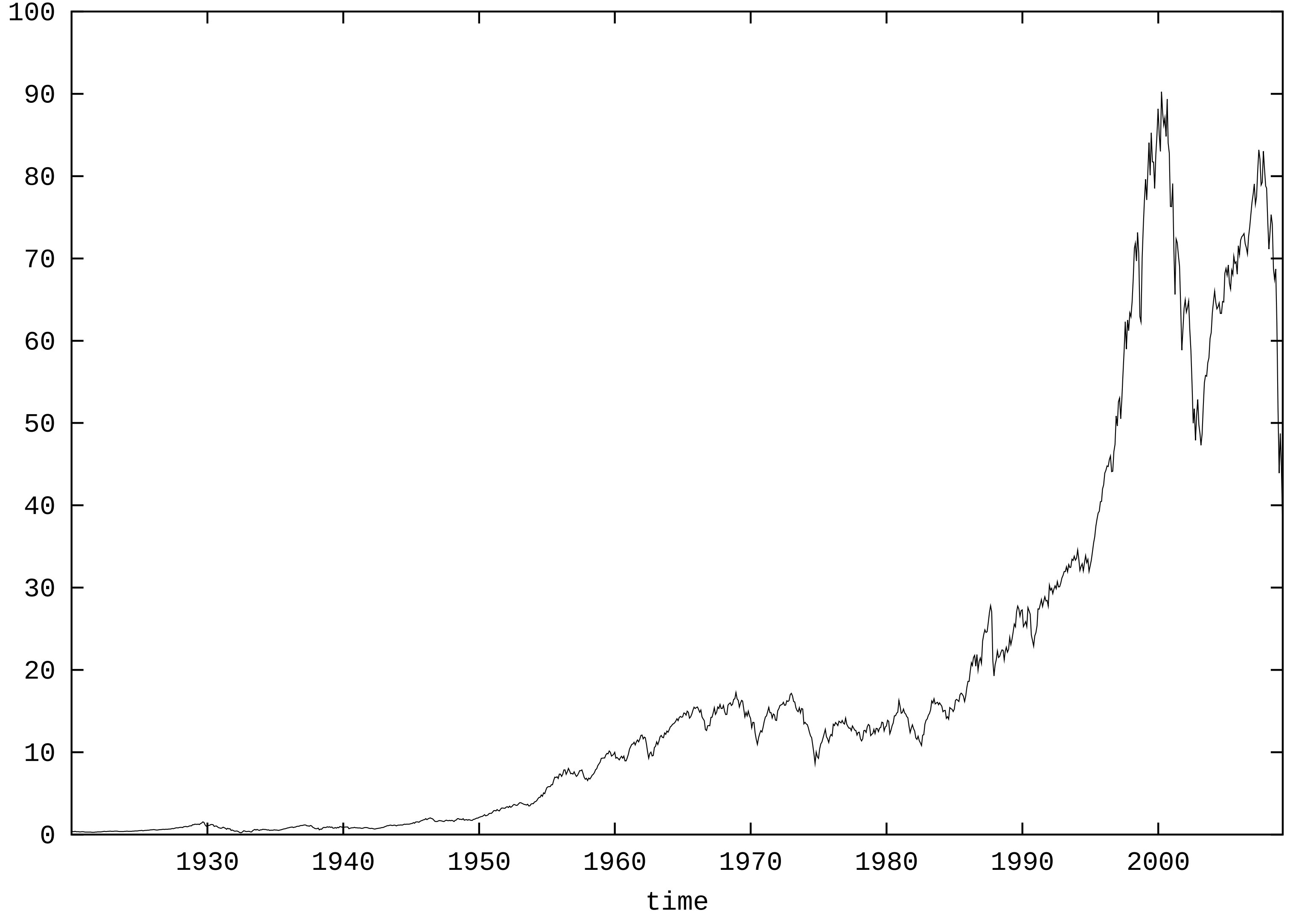

Figure 7.1: Discounted S&P500 total return index.

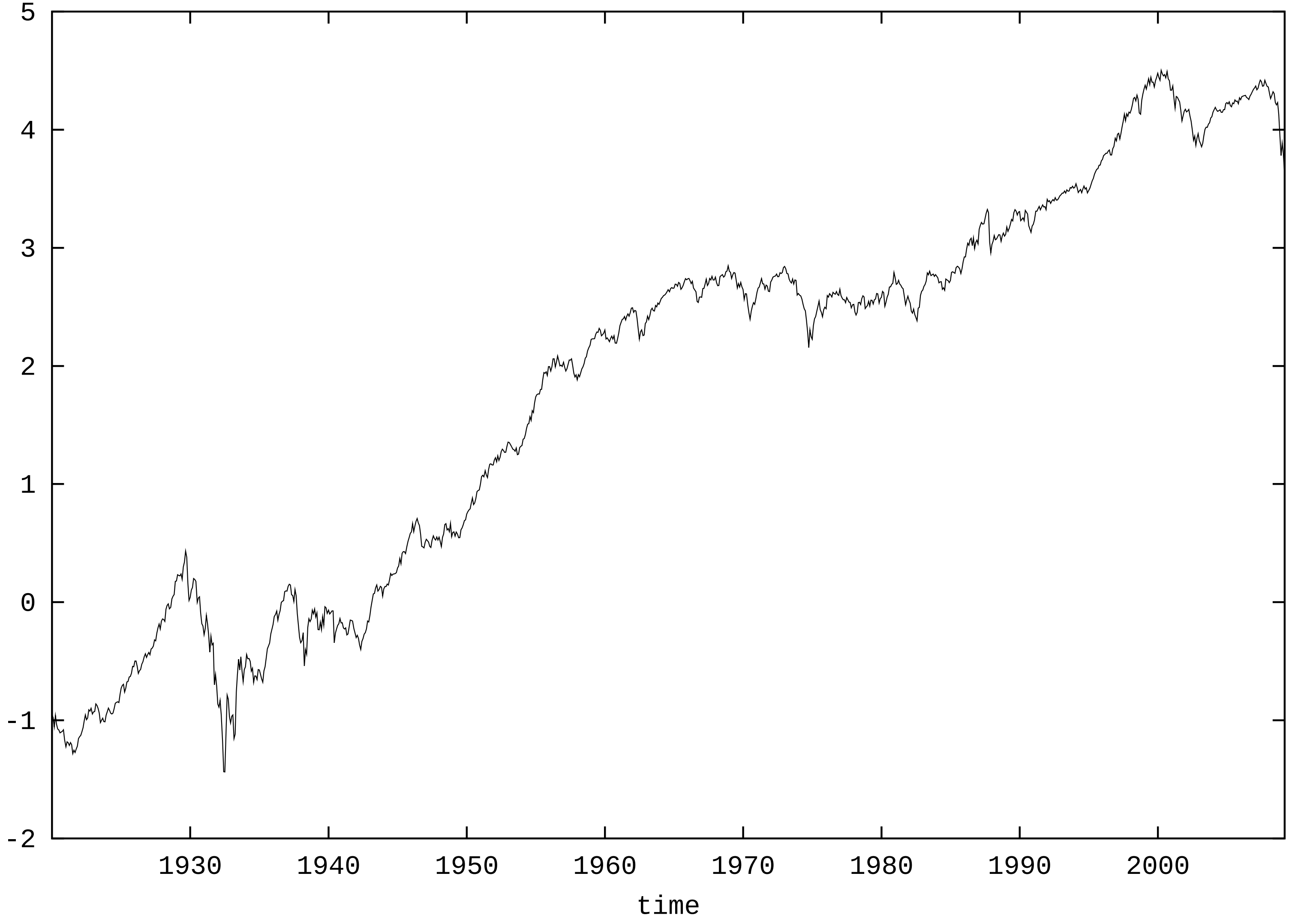

Figure 7.2: Logarithm of discounted S&P500.

- **benchmarked price**

  $\hat{U}_t = \frac{U_t}{V_t^*}$

  $\hat{U}$ nonnegative $\Longrightarrow$

$$\hat{U}_t \geq E_t\left(\hat{U}_s\right)$$

  $t \leq s$

  supermartingales (no upward trend)

  Pl. (2002)

  no strong arbitrage

- benchmarked securities that form martingales are called fair.

$$\hat{U}_t = E_t\left(\hat{U}_s\right)$$

$t \leq s$

- **Law of the Minimal Price**

  Pl. (2008)

  Fair prices are minimal prices

  V nonnegative fair portfolio
  V' nonnegative portfolio

$$V_T = V'_T$$

supermartingale property
$\Longrightarrow$

$$V_t \leq V'_t$$

$t \in [0, T]$,

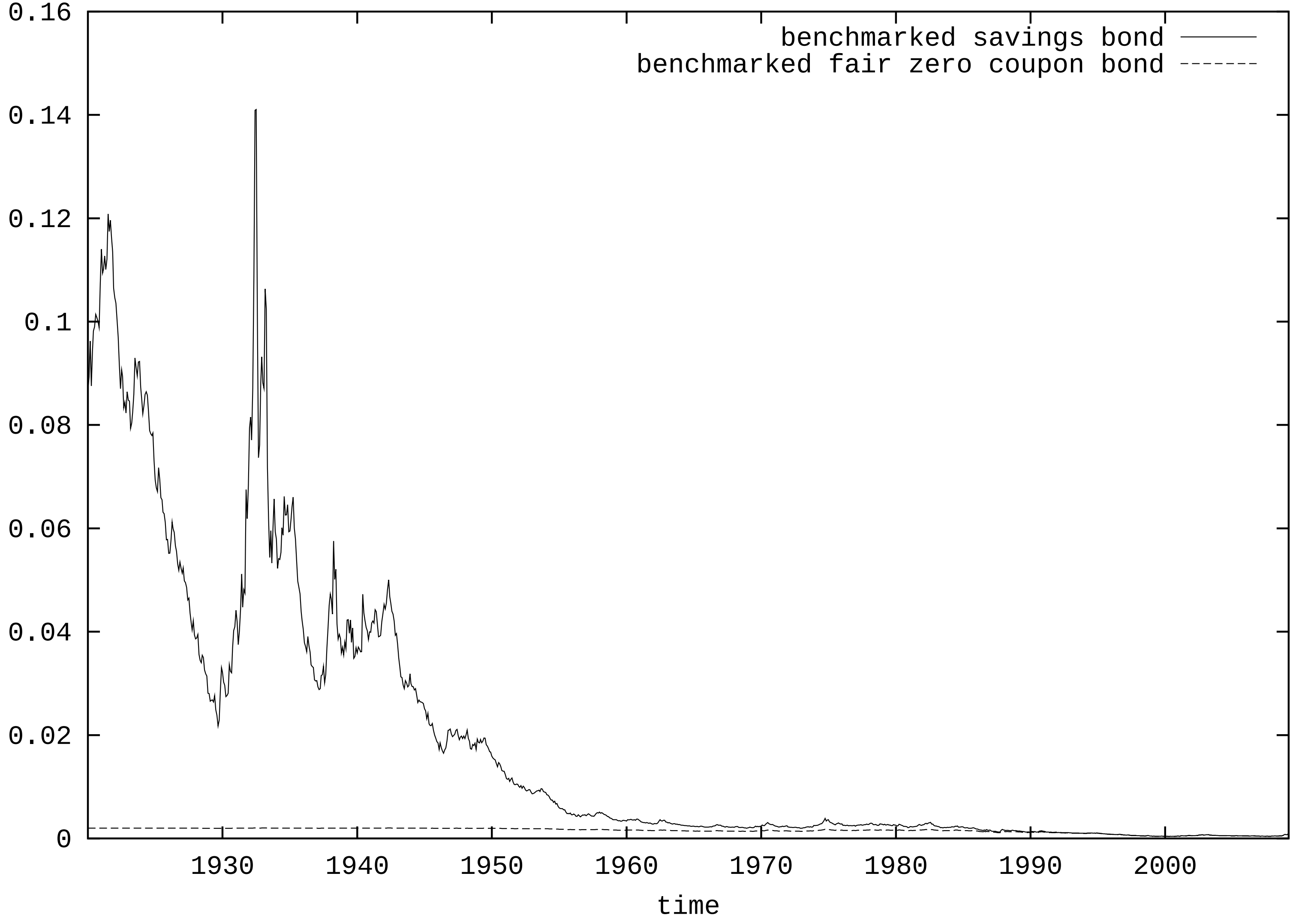

Figure 7.3: Benchmarked savings bond and benchmarked fair zero coupon bond.

- **real world pricing formula**

  for

$$E_t\left(\frac{H_T}{V_T^*}\right) < \infty$$

$$U_H(t) = V_t^* \, E_t\left(\frac{H_T}{V_T^*}\right)$$

$t \in [0, T]$

no risk neutral probability needs to exist

- **actuarial pricing formula**

  when H_T is independent of V_T^*

  $\Longrightarrow$

$$U_H(t) = P(t,T)\, E_t(H_T)$$

  zero coupon bond

$$P(t,T) = V_t^*\, E_t\left(\frac{1}{V_T^*}\right)$$

- **standard risk neutral pricing**

Ross (1976), Harrison & Pliska (1983)

complete market

candidate Radon-Nikodym derivative

$$\Lambda_t = \left.\frac{dQ}{dP}\right|_{\mathcal{A}_t} = \frac{B_t V_0^*}{B_0 V_t^*}$$

supermartingale $\Longrightarrow$

$$1 = \Lambda_0 \geq E_0\left(\Lambda_T\right)$$

$\Longrightarrow$

$$\boldsymbol{U_H(0) = E\left(\frac{V_0^*}{V_T^*}\, H_T\right)}$$

$$\boldsymbol{= E\left(\Lambda_T\, \frac{B_0}{B_T}\, H_T\right) \leq \frac{E\left(\Lambda_T\, \frac{B_0}{B_T}\, H_T\right)}{E(\Lambda_T)}}$$

Only in the special case when Λ_T martingale

$\Longrightarrow$ risk neutral pricing formula:

$$U_H(0) = E_Q\left(\frac{B_0}{B_T}\, H_T\right)$$

Ignores any real trend !

- **Trends ignored also when using:**

  stochastic discount factor, Cochrane (2001);

  deflator, Duffie (2001);

  pricing kernel, Constantinides (1992);

  state price density, Ingersoll (1987);

  NP as Long (1990)

- **Example zero coupon bond**

  benchmarked fair zero coupon

$$\hat{P}(t,T) = \frac{P(t,T)}{V_t^*} = E_t\left(\frac{1}{V_T^*}\right)$$

martingale (no trend)

for r_t - deterministic

$$P(t,T) = V_t^*\, E_t\left(\frac{1}{V_T^*}\right) = \exp\left\{-\int_t^T r_s\, ds\right\} E_t\left(\frac{\bar{V}_t^*}{\bar{V}_T^*}\right),$$

$\bar{V}_t^* = \frac{V_t^*}{B_t}$ - discounted NP

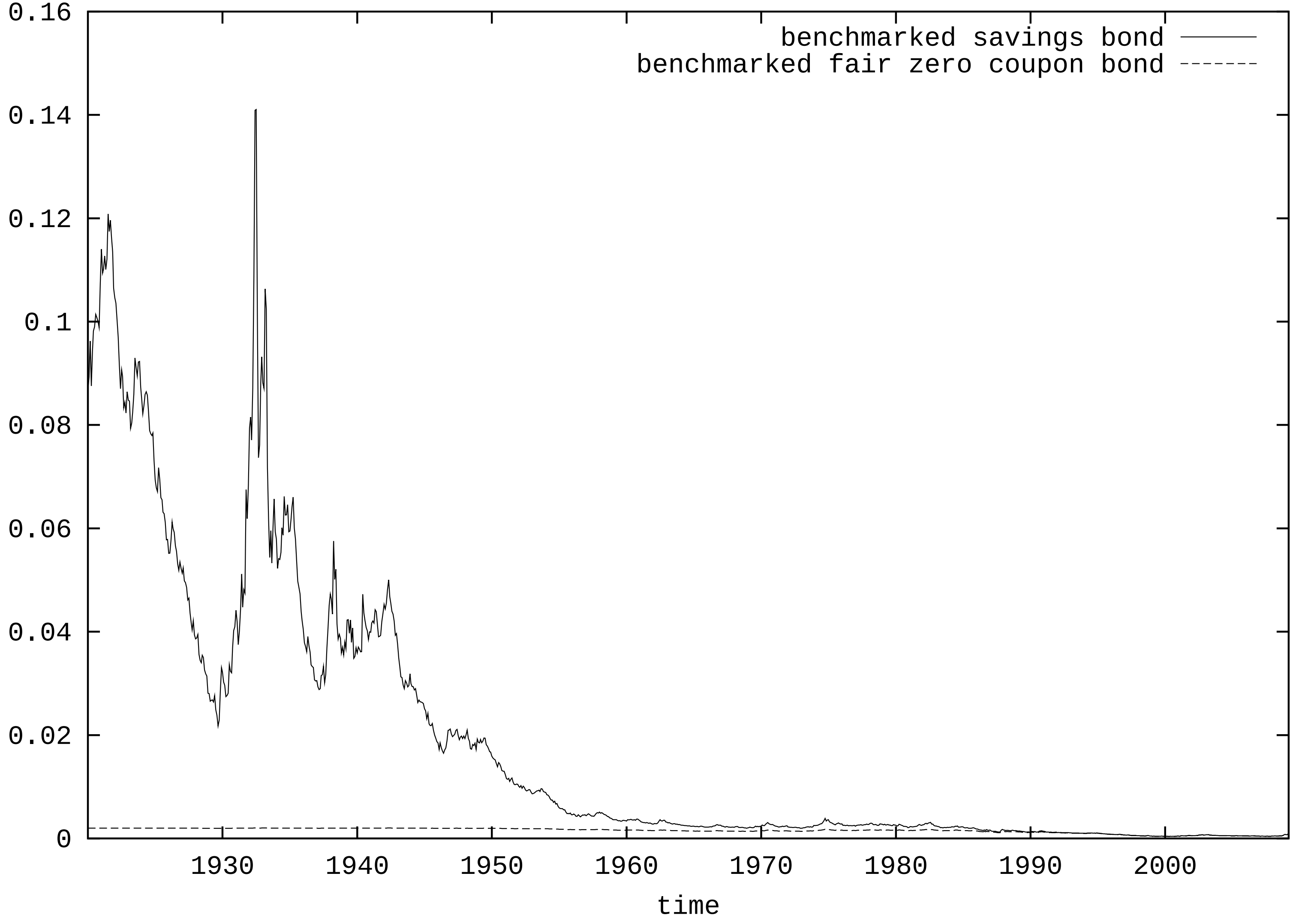

Figure 7.4: Benchmarked savings bond and benchmarked fair zero coupon bond.

$\Longrightarrow$ downward trend reflects equity premium

$\Longrightarrow$ $\boldsymbol{\Lambda}$ strict supermartingale (downward trend), then

$$P(t,T) < \exp\left\{-\int_t^T r_s\, ds\right\} = \frac{B_t}{B_T}$$

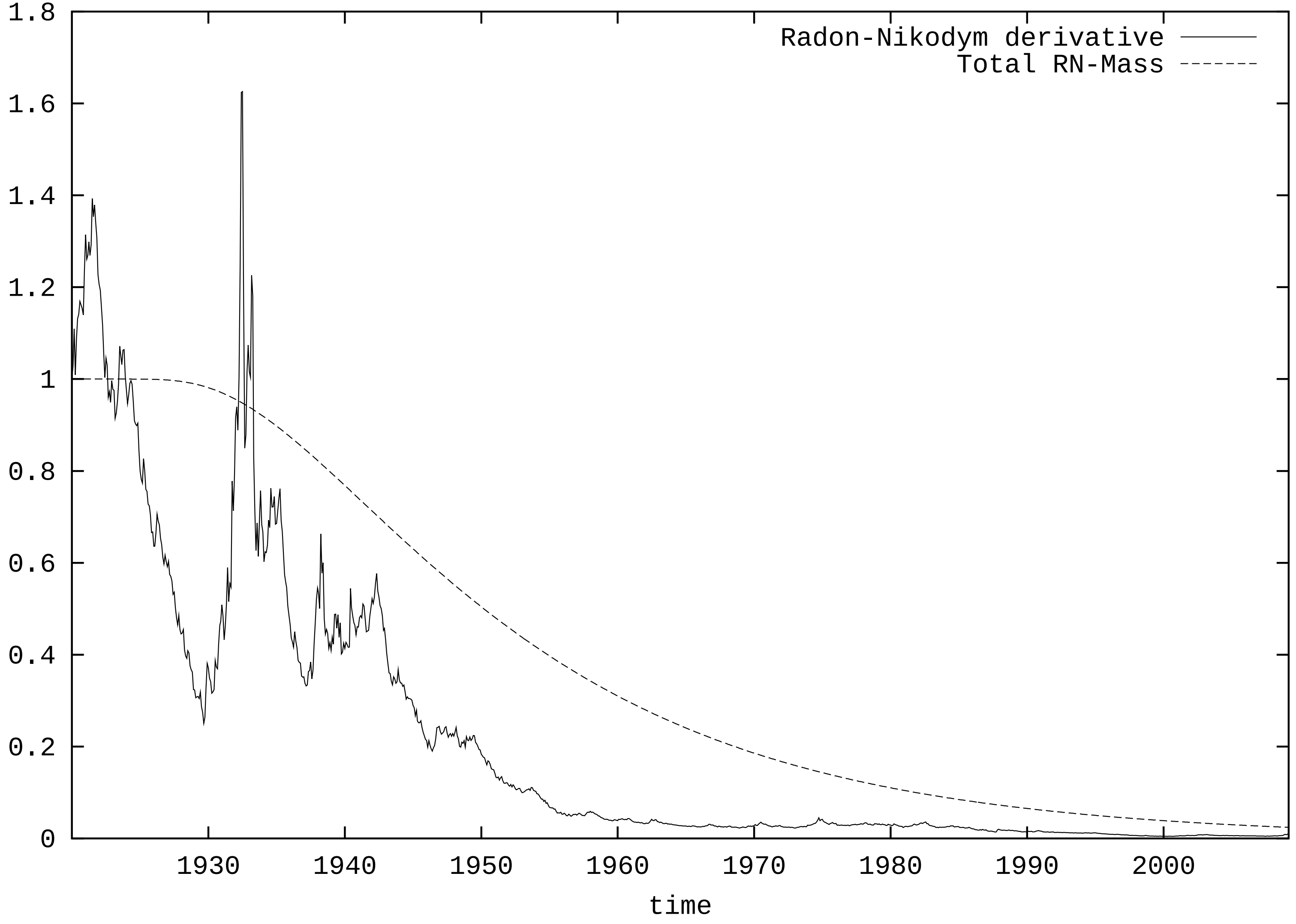

Figure 7.5: Radon-Nikodym derivative and total mass of putative risk neutral measure.

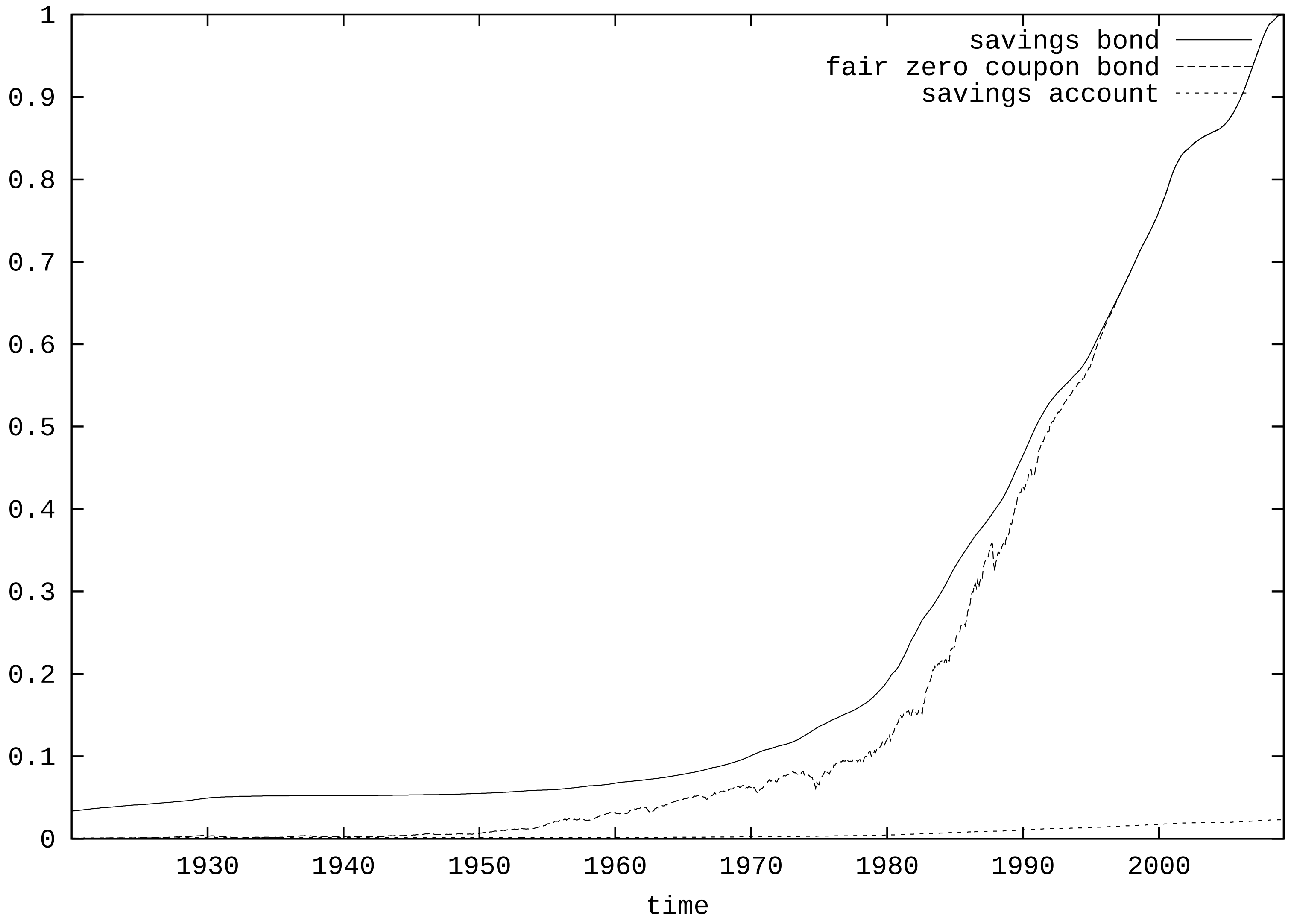

Figure 7.6: Savings bond, fair zero coupon bond and savings account.

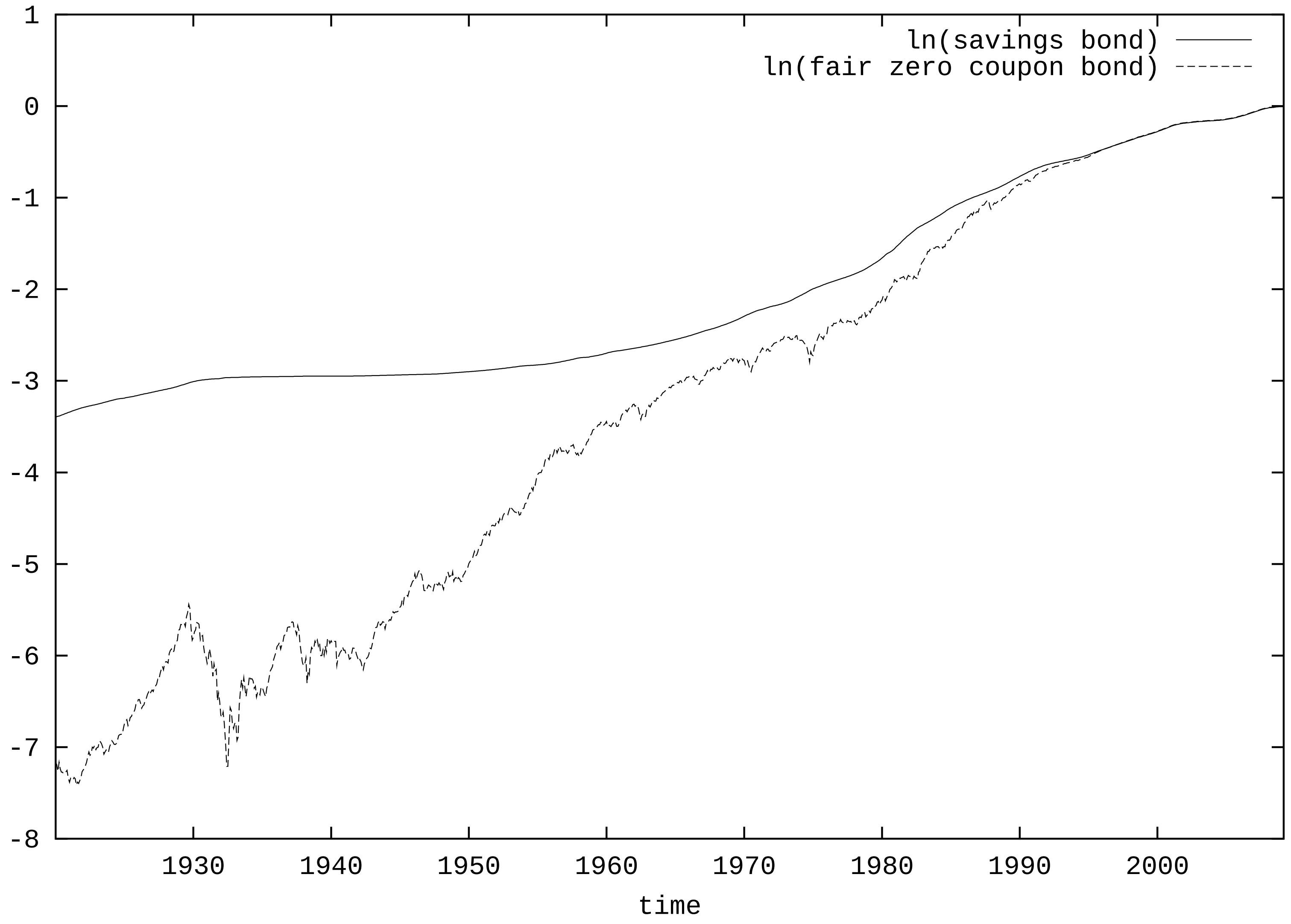

Figure 7.7: Logarithms of savings bond and fair zero coupon bond.

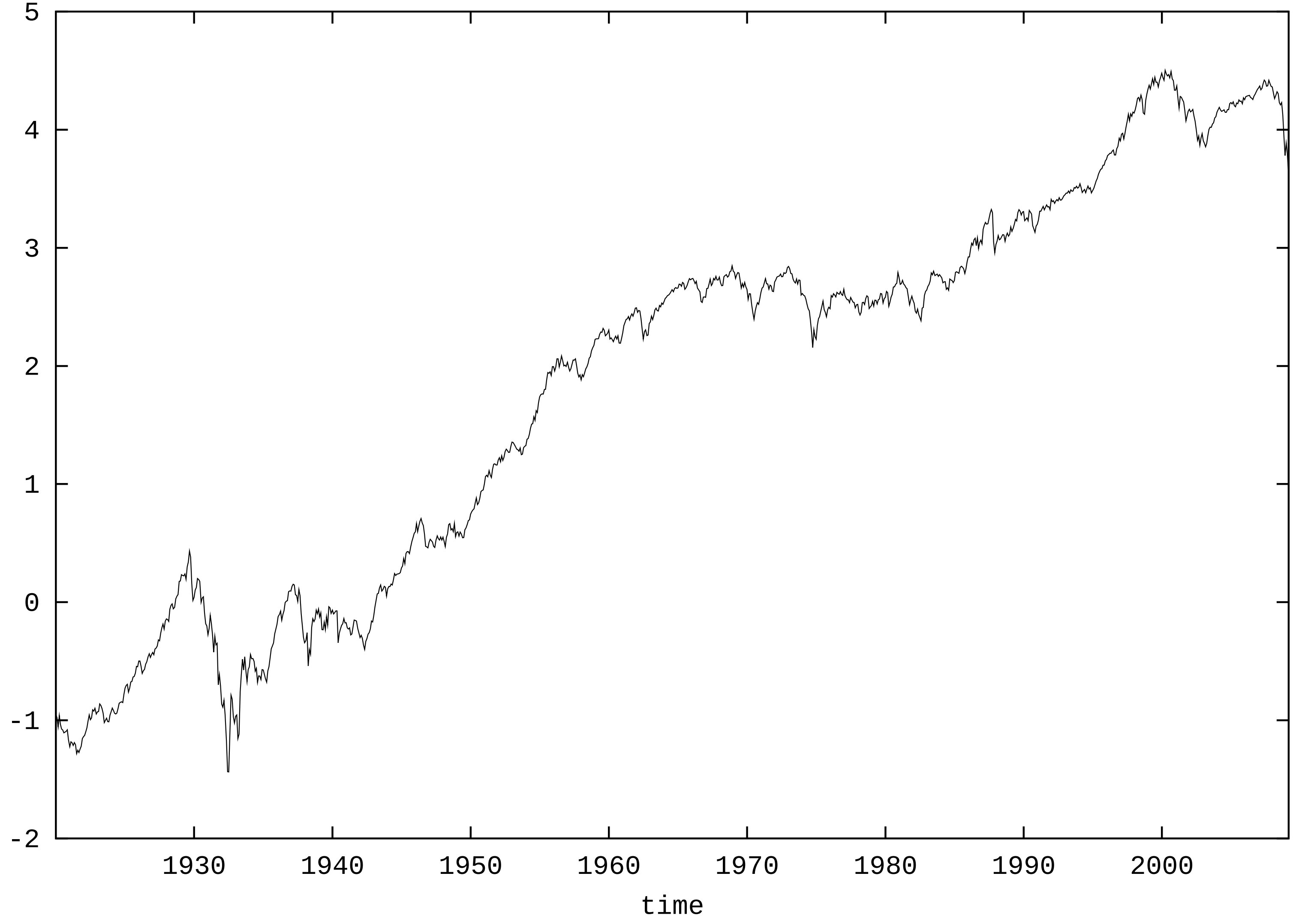

Figure 7.8: Logarithm of discounted S&P500.

- discounted NP

$$\bar{V}_t^* = \frac{V_t^*}{B_t}$$

$$\begin{aligned} d\bar{V}_t^* &= \bar{V}_t^* \, \theta_t \, (\theta_t \, dt + dW_t) \\ &= \alpha_t \, dt + \sqrt{\alpha_t \, \bar{V}_t^*} \, dW_t \end{aligned}$$

- discounted NP drift

$$\alpha_t := \bar{V}_t^* \theta_t^2$$

$\Longrightarrow$ volatility

$$\theta_t = \sqrt{\frac{\alpha_t}{\bar{V}_t^*}}$$

reflects leverage effect

- **minimal market model**

  Pl. (2001, 2002)

  MMM

  discounted NP drift

  assume

$$\alpha_t = \alpha_0 \exp\{\eta t\}$$

  $\alpha_0 > 0$

  net growth rate $\eta > 0$

  $\Longrightarrow$ MMM

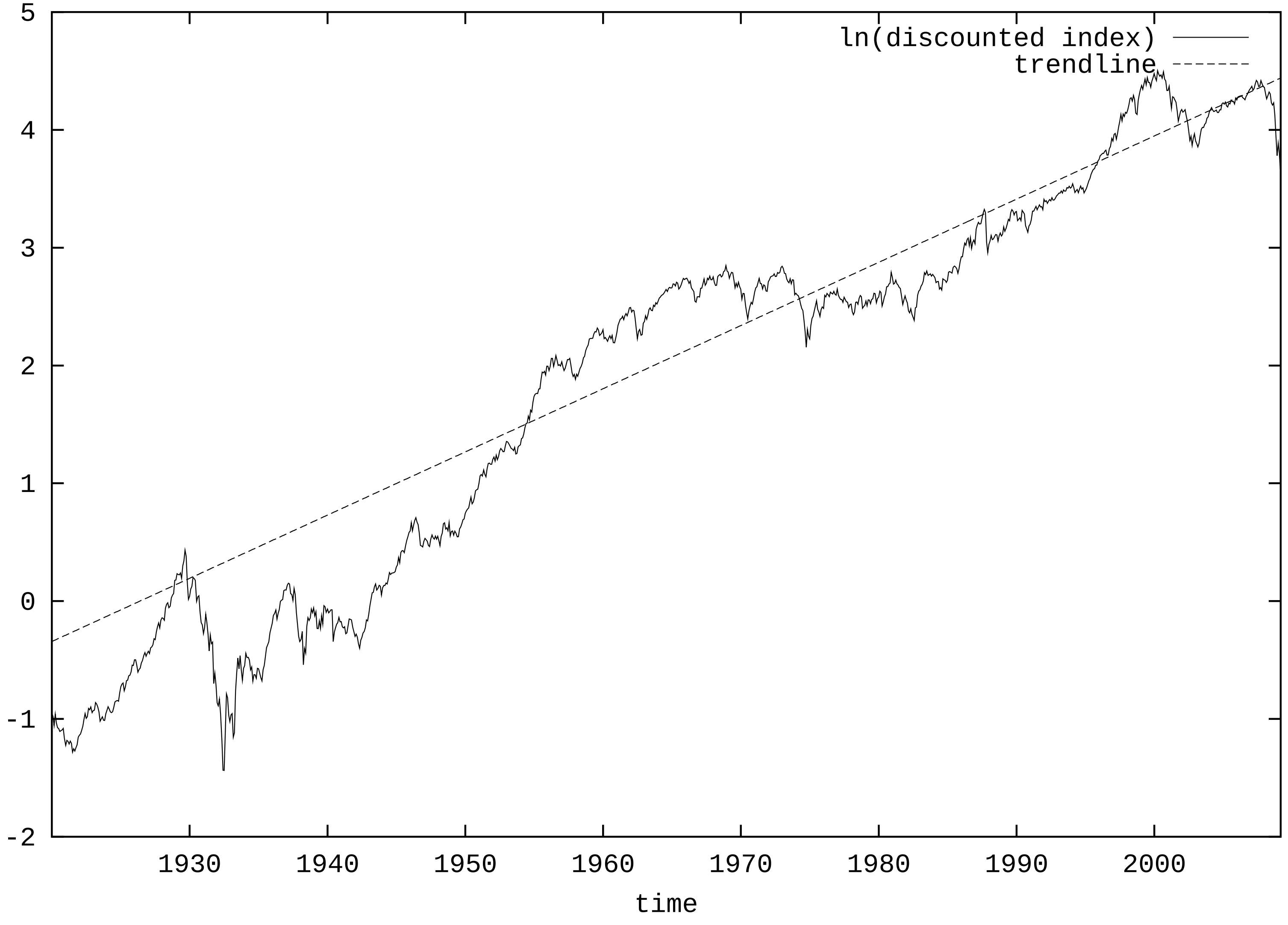

Figure 7.9: Logarithm of discounted S&P500 and trendline.

- **normalized NP**

$$\boldsymbol{Y_t = \frac{\bar{V}_t^*}{\alpha_t}}$$

$$\boldsymbol{dY_t = (1 - \eta Y_t)\, dt + \sqrt{Y_t}\, dW_t}$$

square root process of dimension four

Only one parameter !

- **volatility of NP**

$$\theta_t = \frac{1}{\sqrt{Y_t}} = \sqrt{\frac{\alpha_t}{\bar{V}_t^*}}$$

- **Zero coupon bond under the MMM**

r_t - deterministic

$$P(t,T) = \exp\left\{-\int_t^T r_s\,ds\right\}\left(1 - \exp\left\{-\frac{\bar{V}_t^*}{2(\varphi(T)-\varphi(t))}\right\}\right)$$

Explicit formula !

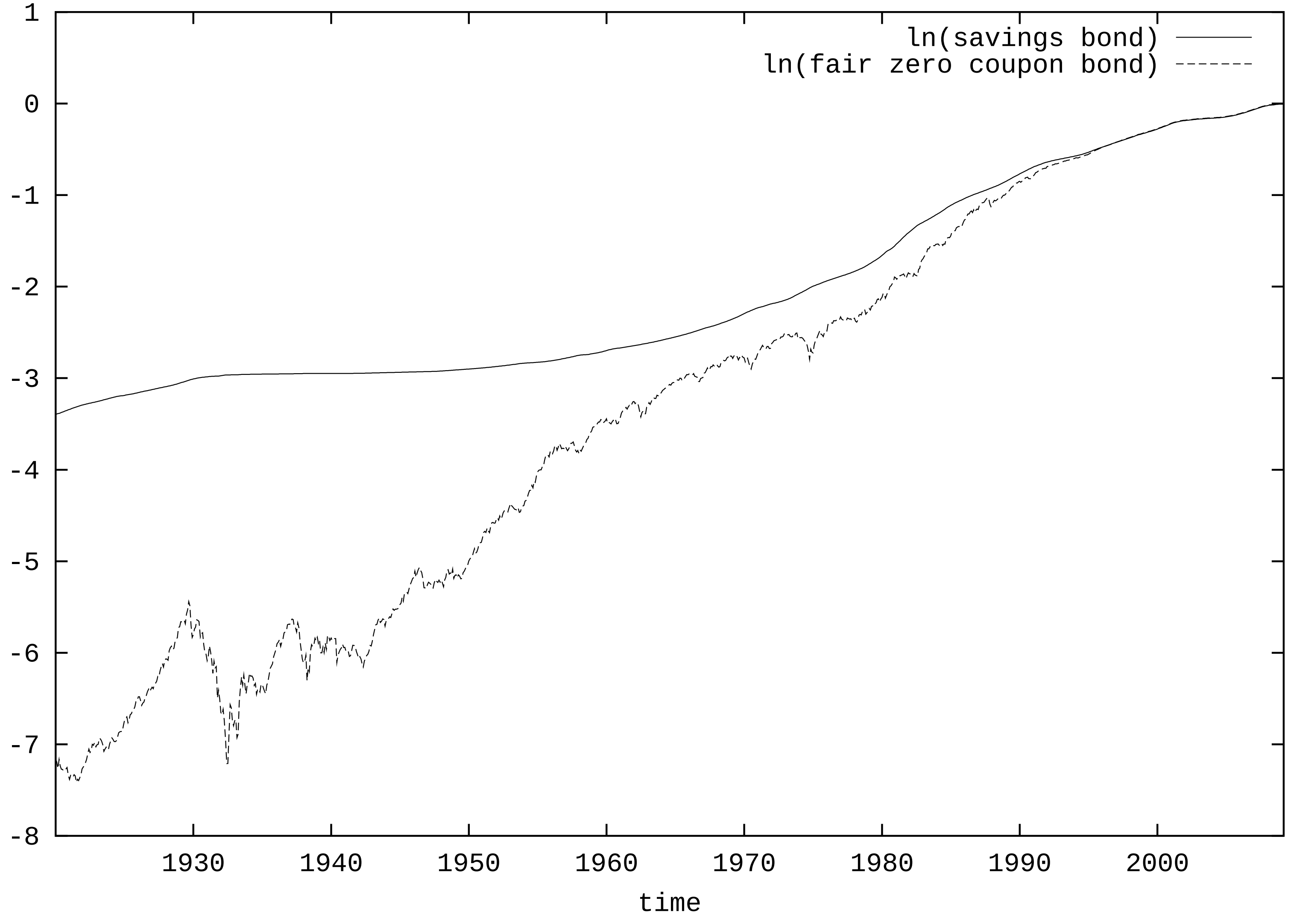

Figure 7.10: Logarithms of savings bond and fair zero coupon bond.

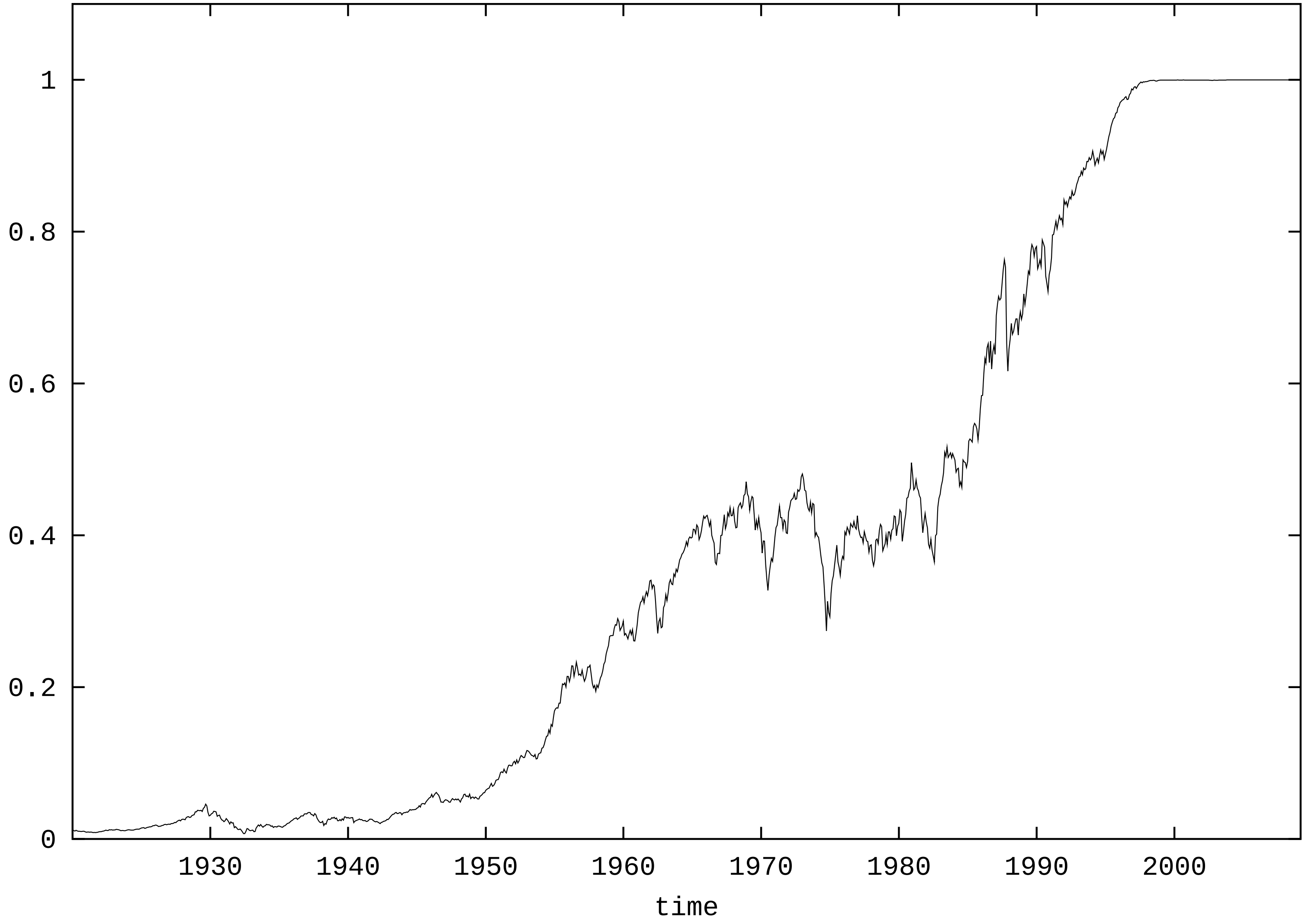

Figure 7.11: Fraction invested in the savings account.

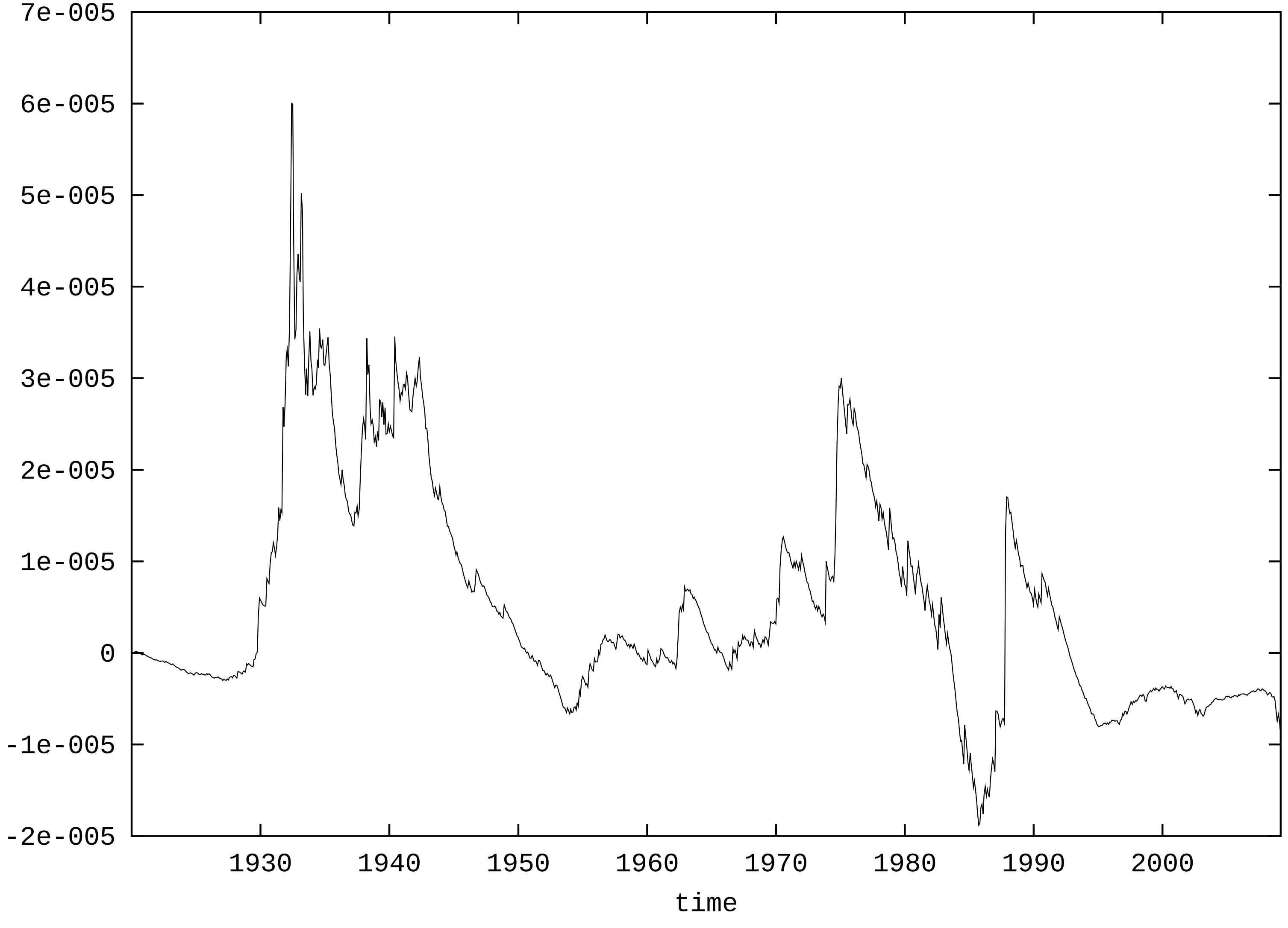

Figure 7.12: Benchmarked profit and loss.

since

$$E_t\left(\frac{\Lambda_T}{\Lambda_t}\right) = E_t\left(\frac{\bar{V}_t^*}{\bar{V}_T^*}\right) = \left(1 - \exp\left\{-\frac{\bar{V}_t^*}{2(\varphi(T) - \varphi(t))}\right\}\right)$$

$$\Longrightarrow \quad P(t,T) < \frac{B_t}{B_T}$$

Overpricing by savings bond

no equivalent risk neutral probability measure

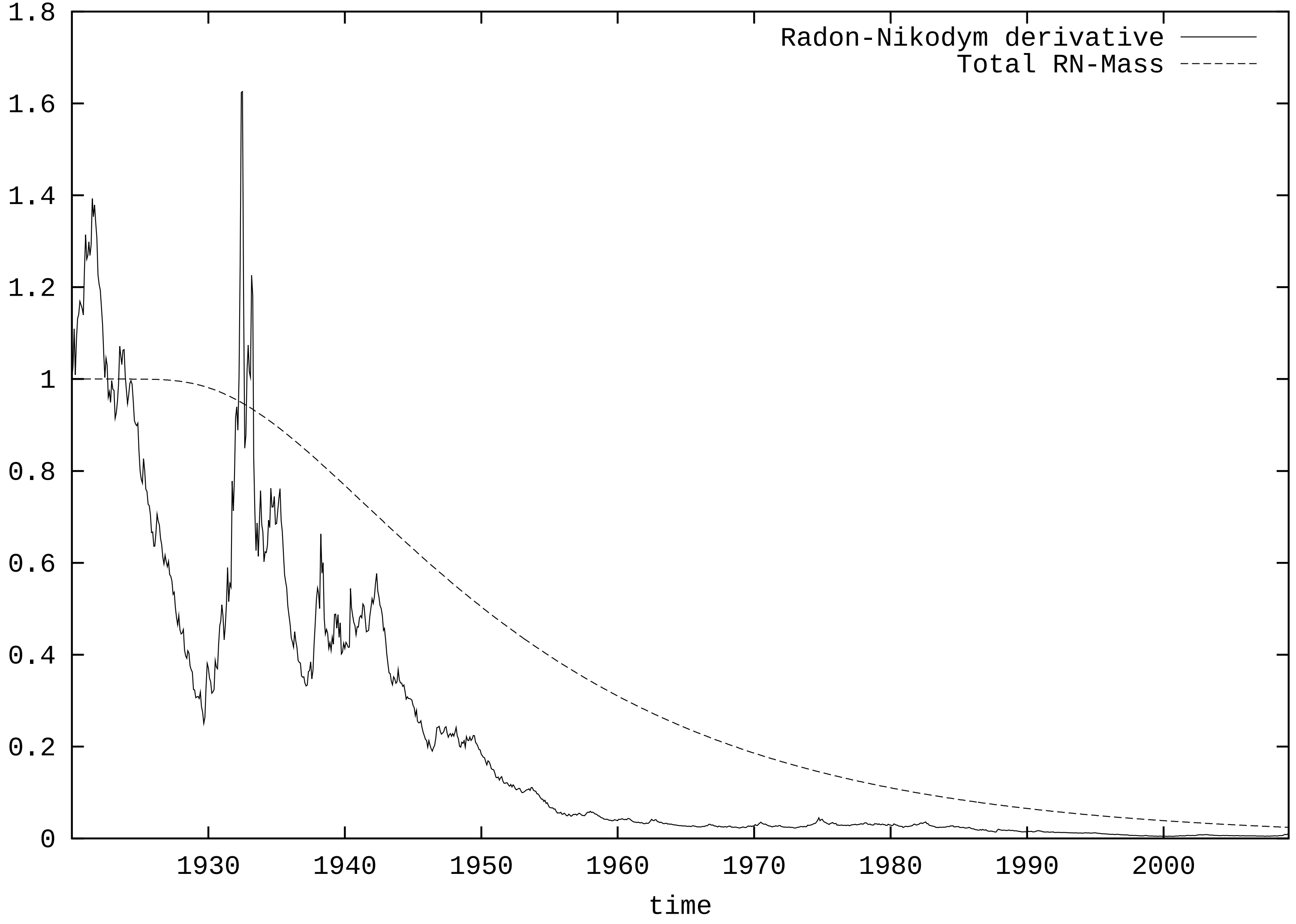

Figure 7.13: Radon-Nikodym derivative and total mass of putative risk neutral measure.

- **European put option under the MMM**

  Hulley, Miller & Pl. (2005)

$$p(t, V_t^*, T, K, r) = V_t^* E_t \left(\frac{(K - V_\tau^*)^+}{V_\tau^*} \right)$$

$$\begin{aligned} p(t, V_t^*, T, K, r) &= -V_t^* \chi^2(d_1; 4, l_2) \\ &\quad + K e^{-r(T-t)} \left(\chi^2(d_1; 0, l_2) - \exp\{-l_2/2\} \right) \end{aligned}$$

with

$$d_1 = \frac{4\eta K \exp\{-r(T-t)\}}{B_t \alpha_t (\exp\{\eta(T-t)\} - 1)}$$

and

$$l_2 = \frac{4\eta V_t^*}{B_t \alpha_t (\exp\{\eta(T-t)\} - 1)}$$

$\chi^2(x; n, l)$ non-central chi-square distribution function

$n \geq 0$ degrees of freedom

non-centrality parameter $l > 0$

$$\chi^2(x; n, l) = \sum_{k=0}^{\infty} \frac{\exp\left\{-\frac{l}{2}\right\} \left(\frac{l}{2}\right)^k}{k!} \left(1 - \frac{\Gamma\left(\frac{x}{2}; \frac{n+2k}{2}\right)}{\Gamma\left(\frac{n+2k}{2}\right)}\right)$$

- **putative risk neutral price**

$$\tilde{p}(t, V_t^*, T, K, r) = p(t, V_t^*, T, K, r) + K e^{-r(T-t)} \exp\left\{ - \frac{\bar{V}_t^*}{2(\varphi(T) - \varphi(t))} \right\}$$

risk neutral is overpricing

for $\bar{V}_t^* \to 0 \quad \Longrightarrow \quad \tilde{p}(t, V_t^*, T, K, r) \to 0$

and

$$\tilde{p}(t, V_t^*, T, K, r) \to K\, e^{-r(T-t)} > 0$$

risk neutral ignores trends

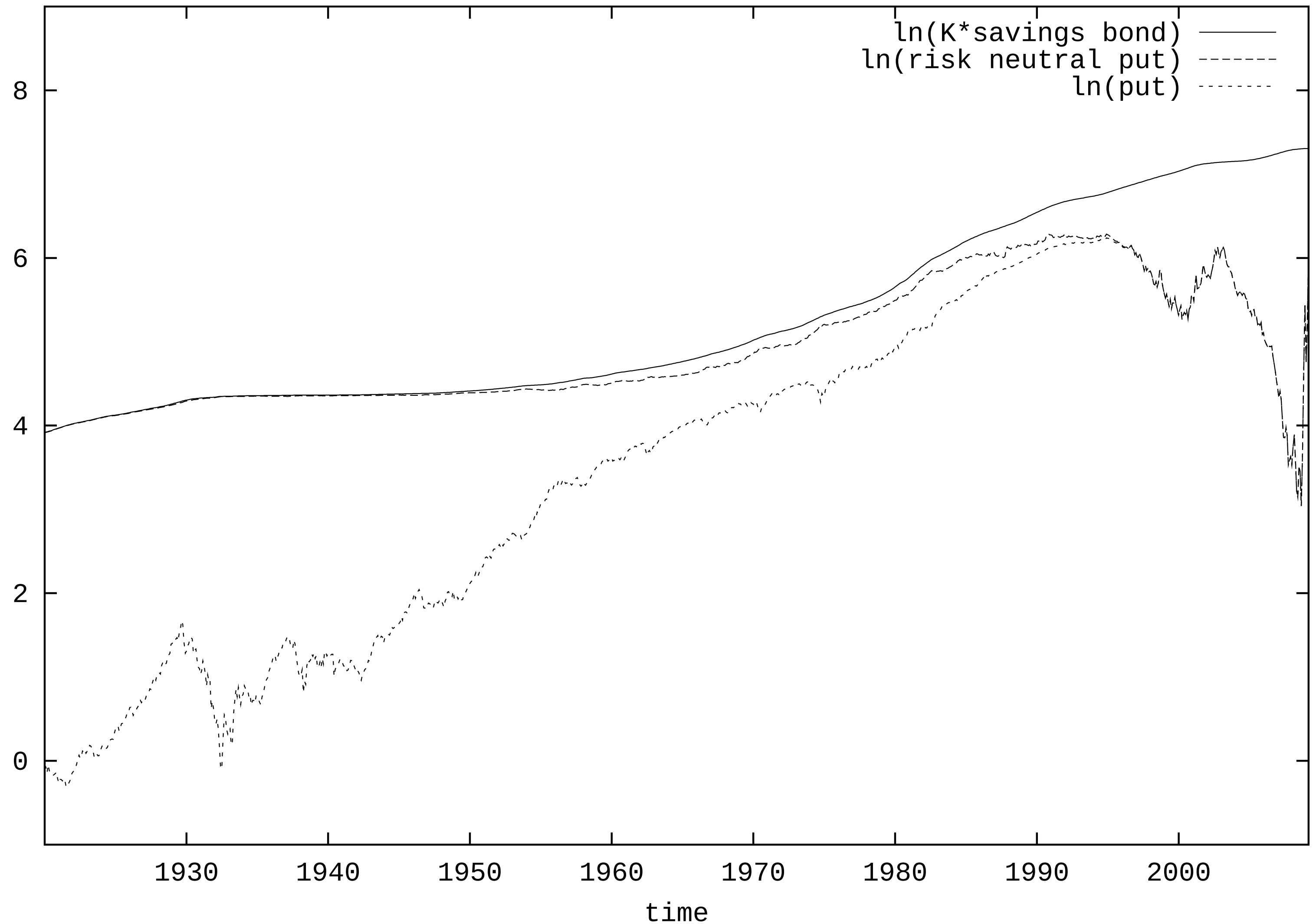

Figure 7.14: Logarithms of savings bond times $\boldsymbol{K}$, risk neutral put and fair put.

Guaranteed Minimum Death Benefit (GMDB)

- payout to the policyholder

$$\max(e^{g\tau}V_0, V_\tau)$$

time of death τ

$g \geq 0$ is the guaranteed instantaneous growth rate

V_0 is the initial account value

V_τ is the unit value of the policyholder's account at time of death τ

- insurance charges $\xi \geq 0$

$\Longrightarrow$ policyholder's unit value

$$V_t = e^{-\xi t} V_t^*$$

$\Longrightarrow$

$$\max(e^{g\tau} V_0, V_\tau) = \max(e^{g\tau} V_0, e^{-\xi\tau} V_\tau^*)$$

$$= e^{-\xi\tau} \max(e^{(g+\xi)\tau} V_0, V_\tau^*)$$

insurance company invests the entire fund value V in the NP V^*

$\Longrightarrow$ payoff

$$H_T = GMDB_T = e^{-\xi T}\left[(e^{(g+\xi)T}V_0^* - V_T^*)^+ + V_T^*\right]$$

fair value $GMDB_0$ of the total claim

real world pricing formula

$$GMDB_0 = V_0^* \, E\left(\frac{GMDB_T}{V_T^*}\right)$$

$$= e^{-\xi T}\left(p(0, V_0^*, T, e^{(g+\xi)T} V_0^*, r) + V_0^*\right)$$

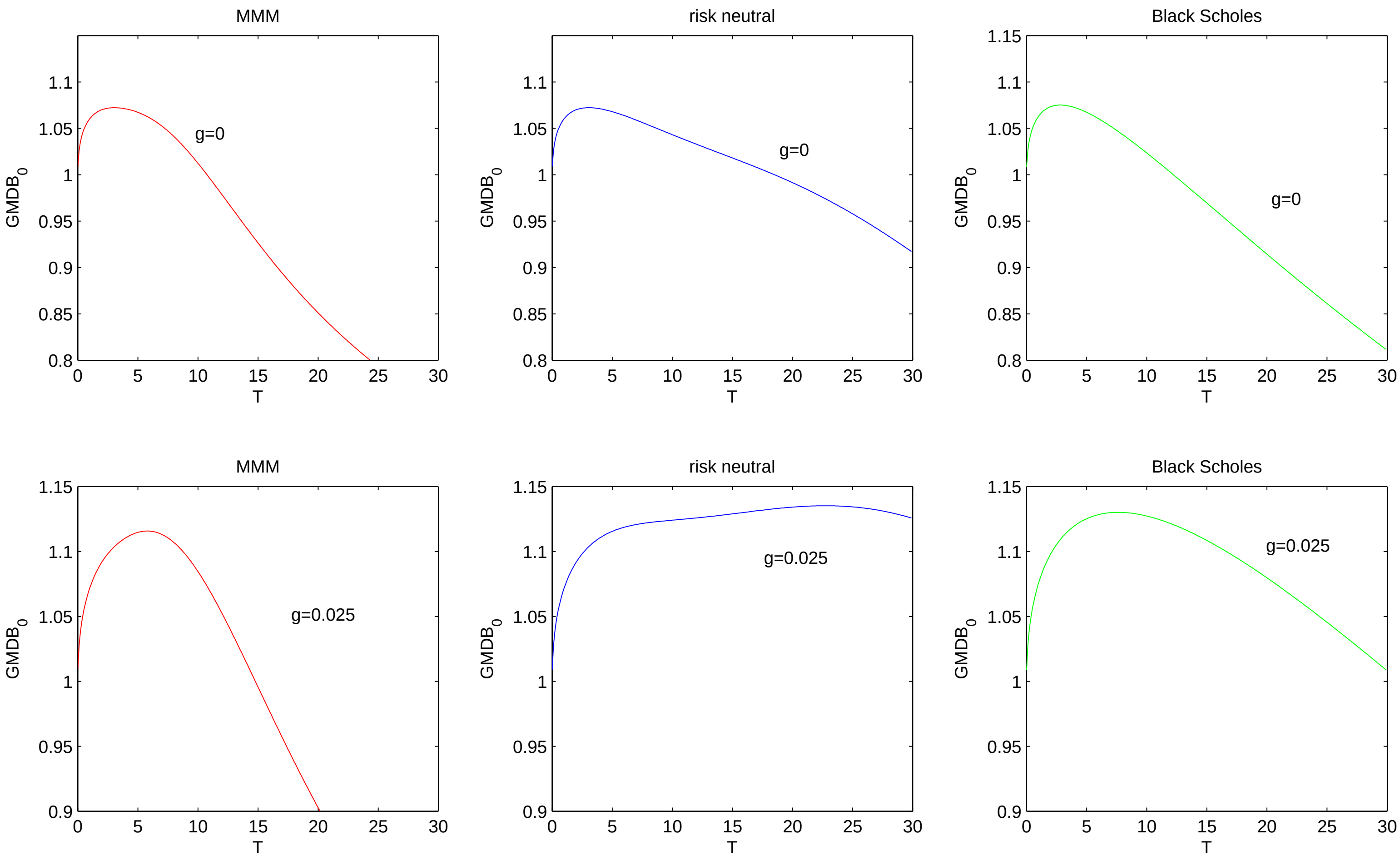

Figure 7.15: Present value of the GMDB under the real world pricing formula (left), the risk neutral pricing formula (middle) and the Black Scholes formula (right) for $\eta = 0.05$, $\alpha_0 = 0.05$, $r = 0.05$, $\xi = 0.01$ and $Y_0 = 20$.

- **lifetime τ is stochastic**

$$GMDB_0 = V_0^* \, E\left(E\left(\frac{GMDB_\tau}{V_\tau^*} \,\middle|\, \mathcal{F}_\tau\right)\right)$$

$$GMDB_0 = \int_0^T \left(p(0, V_0^*, t, e^{(g+\xi)t} V_0^*, r) + V_0^*\right) e^{-\xi t} f_\tau(t) \, dt$$

$f_\tau(\cdot)$ - mortality density

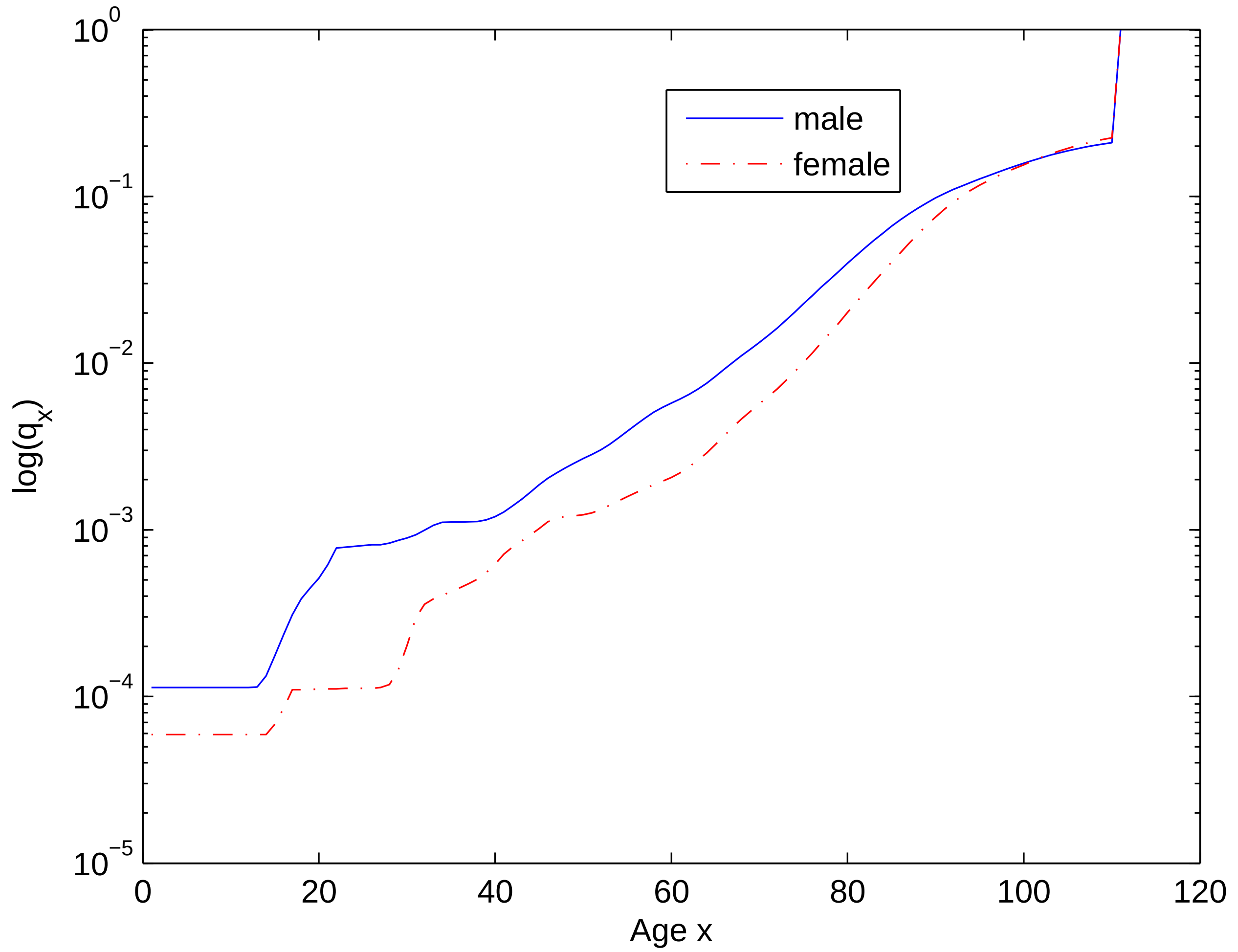

Figure 7.16: German mortality data

rational investors will lapse when embedded put-option out of the money

GMDBs have stochastic maturity

Titanic options, Milevsky & Posner (2001)

- **roll-up GMDB payoff**

$$H_t = \begin{cases} (1-\beta_t)V_t^*, & \text{if lapsed at time } t, \\ \max(e^{g\tau}V_0, V_\tau^*), & \text{if death occurs at time } t = \tau \end{cases}$$

surrender charge β_t

$$\beta_t = \begin{cases} (8 - \lceil t \rceil)\%, & t \leq 7, \\ 0, & t > 7 \end{cases}$$

no credit risk

no accumulation phase

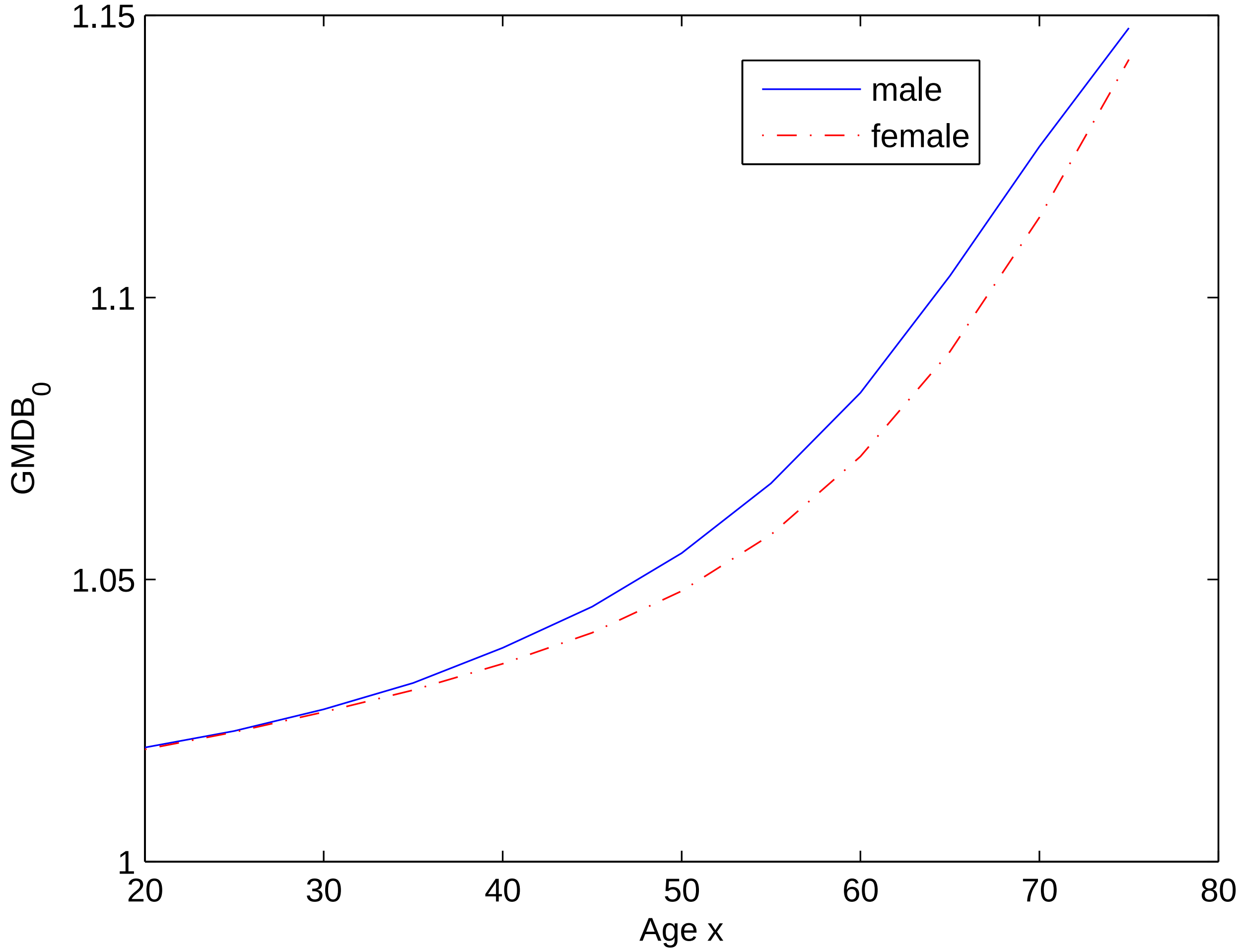

Figure 7.17: Value of the GMDB under the MMM for male and female policyholders aged x, assuming an irrational lapsation of $l = 1\%$.

8 Markets with Event Risk

Jump Diffusion Markets

- filtered probability space $(\Omega, \mathcal{A}, \underline{\mathcal{A}}, P)$

- **continuous trading uncertainty**

  $m \in \{1, 2, \ldots, d\}$

  Wiener processes $\tilde{W}^k = \{\tilde{W}^k_t,\, t \in [0, \infty)\},\ k \in \{1, 2, \ldots, m\}$

- **events**

  counting process $p^k = \{p^k_t,\ t \in [0, \infty)\}$

  intensity $h^k = \{h^k_t,\ t \in [0, \infty)\}$

$$h^k_t > 0$$

$$\int_0^t h^k_s\, ds < \infty$$

- **jump martingale**

$$dq^k_t = \left(dp^k_t - h^k_t\, dt\right) \left(h^k_t\right)^{-\frac{1}{2}}$$

  $k \in \{1, 2, \ldots, d-m\}$

$$\boldsymbol{E}\left(\left(q_{t+\Delta}^{k}-q_{t}^{k}\right)^{2} \,\Big|\, \mathcal{A}_{t}\right)=\boldsymbol{\Delta}$$

- **trading uncertainty**

$$W=\{W_t=(\tilde{W}_t^1,\ldots,\tilde{W}_t^m,q_t^1,\ldots,q_t^{d-m})^\top,\ t\in[0,\infty)\}$$

Primary Security Accounts

$$dS_t^j = S_{t-}^j \left(a_t^j \, dt + \sum_{k=1}^{d} b_t^{j,k} \, dW_t^k \right)$$

$j \in \{1, 2, \ldots, d\}$

Assumption 8.1

$$\boldsymbol{b_t^{j,k} \geq -\sqrt{h_t^{k-m}}}$$

for all $\boldsymbol{t} \in [\mathbf{0}, \infty)$, $\boldsymbol{j} \in \{\mathbf{1}, \mathbf{2}, \ldots, \boldsymbol{d}\}$ *and* $\boldsymbol{k} \in \{\boldsymbol{m} + \mathbf{1}, \ldots, \boldsymbol{d}\}$.

Assumption 8.2 $\boldsymbol{b_t}$ *is* invertible

- **market price of risk**

$$\boldsymbol{\theta_t = (\theta_t^1, \ldots, \theta_t^d)^\top = b_t^{-1}\,[a_t - r_t\,1]}$$

$$\boldsymbol{a_t = (a_t^1, \ldots, a_t^d)^\top}$$

$$\boldsymbol{1 = (1, \ldots, 1)^\top}$$

$$\Longrightarrow$$

$$\boldsymbol{dS_t^j = S_{t-}^j \left(r_t\,dt + \sum_{k=1}^{d} b_t^{j,k}\,(\theta_t^k\,dt + dW_t^k) \right)}$$

$$\boldsymbol{j \in \{0, 1, \ldots, d\}}$$

- **strategy**

$$\delta = \{\delta_t = (\delta_t^0, \ldots, \delta_t^d)^\top,\ t \in [0, \infty)\}$$

- **portfolio**

$$S_t^\delta = \sum_{j=0}^{d} \delta_t^j\, S_t^j$$

- self-financing if

$$dS_t^\delta = \sum_{j=0}^{d} \delta_t^j\, dS_t^j$$

Growth Optimal Portfolio

- **fraction**

$$\pi^j_{\delta,t} = \delta^j_t \, \frac{S^j_t}{S^\delta_t}$$

$$\pi_{\delta,t} = (\pi^1_{\delta,t}, \ldots, \pi^d_{\delta,t})^\top$$

$$dS^\delta_t = S^\delta_{t-} \Big\{ r_t \, dt + \pi^\top_{\delta,t-} \, b_t \, (\theta_t \, dt + dW_t) \Big\}$$

strictly positive $\Longleftrightarrow$

$$\sum_{j=1}^d \pi^j_{\delta,t} \, b^{j,k}_t > -\sqrt{h^{k-m}_t}$$

$k \in \{m+1, \ldots, d\}$

$$d\ln(S_t^\delta) = g_t^\delta\,dt + \sum_{k=1}^{m}\sum_{j=1}^{d}\pi_{\delta,t}^j\,b_t^{j,k}\,dW_t^k$$

$$+ \sum_{k=m+1}^{d}\ln\left(1 + \sum_{j=1}^{d}\pi_{\delta,t-}^j\,\frac{b_t^{j,k}}{\sqrt{h_t^{k-m}}}\right)\sqrt{h_t^{k-m}}\,dW_t^k$$

- **growth rate**

$$g_t^\delta = r_t + \sum_{k=1}^{m}\left[\sum_{j=1}^{d}\pi_{\delta,t}^j\,b_t^{j,k}\,\theta_t^k - \frac{1}{2}\left(\sum_{j=1}^{d}\pi_{\delta,t}^j\,b_t^{j,k}\right)^2\right]$$

$$+ \sum_{k=m+1}^{d}\left[\sum_{j=1}^{d}\pi_{\delta,t}^j b_t^{j,k}\left(\theta_t^k - \sqrt{h_t^{k-m}}\right) + \ln\left(1 + \sum_{j=1}^{d}\pi_{\delta,t}^j\,\frac{b_t^{j,k}}{\sqrt{h_t^{k-m}}}\right)h_t^{k-m}\right]$$

Assumption 8.3

$$\sqrt{h_t^{k-m}} > \theta_t^k$$

$k \in \{m+1, \ldots, d\}$

- **fractions of candidate benchmark**

$$\pi_{\delta_*,t} = (\pi^1_{\delta_*,t}, \ldots, \pi^d_{\delta_*,t})^\top = \left(c_t^\top \, b_t^{-1}\right)^\top$$

$$c_t^k = \begin{cases} \theta_t^k & \text{for} \quad k \in \{1, 2, \ldots, m\} \\ \frac{\theta_t^k}{1 - \theta_t^k \, (h_t^{k-m})^{-\frac{1}{2}}} & \text{for} \quad k \in \{m+1, \ldots, d\} \end{cases}$$

if $\frac{\theta_t^k}{\sqrt{h_t^{k-m}}} \ll 1$

$$\Longrightarrow \quad c_t^k \approx \theta_t^k$$

- **candidate benchmark**

$$
\begin{aligned}
dS_t^{\delta_*} &= S_{t-}^{\delta_*}\Big(r_t\,dt + c_t^\top\,(\theta_t\,dt + dW_t)\Big) \\
&= S_{t-}^{\delta_*}\left(r_t\,dt + \sum_{k=1}^{m}\theta_t^k\,(\theta_t^k\,dt + dW_t^k)\right. \\
&\qquad \left. + \sum_{k=m+1}^{d}\frac{\theta_t^k}{1-\theta_t^k\,(h_t^{k-m})^{-\frac{1}{2}}}\,(\theta_t^k\,dt + dW_t^k)\right)
\end{aligned}
$$

Definition 8.4 $S^{\underline{\delta}} \in \mathcal{V}^+$ *NP* *if* $g_t^{\delta} \leq g_t^{\underline{\delta}}$ *for all* $S^{\delta} \in \mathcal{V}^+$.

$\Longrightarrow$

S^{δ_*} is a NP.

$\Longrightarrow$

$$g_t^{\delta_*} = r_t + \frac{1}{2}\sum_{k=1}^{m}(\theta_t^k)^2 - \sum_{k=m+1}^{d} h_t^{k-m}\left(\ln\left(1 + \frac{\theta_t^k}{\sqrt{h_t^{k-m}} - \theta_t^k}\right) + \frac{\theta_t^k}{\sqrt{h_t^{k-m}}}\right)$$

for $\frac{\theta_t^k}{\sqrt{h_t^{k-m}}} \ll 1$ asymptotically

$$g_t^{\delta_*} \approx r_t + \frac{1}{2} \sum_{k=1}^{d} (\theta_t^k)^2 = r_t + \frac{|\theta_t|^2}{2}$$

$$d \ln(S_t^{\delta_*}) \approx g_t^{\delta_*}\, dt + \sum_{k=1}^{d} \theta_t^k\, dW_t^k$$

$$dS_t^{\delta_*} \approx S_t^{\delta_*} \left(r_t + \sum_{k=1}^{d} \theta_t^k \left(\theta_t^k\, dt + dW_t^k \right) \right)$$

- **jump diffusion market (JDM)** $\mathcal{S}^{\text{JD}}_{(d)} = \{S, a, b, \vec{r}, h, \underline{\mathcal{A}}, P\}$

- **benchmarked portfolio**

$$\hat{S}^{\delta}_t = \frac{S^{\delta}_t}{S^{\delta_*}_t}$$

$$d\hat{S}^{\delta}_t = \sum_{k=1}^{m} \left(\sum_{j=1}^{d} \delta^j_t \, \hat{S}^j_t \, b^{j,k}_t - \hat{S}^{\delta}_t \, \theta^k_t \right) dW^k_t$$

$$+ \sum_{k=m+1}^{d} \left(\left(\sum_{j=1}^{d} \delta^j_t \, \hat{S}^j_{t-} \, b^{j,k}_t \right) \left(1 - \frac{\theta^k_t}{\sqrt{h^{k-m}_t}} \right) - \hat{S}^{\delta}_{t-} \, \theta^k_t \right) dW^k_t$$

driftless

- **benchmarked savings account**

$$d\hat{S}^0_t = -\hat{S}^0_{t-} \sum_{k=1}^{d} \theta^k_t \, dW^k_t$$

driftless

benchmarked portfolio $\hat{S}^\delta$ driftless $\Longrightarrow$ $(\underline{\mathcal{A}}, P)$-local martingale

Ansel & Stricker (1994)

$\Longrightarrow$

Theorem 8.5 *In a JDM a nonnegative* $\hat{S}^\delta$ *is an* $(\underline{\mathcal{A}}, P)$*-supermartingale*

$$\hat{S}^\delta_t \geq E\left(\hat{S}^\delta_\tau \,\middle|\, \mathcal{A}_t\right)$$

$\tau \in [0, \infty)$ *and* $t \in [0, \tau]$.

A nonnegative portfolio is a strong ***arbitrage*** if it can get out of zero.

Corollary 8.6

A JDM does not allow nonnegative portfolios that permit strong arbitrage.

Law of the Minimal Price

Corollary 8.7 *$\mathcal{A}_\tau$-measurable payoff H with $E(\frac{H}{S_\tau^{\delta_*}}|\mathcal{A}_0) < \infty$. If there exists a fair nonnegative portfolio S^δ that replicates the payoff*

$$S_\tau^\delta = H,$$

then this is the minimal nonnegative replicating portfolio.

$\Longrightarrow$

- fair portfolios provide the cheapest hedge

H $\mathcal{A}_\tau$-measurable payoff, with $E(\frac{H}{S_\tau^{\delta_*}}) < \infty$

$\Longrightarrow$

real world pricing formula

$$U_H(t) = S_t^{\delta_*}\, E\left(\frac{H}{S_\tau^{\delta_*}} \,\middle|\, \mathcal{A}_t\right)$$

- equivalent to risk **neutral pricing** as long as candidate Radon-Nikodym

  derivative $\frac{\hat{S}_t^0}{\hat{S}_0^0}$ forms $(\underline{\mathcal{A}}, P)$-martingale

- **zero coupon bond**

$$P(t,T) = E\left(\frac{S_t^{\delta_*}}{S_T^{\delta_*}} \,\Big|\, \mathcal{A}_t\right)$$

- **benchmarked zero coupon bond**

$\hat{P}(t,T) = \frac{P(t,T)}{S_t^{\delta_*}}$

$$d\hat{P}(t,T) = -\hat{P}(t-,T) \sum_{k=1}^{d} \sigma^k(t,T)\, dW_t^k$$

$\Longrightarrow$

$$\ln(\hat{P}(t,T)) = \ln(\hat{P}(0,T)) - \sum_{k=1}^{m}\left(\int_0^t \sigma^k(s,T)dW_s^k + \frac{1}{2}\int_0^t (\sigma^k(s,T))^2\,ds\right)$$

$$+ \sum_{k=m+1}^{d}\left(\int_0^t \sigma^k(s,T)\,\sqrt{h_s^{k-m}}\,ds + \int_0^t \ln\left(1 - \frac{\sigma^k(s,T)}{\sqrt{h_s^{k-m}}}\right)dp_s^k\right)$$

- **forward rate**

$$f(t,T) = -\frac{\partial}{\partial T} \ln(P(t,T)) = -\frac{\partial}{\partial T} \ln(\hat{P}(t,T))$$

$$\Longrightarrow$$

- **forward rate equation**

$$f(t,T) \;=\; f(0,T) + \sum_{k=1}^{m} \int_0^t \left(\frac{\partial}{\partial T} \sigma^k(s,T) \right) \left(\sigma^k(s,T)\, ds + dW_s^k \right)$$

$$+ \sum_{k=m+1}^{d} \int_0^t \frac{1}{1 - \frac{\sigma^k(s,T)}{\sqrt{h_s^{k-m}}}} \; \frac{\partial}{\partial T} \sigma^k(s,T) \left(\sigma^k(s,T)\, ds + dW_s^k \right)$$

similar to HJM equation

Sequences of Diversified Portfolios

Sequence $(S^{\delta}_{(d)})_{d\in\mathcal{N}}$ is a sequence of DPs if

$$\left|\pi^{j}_{\delta,t}\right| \leq \frac{K_2}{d^{\frac{1}{2}+K_1}}$$

for all $j \in \{0, 1, \ldots, d\}$, $t \in [0, \infty)$ and $d \in \{d_0, d_0 + 1, \ldots\}$.

- **benchmarked primary security account**

$$d\hat{S}^j_{(d)}(t) = -\hat{S}^j_{(d)}(t-) \sum_{k=1}^{d} \sigma^{j,k}_{(d)}(t)\, dW^k_t$$

- **kth total specific volatility**

$$\hat{\sigma}^k_{(d)}(t) = \sum_{j=0}^{d} |\sigma^{j,k}_{(d)}(t)|$$

Sequence of JDMs *regular* if

$$E\left(\left(\hat{\sigma}^k_{(d)}(t)\right)^2\right) \leq K_5$$

for all $t \in [0, \infty), d \in \mathcal{N}$ and $k \in \{1, 2, \ldots, d\}$.

Sequence of Approximate NPs

- **tracking rate**

$$R^{\delta}_{(d)}(t) = \sum_{k=1}^{d} \left(\sum_{j=0}^{d} \pi^{j}_{\delta,t}\, \sigma^{j,k}_{(d)}(t) \right)^{2}$$

$S^{\delta}_{(d)}$ is NP $\quad \Longleftrightarrow \quad \hat{S}^{\delta}_{(d)} \equiv$ constant

$\Longleftrightarrow$

$$R^{\delta}_{(d)}(t) = 0$$

$(S^{\delta}_{(d)})_{d \in \mathcal{N}}$ is a sequence of *approximate NPs* if

$$\lim_{d \to \infty} R^{\delta}_{(d)}(t) \stackrel{P}{=} 0$$

- **expected tracking rate**

$$e^{\delta}_{(d)}(t) = E\left(R^{\delta}_{(d)}(t)\right)$$

$(S^{\delta}_{(d)})_{d \in \mathcal{N}}$ has vanishing expected tracking rate, if

$$\lim_{d \to \infty} e^{\delta}_{(d)}(t) = 0$$

$\Longrightarrow$

$$\lim_{d\to\infty} P\left(R^{\delta}_{(d)}(t) > \varepsilon\right) \le \lim_{d\to\infty} \frac{1}{\varepsilon}\, e^{\delta}_{(d)}(t) = 0$$

Lemma 8.8 *For a sequence of JDMs, any sequence of strictly positive portfolios with vanishing expected tracking rate is a sequence of approximate NPs.*

Diversification Theorem

Platen (2005a)

For a regular sequence of JDMs $(\mathcal{S}^{\text{JD}}_{(d)})_{d\in\mathcal{N}}$, each sequence $(S^{\delta}_{(d)})_{d\in\mathcal{N}}$ of DPs is a sequence of approximate NPs. Moreover

$$e^{\delta}_{(d)}(t) \leq \frac{(K_2)^2\, K_5}{d^{2K_1}}.$$

Diversification in an MMM Setting

- **savings account**

$$S^0_{(d)}(t) = \exp\{r\,t\}$$

- **discounted NP drift**

$$\alpha^{\delta_*}_t = \alpha_0\ \exp\{\eta\,t\}$$

- **jth benchmarked primary security account**

$$\hat{S}^j_{(d)}(t) = \frac{1}{Y^j_t\ \alpha^{\delta_*}_t}$$

- **SR process**

$$dY_t^j = \left(1 - \eta\, Y_t^j\right) dt + \sqrt{Y_t^j}\, dW_t^j$$

- **NP**

$$S_{(d)}^{\delta_*}(t) = \frac{S_{(d)}^0(t)}{\hat{S}_{(d)}^0(t)}$$

- j**th primary security account**

$$S_{(d)}^j(t) = \hat{S}_{(d)}^j(t)\, S_{(d)}^{\delta_*}(t)$$

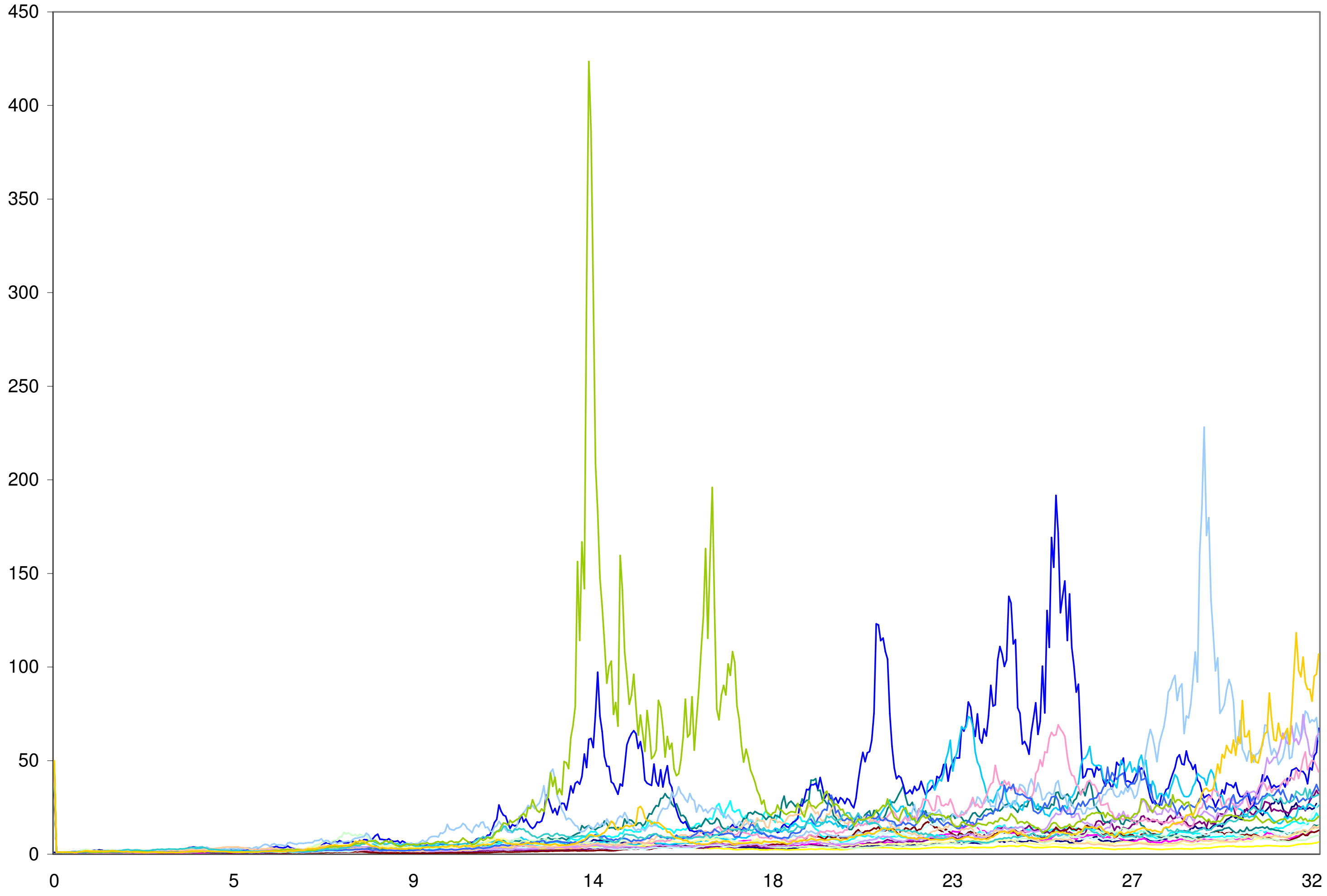

Figure 8.1: Primary security accounts under the MMM.

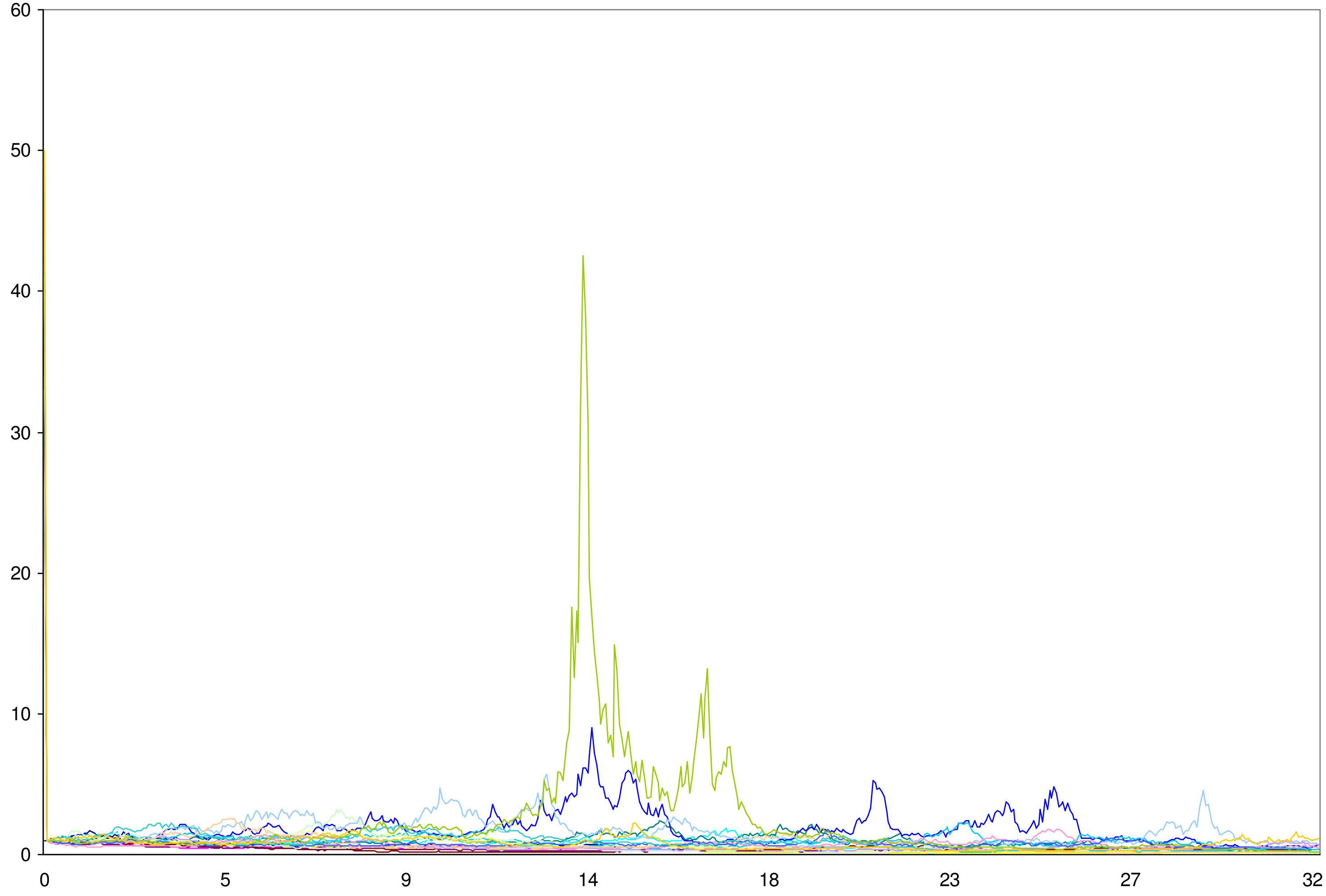

Figure 8.2: Benchmarked primary security accounts.

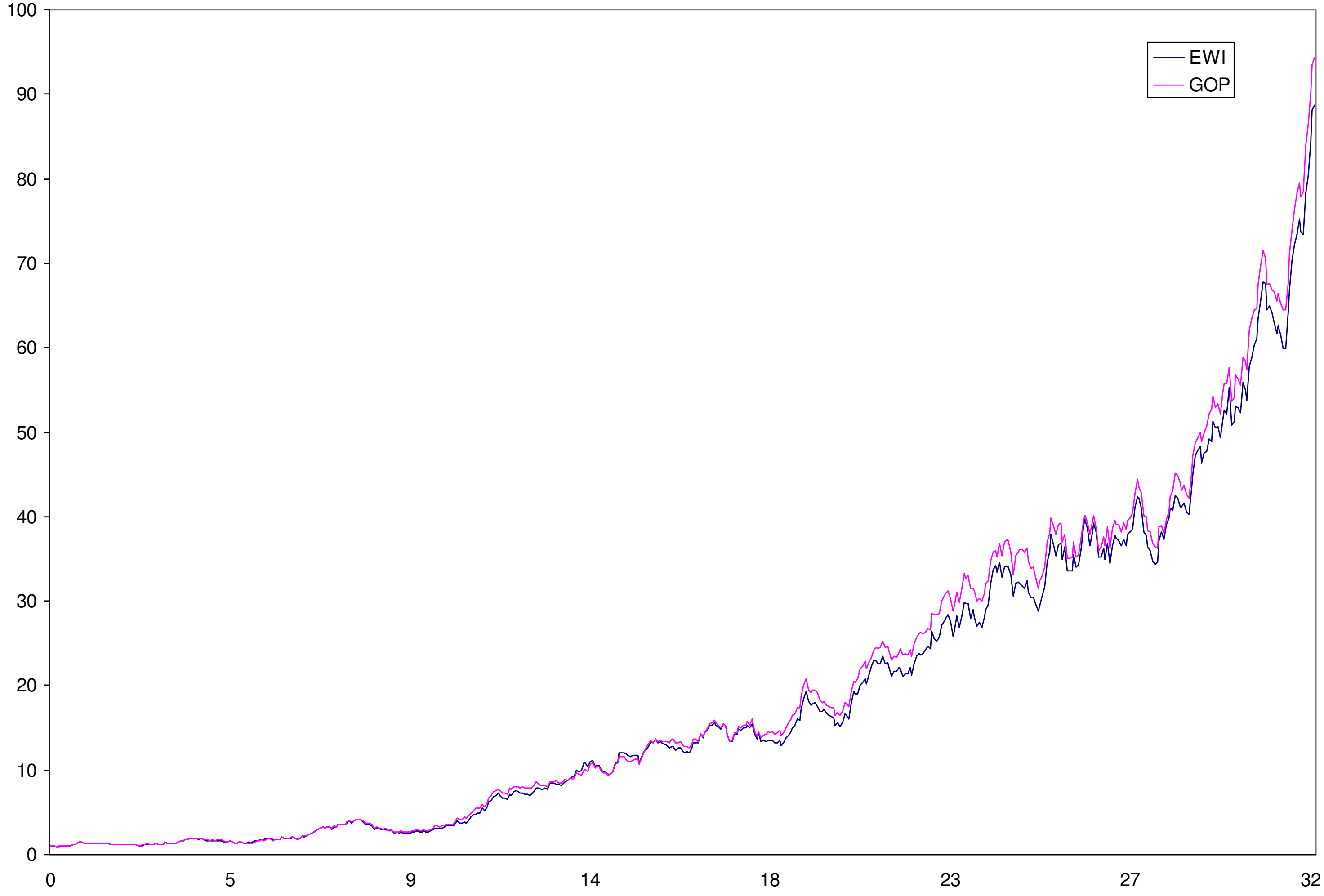

Figure 8.3: NP and EWI.

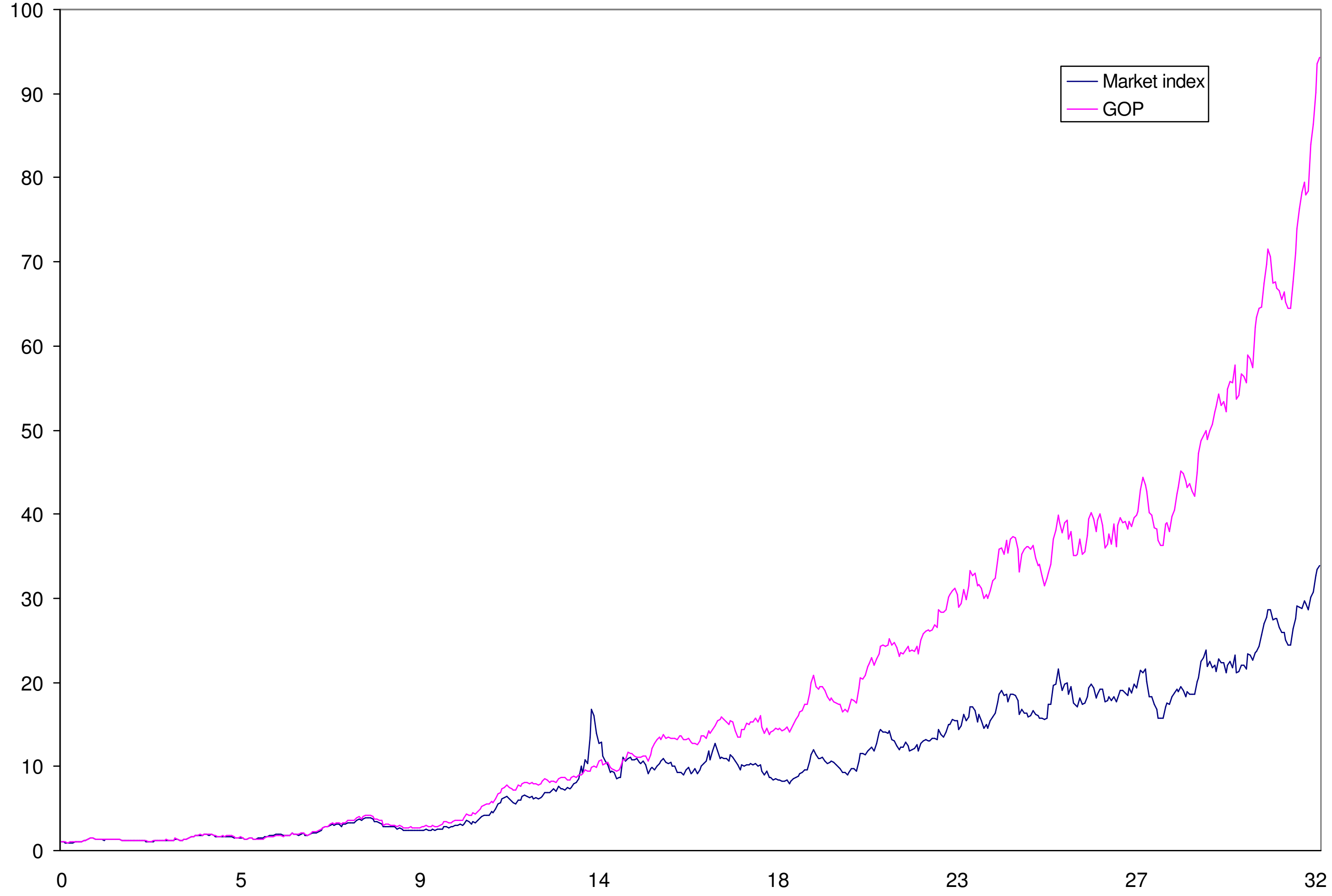

Figure 8.4: NP and market index.

Real World Pricing for Two Market Models

Specifying a Continuous NP

market prices of event risks are zero

$$\theta_t^k = 0$$

NP

$$dS_t^{\delta_*} = S_t^{\delta_*} \left(r_t \, dt + \sum_{k=1}^{m} \theta_t^k \left(\theta_t^k \, dt + dW_t^k \right) \right)$$

$$S_0^{\delta_*} = 1$$

Benchmarked Primary Security Accounts

$$d\hat{S}_t^j = -\hat{S}_{t-}^j \sum_{k=1}^{d} \sigma_t^{j,k}\, dW_t^k$$

$$\hat{S}_t^j = S_0^j \exp\left\{-\frac{1}{2}\int_0^t \sum_{k=1}^{m} \left(\sigma_s^{j,k}\right)^2 ds - \sum_{k=1}^{m}\int_0^t \sigma_s^{j,k}\, dW_s^k\right\}$$

$$\times \exp\left\{\int_0^t \sum_{k=m+1}^{d} \sigma_s^{j,k} \sqrt{h_s^{k-m}}\, ds\right\} \prod_{k=m+1}^{d} \prod_{l=1}^{p_t^{k-m}} \left(1 - \frac{\sigma_{\tau_l^k-}^{j,k}}{\sqrt{h_{\tau_l^k-}^{k-m}}}\right)$$

pivotal objects of study

- **simplifying notation:**

$$|\sigma_t^j| = \sqrt{\sum_{k=1}^{m} \left(\sigma_t^{j,k}\right)^2}$$

- **aggregate continuous noise processes**

$$\hat{W}_t^j = \sum_{k=1}^{m} \int_0^t \frac{\sigma_s^{j,k}}{|\sigma_s^j|} \, dW_s^k$$

- **generalized volatility matrix** $b_t = [b_t^{j,k}]_{j,k=1}^d$ invertible

$$b_t^{j,k} = \theta_t^k - \sigma_t^{j,k}$$

for $k \in \{1, 2, \ldots, m\}$

$$b_t^{j,k} = -\sigma_t^{j,k}$$

$k \in \{m+1, \ldots, d\}$

- **constant intensities**

$$h_t^k = h^k > 0$$

$$\sigma_t^{j,k} = \sigma^{j,k} \leq \sqrt{h^{k-m}}$$

- benchmarked jth primary security account:

$$\hat{S}_t^j = \hat{S}_t^{j,\mathrm{c}} \, S_t^{j,\mathrm{d}}$$

- continuous part

$$\hat{S}_t^{j,\mathrm{c}} = S_0^j \exp\left\{-\frac{1}{2}\int_0^t |\sigma_s^j|^2\, ds - \int_0^t |\sigma_s^j|\, d\hat{W}_s^j\right\}$$

- compensated jump part

$$S_t^{j,\mathrm{d}} = \exp\left\{\sum_{k=m+1}^{d} \sigma^{j,k}\sqrt{h^{k-m}}\, t\right\} \prod_{k=m+1}^{d}\left(1 - \frac{\sigma^{j,k}}{\sqrt{h^{k-m}}}\right)^{p_t^{k-m}}$$

Merton Model

$$r_t = r,\ \sigma_t^{j,k} = \sigma^{j,k}$$

$$\hat{S}_t^{j,\mathrm{c}} = S_0^j \exp\left\{-\frac{1}{2}\left|\sigma^j\right|^2 t - \left|\sigma^j\right| \hat{W}_t^j\right\}$$

Girsanov's theorem, see Section 9.5

A Minimal Market Model with Jumps

Hulley, Miller & Pl. (2005)

- **discounted NP drift**

$$\alpha^j(t) = \alpha_0^j \exp\{\eta^j t\}$$

η^j - net growth rate

- **jth square root process**

$$dY_t^j = \left(1 - \eta^j Y_t^j\right)\, dt + \sqrt{Y_t^j}\, d\hat{W}_t^j$$

- continuous part

$$\hat{S}_t^{j,\mathrm{c}} = \frac{1}{\alpha^j(t) Y_t^j}$$

- **time transformation**

$$\varphi^j(t) = \varphi^j_0 + \frac{1}{4}\int_0^t \alpha^j(s)\,ds$$

- **squared Bessel process of dimension four**

$$X^j_{\varphi^j(t)} = \alpha^j(t) Y^j_t = \frac{1}{\hat{S}^{j,c}_t}$$

$\hat{S}^0$ is a strict local martingale

Zero Coupon Bonds

$$P(t,T) = S_t^{\delta_*} \, E\left(\frac{1}{S_T^{\delta_*}} \,\middle|\, \mathcal{A}_t \right) = \frac{1}{\hat{S}_t^0} \, E\left(\exp\left\{ -\int_t^T r_s \, ds \right\} \hat{S}_T^0 \,\middle|\, \mathcal{A}_t \right)$$

- **MM case**

$$P(t,T) = \exp\{-r(T-t)\} \frac{1}{\hat{S}_t^0} \, E\left(\hat{S}_T^0 \,\middle|\, \mathcal{A}_t \right) = \exp\{-r(T-t)\}$$

- **MMM case**

$$\lambda_t^j = \frac{1}{\hat{S}_t^j (\varphi^j(t) - \varphi^j(T))}$$

$$\hat{S}_T^0 = \hat{S}_T^{0,c}$$

$$P(t,T) \;=\; E\left(\exp\left\{-\int_t^T r_s\, ds\right\} \;\Big|\; \mathcal{A}_t\right) \frac{1}{\hat{S}_t^0}\, E\left(\hat{S}_T^0 \;\Big|\; \mathcal{A}_t\right)$$

$$= \; E\left(\exp\left\{-\int_t^T r_s\, ds\right\} \;\Big|\; \mathcal{A}_t\right) \left(1 - \exp\left\{-\frac{1}{2}\lambda_t^0\right\}\right)$$

Forward Contracts

$$S_t^{\delta_*} \, E\left(\frac{F^j(t,T) - S_T^j}{S_T^{\delta_*}} \,\Big|\, \mathcal{A}_t \right) = 0$$

forward price

$$F^j(t,T) = \frac{S_t^{\delta_*} E\left(\hat{S}_T^j \,\Big|\, \mathcal{A}_t \right)}{S_t^{\delta_*} \, E\left(\frac{1}{S_T^{\delta_*}} \,\Big|\, \mathcal{A}_t \right)} = \begin{cases} \frac{S_t^j}{P(t,T)} \frac{1}{\hat{S}_t^j} \, E\left(\hat{S}_T^j \,\big|\, \mathcal{A}_t \right) & \text{if } S_t^j > 0 \\ 0 & \text{if } S_t^j = 0 \end{cases}$$

- **MM case**

$$F^j(t,T) = S_t^j \, \exp\{r(T-t)\}$$

standard expression

- **MMM case**

$$\frac{1}{\hat{S}_t^j} E\left(\hat{S}_T^j \,\Big|\, \mathcal{A}_t\right) = \frac{1}{\hat{S}_t^{j,\mathrm{c}}} E\left(\hat{S}_T^{j,\mathrm{c}} \,\Big|\, \mathcal{A}_t\right) \frac{1}{S_t^{j,\mathrm{d}}} E\left(S_T^{j,\mathrm{d}} \,\Big|\, \mathcal{A}_t\right)$$

$$= 1 - \exp\left\{-\frac{1}{2}\,\lambda_t^j\right\}$$

forward price

$$F^j(t,T) = S_t^j \, \frac{1-\exp\left\{-\frac{1}{2}\,\lambda_t^j\right\}}{1-\exp\left\{-\frac{1}{2}\,\lambda_t^0\right\}} \left(E\left(\exp\left\{-\int_t^T r_s\,ds\right\} \,\Big|\, \mathcal{A}_t\right)\right)^{-1}$$

Asset-or-Nothing Binaries

Ingersoll (2000), Buchen (2004) and Buchen & Konstandatos (2005)

$r_t = r$

$$
\begin{aligned}
A^{j,k}(t,T,K) &= S_t^{\delta_*}\, E\left(1_{\{S_T^j \geq K\}} \frac{S_T^j}{S_T^{\delta_*}} \,\middle|\, \mathcal{A}_t \right) \\
&= \frac{S_t^j}{\hat{S}_t^j}\, E\left(1_{\{\hat{S}_T^j \geq K (S_T^0)^{-1} \hat{S}_T^0\}} \hat{S}_T^j \,\middle|\, \mathcal{A}_t \right) \\
&= \frac{S_t^j}{\hat{S}_t^{j,c}}\, E\Bigg(1_{\{\hat{S}_T^{j,c} \geq g(p_T^{k-m} - p_t^{k-m}) \hat{S}_T^0\}} \\
&\quad \times \exp\left\{ \sigma^{j,k} \sqrt{h^{k-m}}\, (T-t) \right\} \left(1 - \frac{\sigma^{j,k}}{\sqrt{h^{k-m}}} \right)^{p_T^{k-m} - p_t^{k-m}} \hat{S}_T^{j,c} \,\Bigg|\, \mathcal{A}_t \Bigg)
\end{aligned}
$$

$$A^{j,k}(t,T,K) =$$

$$= \sum_{n=0}^{\infty} \exp\left\{-h^{k-m}(T-t)\right\} \frac{(h^k(T-t))^n}{n!} \exp\left\{\sigma^{j,k}\sqrt{h^{k-m}}(T-t)\right\}$$

$$\times \left(1 - \frac{\sigma^{j,k}}{\sqrt{h^{k-m}}}\right)^n \frac{S_t^j}{\hat{S}_t^{j,\mathrm{c}}} \, E\left(1_{\{\hat{S}_T^{j,c} \geq g(n)\,\hat{S}_T^0\}} \hat{S}_T^{j,\mathrm{c}} \,\Big|\, \mathcal{A}_t\right)$$

for all $t \in [0, T]$, where

$$g(n) = \frac{K}{S_t^0 \, S_t^{j,\mathrm{d}}} \exp\left\{-\left(r + \sigma^{j,k}\sqrt{h^{k-m}}\right)(T-t)\right\}\left(1 - \frac{\sigma^{j,k}}{\sqrt{h^{k-m}}}\right)^{-n}$$

for all $n \in \mathcal{N}$

MM case

$$A^{j,k}(t,T,K) \;=\; \sum_{n=0}^{\infty} \exp\left\{-h^{k-m}\,(T-t)\right\} \frac{\left(h^{k-m}\,(T-t)\right)^n}{n!}$$

$$\times \exp\left\{\sigma^{j,k}\sqrt{h^{k-m}}\,(T-t)\right\}\left(1-\frac{\sigma^{j,k}}{\sqrt{h^{k-m}}}\right)^n S_t^j\, N(d_1(n))$$

for all $t \in [0,T]$, where

$$d_1(n) = \frac{\ln\left(\frac{S_t^j}{K}\right) + \left(r + \sigma^{j,k}\sqrt{h^{k-m}} + n\frac{\ln\left(1-\frac{\sigma^{j,k}}{\sqrt{h^{k-m}}}\right)}{T-t} + \frac{1}{2}\left(\hat{\sigma}^{0,j}\right)^2\right)(T-t)}{\hat{\sigma}^{0,j}\,\sqrt{T-t}}$$

$$\hat{\sigma}^{i,j} = \sqrt{|\sigma^i|^2 - 2\,\varrho^{i,j}\,|\sigma^i|\,|\sigma^j| + |\sigma^j|^2}$$

$\varrho^{i,j}$ - correlation

MMM case

$\hat{S}^0$ and $\hat{S}^{j,c}$ are independent

$$A^{j,k}(t,T,K) \;=\; \sum_{n=0}^{\infty} \exp\left\{-h^{k-m}(T-t)\right\} \frac{(h^{k-m}(T-t))^n}{n!}$$

$$\times \exp\left\{\sigma^{j,k}\sqrt{h^{k-m}}(T-t)\right\} \left(1 - \frac{\sigma^{j,k}}{\sqrt{h^{k-m}}}\right)^n$$

$$\times S_t^j \left(G''_{0,4}\left(\frac{\varphi^0(T)-\varphi^0(t)}{g(n)}; \lambda_t^j, \lambda_t^0\right) - \exp\left\{-\frac{1}{2}\lambda_t^j\right\}\right)$$

$G''_{0,4}(x; \lambda, \lambda')$ equals probability $P(\frac{Z}{Z'} \leq x)$

non-central chi-square distributed random variable $Z \sim \chi^2(0, \lambda)$

non-central chi-square distributed random variable $Z' \sim \chi^2(4, \lambda')$

Bond-or-Nothing Binaries

MM case

$$B^{j,k}(t,T,K) \;=\; \sum_{n=0}^{\infty} \exp\{-h^{k-m}\,(T-t)\}\,\frac{(h^{k-m}\,(T-t))^n}{n!}$$

$$\times\, K\exp\{-r(T-t)\}\,N(d_2(n))$$

for all $t \in [0,T]$, where

$$d_2(n) \;=\; \frac{\ln\!\left(\frac{S^j_t}{K}\right) + \left(r + \sigma^{j,k}\,\sqrt{h^{k-m}} + n\,\frac{\ln\left(1-\frac{\sigma^{j,k}}{\sqrt{h^{k-m}}}\right)}{T-t} - \frac{1}{2}\,\left(\hat{\sigma}^{0,j}\right)^2\right)(T-t)}{\hat{\sigma}^{0,j}\,\sqrt{T-t}}$$

$$=\; d_1(n) - \hat{\sigma}^{0,j}\,\sqrt{T-t}$$

MMM case

$$B^{j,k}(t,T,K)$$

$$= \sum_{n=0}^{\infty} \exp\{-h^{k-m}(T-t)\} \frac{(h^{k-m}\,(T-t))^n}{n!}$$

$$\times K \exp\{-r(T-t)\} \left(1 - G''_{0,4}\left((\varphi^j(T) - \varphi^j(t))g(n); \lambda^0_t, \lambda^j_t\right)\right)$$

European Call Options

$$
\begin{aligned}
c^{j,k}_{T,K}(t) &= S^{\delta_*}_t E\left(\frac{\left(S^j_T - K\right)^+}{S^{\delta_*}_T} \,\Bigg|\, \mathcal{A}_t \right) \\
&= S^{\delta_*}_t E\left(1_{\{S^j_T \geq K\}} \frac{S^j_T - K}{S^{\delta_*}_T} \,\Bigg|\, \mathcal{A}_t \right) \\
&= A^{j,k}(t,T,K) - B^{j,k}(t,T,K)
\end{aligned}
$$

MM case

$$
\begin{aligned}
c^{j,k}_{T,K}(t) \;=\; \sum_{n=0}^{\infty} \exp\{-h^{k-m}(T-t)\}\,\frac{(h^{k-m}(T-t))^n}{n!} \\
\times\left(\exp\left\{\sigma^{j,k}\sqrt{h^{k-m}}\,(T-t)\right\}\left(1-\frac{\sigma^{j,k}}{\sqrt{h^{k-m}}}\right)^n S^j_t\,N(d_1(n))\right. \\
\left. -\,K\exp\{-r(T-t)\}N(d_2(n))\right)
\end{aligned}
$$

MMM case

$$
\begin{aligned}
c^{j,k}_{T,K}(t) \;=\; & \sum_{n=0}^{\infty} \exp\{-h^{k-m}(T-t)\} \frac{(h^{k-m}(T-t))^n}{n!} \\
& \times \Bigg[\exp\Big\{\sigma^{j,k}\sqrt{h^{k-m}}\,(T-t)\Big\} \left(1 - \frac{\sigma^{j,k}}{\sqrt{h^{k-m}}}\right)^n \\
& \times S^j_t \left(G''_{0,4}\left(\left(\frac{\varphi^0(T) - \varphi^0(t)}{g(n)} \right); \lambda^j_t, \lambda^0_t \right) - \exp\left\{-\frac{1}{2}\lambda^j_t\right\} \right) \\
& - K \exp\{-r(T-t)\} \Big(1 - G''_{0,4}\Big((\varphi^j(T) - \varphi^j(t)) g(n); \lambda^0_t, \lambda^j_t\Big)\Big) \Bigg]
\end{aligned}
$$

Defaultable Zero Coupon Bonds

maturity T

defaults at the first jump time τ_1^{k-m} of p^{k-m}

zero recovery

$$\tilde{P}^{k-m}(t,T) = S_t^{\delta_*} E\left(\frac{1_{\{\tau_1^{k-m} > T\}}}{S_T^{\delta_*}} \,\Big|\, \mathcal{A}_t \right)$$

$$= S_t^{\delta_*} E\left(\frac{1}{S_T^{\delta_*}} \,\Big|\, \mathcal{A}_t \right) E\left(1_{\{\tau_1^{k-m} > T\}} \,\Big|\, \mathcal{A}_t \right)$$

$$= P(t,T) P(p_T^{k-m} = 0 \,|\, \mathcal{A}_t)$$

- **conditional probability of survival**

$$
\begin{aligned}
P\left(p_T^{k-m} = 0 \,\middle|\, \mathcal{A}_t\right) &= E\left(1_{\{p_t^{k-m}=0\}}\, 1_{\{p_T^{k-m}-p_t^{k-m}=0\}} \,\middle|\, \mathcal{A}_t\right) \\
&= 1_{\{p_t^{k-m}=0\}}\, P\left(p_T^{k-m} - p_t^{k-m} = 0 \,\middle|\, \mathcal{A}_t\right) \\
&= 1_{\{p_t^{k-m}=0\}}\, E\left(\exp\left\{-\int_t^T h_s^{k-m}\, ds\right\} \,\middle|\, \mathcal{A}_t\right)
\end{aligned}
$$

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