Lectures for the Course on Foundations of Mathematical Finance

Second Part: An Orlicz space approach to utility maximization and indifference pricing

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A key result

In general financial markets, the indifference price is (except for the sign) a convex risk measure on the Orlicz space $L^{\widehat{u}}$ naturally induced by the utility function u of the agent.

It is continuous and subdifferential on the interior $\mathcal{B}$ of its proper domain, which is considerably large as it coincides with $-\mathrm{int}(\mathrm{Dom}(I_u))$, i.e. the opposite of the interior in $L^{\widehat{u}}$ of the proper domain of the integral functional $I_u(f) = E[u(f)]$.

The optimization problem with random endowment

Consider the maximization problem

$$\sup_{H\in\mathcal{H}^W} E\left[u\left(x+(H\cdot S)_T - B\right)\right]$$

1) $u:\mathbb{R}\to\mathbb{R}\cup\{-\infty\}$ is concave and increasing (but not constant on $\mathbb{R}$)
2) S is a general $\mathbb{R}^d-$valued càdlàg semimartingale

3) B is a $\mathcal{F}_T$ measurable rv, the payoff of a claim; $x\in\mathbb{R}$

4) A predictable S-integrable proc. H is in $\mathcal{H}^W$ if there is $c>0$

$$(H\cdot S)_t \geq -cW \quad \forall t\leq T.$$

Key aspects

- S is not necessarily locally bounded
- Control of the integrals by a loss bound random variable $W \in L^0_+$:

$$\int_0^t H_s dS_s = (H \cdot S)_t \geq -cW \quad \forall t \leq T.$$

- Weak assumptions on the claim B
- Orlicz space duality.

Generic notations

In order to present some key issues related to the utility maximization problem, for the moment we generically set:

- $\mathcal{H}$ is the class of admissible integrands
- $K = \{(H.S)_T \,|H \in \mathcal{H}\}$
- $\mathcal{M}$ is the convex set of pricing measures (martingale probability measures)

Key issues

- Duality relation and optimal $Q^* \in \mathcal{M}$ of the dual problem

$$\sup_{H \in \mathcal{H}} Eu(B+(H.S)_T) = \min_{\lambda>0,\ Q\in\mathcal{M}} \left\{ \lambda Q(B) + E\left[\Phi\left(\lambda \frac{dQ}{dP}\right)\right]\right\}$$

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- Optimal $f^* \in \mathbb{K} \supseteq K$ of the primal problem

$$\sup_{H \in \mathcal{H}} Eu(B+(H.S)_T) = \sup_{k \in \mathbb{K}} Eu(B+k) = Eu(B+f^*)$$

continues

- Representation of f^* as stochastic integral

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- Supermartingale property of the optimal wealth process $(H^*.S)$ is a $Q-$ supermartingale wrt all $Q \in \mathcal{M}$ and a Q^*-martingale
- Indifference Price

Steps to solve the utility maximization problem

1) Find a "good" $\mathcal{H}$ and set:

$$K = \{ (H \cdot S)_T \mid \ H \in \mathcal{H} \}$$

2) Find a "good" Topological Vector Space L, such that:

$$\sup_{k \in K} E[u(B+k)] = \sup_{k \in C} E[u(B+k)]$$

where

$$C \triangleq (K - L^0_+) \cap L$$

3) Apply the duality (L, L'), compute the polar $C^0 \subseteq L'$ and solve the dual problem over C^0.
4) Using (3), solve the primal problem.

Orlicz duality

MESSAGE

The good topological vector space is the Orlicz space $L^{\widehat{u}}$ associated to the utility function u.

The good duality is the Orlicz space duality $(L^{\widehat{u}}, (L^{\widehat{u}})')$.

How and why Orlicz spaces

Very simple observation: u is concave, so the steepest behavior is on the left tail.
Economically, this reflects the risk aversion of the agent: the losses are weighted in a more severe way than the gains.

We will turn the left tail of u into a Young function $\widehat{u}$.
Then, $\widehat{u}$ gives rise to an Orlicz space $L^{\widehat{u}}$, naturally associated to our problem, which allows for an **unified treatment of Utility Maximization**.

Heuristic

Let W be a positive random variable. We will control the losses in the following way:

$$(H.S)_t \geq -cW$$

and we will require:

$$E[u(-W)] > -\infty$$

If we set: $\widehat{u}(x) := -u(-|x|) + u(0)$ then

$$E[\widehat{u}(W)] < \infty$$

which means that $W \in L^{\widehat{u}}$, the Orlicz space induced by the utility function u.

Orlicz function spaces on a probability space

A Young function Ψ is an even, convex function

$$\Psi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$$

with the properties:

1- $\Psi(0) = 0$

2- $\Psi(\infty) = +\infty$

3- $\Psi < +\infty$ in a neighborhood of 0.

The Orlicz space L^{Ψ} on $(\Omega, \mathcal{F}, P)$ is then

$$L^{\Psi} = \{f \in L^0 \mid \exists \alpha > 0 \, E[\Psi(\alpha f)] < +\infty\}$$

It is a Banach space with the gauge norm

$$\|f\|_{\Psi} = \inf \left\{ c > 0 \mid E\left[\Psi\left(\frac{f}{c}\right)\right] \leq 1 \right\}$$

The subspace $\mathbf{M}^\Psi$

Consider the closed subspace of L^Ψ

$$M^\Psi = \{ f \in L^0(P) \mid E\left[\Psi(\alpha f)\right] < +\infty \ \forall \alpha > 0 \}$$

- In general,

$$M^\Psi \subsetneq L^\Psi .$$

  When Ψ is continuous on $\mathbb{R}$, we have $M^\Psi = \overline{L^\infty}^\Psi$

- But when Ψ satisfies the Δ_2 growth-condition (as il the L^p case) the two spaces coincide:

$$M^\Psi = L^\Psi .$$

  and $L^\Psi = \{ f \mid E[\Psi(f)] < +\infty \} = \overline{L^\infty}^\Psi$.

The Orlicz space $L^{\widehat{u}}$

Given $u : (a, +\infty) \to \mathbb{R}$ with $-\infty \leq a < 0$, define the function $\widehat{u} : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$

$$\widehat{u}(x) \triangleq -u(-|x|) + u(0)$$

it is a Young function, so

$$L^{\widehat{u}} = \{f \in L^0(P) \mid \exists \alpha > 0 \; E[\widehat{u}(\alpha f)] < +\infty\}$$

and

$$M^{\widehat{u}} \triangleq \{f \in L^{\widehat{u}} \mid E[\widehat{u}(\alpha f)] < +\infty \; \forall \alpha > 0\}.$$

are well defined.

$\widehat{\Phi}$, the convex conjugate of $\widehat{u}$

$$\widehat{\Phi}(y) \triangleq \sup_{x\in\mathbb{R}} \{xy - \widehat{u}(x)\} = \begin{cases} 0 & \text{if } |y| \le \beta \\ \Phi(|y|) - \Phi(\beta) & \text{if } |y| > \beta \end{cases}$$

$$\Phi(y) \triangleq \sup_{x\in\mathbb{R}} \{u(x) - xy\}$$

$L^{\widehat{\Phi}}$ and $M^{\widehat{\Phi}}$ are the Orlicz spaces associated to $\widehat{\Phi}$.

Note: $\beta > 0$ is the point where Φ attains the minimum

The case $u : (a, +\infty) \to \mathbb{R}$ with $a < 0$ and a **finite**

If a is finite, then we always have:

$$\widehat{u}(x) = +\infty \text{ if } |x| > a.$$

Hence:

$$L^{\widehat{u}} = L^{\infty}, \quad M^{\widehat{u}} = \{0\}$$

and

$$L^{\widehat{\Phi}} = M^{\widehat{\Phi}} = L^1$$

A specific example: if $u(x) = \sqrt{1+x}$ then

$$\widehat{u}(x) = \begin{cases} 1 - \sqrt{1-|x|} - \frac{1}{2}|x| & \text{if } |x| \leq 1 \\ +\infty & \text{if } |x| > 1 \end{cases}$$

The interesting case $u : (-\infty, +\infty) \to \mathbb{R} \quad (a = -\infty)$

When $u : (-\infty, +\infty) \to \mathbb{R}$ then the function $\widehat{u} : \mathbb{R} \to \mathbb{R}$

$$\widehat{u}(x) \triangleq -u(-|x|) + u(0)$$

is a regular Young function (it does not jump to $+\infty$), and

$$M^{\widehat{u}} = \{f \in L^{\widehat{u}} \mid E[\widehat{u}(\alpha f)] < +\infty \, \forall \alpha > 0\} = \overline{L^{\infty}}^{\widehat{u}}.$$

and $L^{\widehat{u}}$ are well defined.

Example: $u(x) = -e^{-x}$

In this case, $\Phi(y) = y \ln y - y + 1$,

$$\widehat{u}(x) = e^{|x|} - |x| - 1$$

$$\widehat{\Phi}(y) = (1 + |y|\ ) \ln(1 + |y|\ ) - |y|$$

and therefore

$$L^{\widehat{u}} = \left\{ f \in L^0(P) \mid \exists \alpha > 0 \text{ s.t. } E\left[e^{\alpha |f|} \right] < +\infty \right\}$$

$$M^{\widehat{u}} = \left\{ f \in L^0(P) \mid \forall \alpha > 0 \; E\left[e^{\alpha |f|} \right] < +\infty \right\}$$

while: $L^{\widehat{\Phi}} = \{ g \mid E[\ (1 + |g|\ ) \ln(|g| + 1)\ ] < +\infty \} = M^{\widehat{\Phi}}$

Also,

$$\frac{dQ}{dP} \in L^{\widehat{\Phi}} \quad \text{iff} \quad H(Q,P) = E\left[\frac{dQ}{dP} \ln \left(\frac{dQ}{dP} \right) \right] < \infty.$$

References (general utility, selected topics)

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	$u:(a,\infty)\to\mathbb{R}$		$u:\mathbb{R}\to\mathbb{R}$ ($a=-\infty$)				
	S general		S loc. bdd		S general		
	$x\in\mathbb{R}$	$B\in L^0$	$x\in\mathbb{R}$	$B\in L^0$	$x\in\mathbb{R}$		$B\in L^0$
					$\widehat{M^u}$	$\widehat{L^u}$	$\widehat{L^u}$
Duality	KS99	CSW01	BF00	BF00	BF05	BF08	BFG08
Optimal	KS99	CSW01	S01	OZ07	BF05	BF08	BFG08
$(H^*.S)_T$	KS99	CSW01	S01	OZ07	BF05		
Supermart.			S03	OZ07	BF07		

KS99=Kramkov-Schachermayer; BF00=Bellini-Frittelli;
CSW01=Cvitanic-Schachermayer-Wang; S01=Schachermayer;
S03=Schachermayer; OZ07=Owen-Zitkovic;
BF05 & BF07 & BF08=Biagini-Frittelli;
BFG08=Biagini-Frittelli-Grasselli 2008.

Definition of $W-$admissible strategies

Let $W \in L^0_+(P)$ be a fixed positive random variable.

The $\mathbb{R}^d-$valued predictable $S-$integrable process H is $W-$admissible, or it belongs to $\mathcal{H}^W$, if there exists a $c \geq 0$ such that, $P-$ a.s.,

$$(H \cdot S)_t \geq -cW \quad \forall t \leq T.$$

The class of these $W-$admissible processes is denoted by

$$\mathcal{H}^W$$

Definition: W is suitable with S

The dual variables are going to be good pricing measures under the following condition on the random variable W:

W is $S-$suitable if $W \geq 1$ and for all $1 \leq i \leq d$ there exists a process $H^i \in L(S^i)$ such that:

- the paths of H^i a.s. never touch zero:

$$P(\{\omega \mid \exists t \geq 0 \, H^i_t(\omega) = 0\}) = 0$$

- for all $t \in [0, T]$,

$$-W \leq (H^i \cdot S^i)_t \leq W \qquad P - a.s.$$

Compatibility conditions

Let $W \in L^0$, $W \geq 0$ and consider the conditions

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Let $W \in L^0$, $W \geq 0$ and consider the conditions

1. $W \in L^\infty$
2. $\forall \alpha > 0 \; E[u(-\alpha W)] > -\infty \quad (W \in M^{\widehat{u}})$
3. $\exists \alpha > 0 \; E[u(-\alpha W)] > -\infty \quad (W \in L^{\widehat{u}})$

Obviously:

$$W \in L^\infty \Rightarrow W \in M^{\widehat{u}} \Rightarrow W \in L^{\widehat{u}}$$

Example for the condition $E[u(-\alpha W)] > -\infty$

One period market, single underlying $S = (S_0, S_1)$ with $S_0 = 0$ and

$$S_1 \text{ 2-sided exponentially distributed: } p_{S_1}(x) = \frac{1}{2}e^{-|x-1|}$$

Then S is non locally bounded and

$$(H \cdot S)_1 = hS_1, \quad h \in \mathbb{R}$$

Hence $\mathcal{H}^1 = \{0\}$.

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Then S is non locally bounded and

$$(H \cdot S)_1 = hS_1, \quad h \in \mathbb{R}$$

Hence $\mathcal{H}^1 = \{0\}$. But

$$\mathcal{H}^W = \mathbb{R} \text{ if we select } W = |S_1|.$$

If u is exponential, $W = |S_1|$ is (only) weakly compatible:

$$E[u(-\alpha W)] = \frac{1}{2}\int -e^{\alpha x}e^{-|x-1|}dx > -\infty \text{ only if } \alpha < 1 \text{ (and NOT } \forall\alpha)$$

Example: the optimal wealth increases if we maximize over $\mathcal{H}^W = \mathbb{R}$

$$\max_{h\in\mathcal{H}^W=\mathbb{R}} E\left[-e^{-(x+hS_1)}\right] = -e^{-x}\frac{1}{4h^*}e^{-h^*}$$

$$> \quad -e^{-x} = \max_{h\in\mathcal{H}^1=\{0\}} E\left[-e^{-(x+hS_1)}\right]$$

where $h^* = \sqrt{2}-1$ and

the optimal claim is $f_x = x + h^* S_1$

the optimal measure is $Q^* \propto e^{-h^* S_1}$

The assumption on W

There exists a suitable W, which is in $L^{\widehat{u}}$

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Remarks

- In the locally bounded case the constant 1 is suitable and trivially is in $L^{\widehat{u}}$ and so the assumption above is automatically satisfied $\Rightarrow$ This theory generalizes the locally bounded case !

The assumption on W

There exists a suitable W, which is in $L^{\widehat{u}}$

Remarks

- In the locally bounded case the constant 1 is suitable and trivially is in $L^{\widehat{u}}$ and so the assumption above is automatically satisfied $\Rightarrow$ This theory generalizes the locally bounded case !
- The compatibility condition $W \in L^{\widehat{u}}$ puts some restrictions on the jumps (as shown in the example).

The domain in the utility maximization problem

Fix a suitable $W \in L^{\widehat{u}}$ and set

$$K^W = \{(H \cdot S)_T \mid H \in \mathcal{H}^W\}, \qquad C^W \triangleq (K^W - L^0_+) \cap L^{\widehat{u}}$$

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Then

$$U^W(B) \triangleq \sup_{k \in K^W} E[u(B+k)] = \sup_{k \in C^W} E[u(B+k)]$$

We can formulate the maximization over the Banach lattice $L^{\widehat{u}}$ naturally induced by the problem!

$u : (-\infty, +\infty) \to \mathbb{R}$

To simplify the notations, from now on we assume that

$$u : (-\infty, +\infty) \to \mathbb{R},$$

but the results hold also for utility functions

$$u : (a, +\infty) \to \mathbb{R}$$

with $a < 0$ and finite.

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with $a < 0$ and finite.

To define the pricing measures we will need the polar of C^W:

$$\left(C^W\right)^\circ = \left\{z \in (L^{\widehat{u}})' \mid z(f) \le 0 \ \forall f \in C^W\right\}$$

and so we recall the dual $(L^{\widehat{u}})'$

On the dual $(L^{\widehat{u}})'$

¿From the general theory of Banach lattices

$$(L^{\widehat{u}})' = L^{\widehat{\Phi}} \oplus (M^{\widehat{u}})^{\perp}$$

($\widehat{\Phi}$ is the conjugate of $\widehat{u}$)

- $L^{\widehat{\Phi}}$ is the band of order-continuous linear functionals (the regular ones)
- $(M^{\widehat{u}})^{\perp}$ is the band of those singular ones, which are lattice orthogonal to the functionals in $L^{\widehat{\Phi}}$.

Decomposition of the dual space

$$(L^{\widehat{u}})' = L^{\widehat{\Phi}} \oplus (M^{\widehat{u}})^{\perp}$$

Hence: if $Q \in (L^{\widehat{u}})'$ then

$$Q = Q_r + Q_s,$$

with

$$|Q_r| \wedge |Q_s| = 0.$$

$$\frac{dQ_r}{dP} \in L^{\widehat{\Phi}} \subseteq L^1 \qquad Q_s(f) = 0 \ \forall f \in M^{\widehat{u}}.$$

The dual variables

$$\left(C^W\right)^\circ = \left\{Q \in (L^{\widehat{u}})'_+ \mid Q(f) \le 0 \ \forall f \in C^W\right\}$$

The set of pricing functionals is:

$$\mathcal{M}^W \triangleq \left\{Q \in \left(C^W\right)^\circ \mid Q(I_\Omega) = 1\right\}$$

$$= \left\{ \begin{array}{c} Q \in (L^{\widehat{u}})' \mid Q_r(I_\Omega) = 1 \text{ and } Q(f) \le 0 \\ \forall f \in L^{\widehat{u}} \text{ s.t. } f \le (H \cdot S)_T, \ H \in \mathcal{H}^W \end{array} \right\}.$$

On the dual variables

$$Q(I_\Omega) = 1 \text{ iff } Q_r(I_\Omega) = 1,$$

since Q_s is null over L^∞ and $Q(f) = E_{Q_r}[f] + Q_s(f)$

$$Q \in \mathcal{M}^W \Rightarrow Q_r \text{ is a true probability.}$$

On the dual variables

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$$Q \in \mathcal{M}^W \Rightarrow Q_r \text{ is a true probability.}$$

Lemma:
The norm of a *nonnegative* singular element $Q_s \in (M^{\widehat{u}})^\perp$ satisfies

$$\|Q_s\|_{(L^{\widehat{u}})^*} := \sup_{N_{\widehat{u}}(f) \leq 1} Q_s(f) = \sup_{f \in \mathrm{Dom}(I_u)} Q_s(-f).$$

$\sigma-$ martingale measures

Proposition:

$$\mathcal{M}^W \cap L^{\widehat{\Phi}} = M_\sigma \cap L^{\widehat{\Phi}}$$

i.e. the regular elements in $\mathcal{M}^W$ are exactly $M_\sigma \cap L^{\widehat{\Phi}}$ where

$$M_\sigma = \{Q \ll P \mid S \text{ is a } \sigma - \text{martingale w.r.to } Q\}$$

Definition:
The semimartingale S is a σ-martingale if there exist:

1) a d-dimensional martingale M

2) a positive (scalar) predictable process φ, which is M^i-integrable for all $i = 1 \cdots d$ and such that $S^i = \varphi \cdot M^i$.

The (weak) assumptions on the claim B

$$\sup_{H \in \mathcal{H}} E[u(-B + (H \cdot S)_T)]$$

We say that $B \in L^0(\mathcal{F}_T)$ is admissible if it satisfies

1. $E[u(-(1+\epsilon)B^+)] > -\infty$, for some $\epsilon > 0$,
2. $E[u(f - B)] < \infty$, for all $f \in L^{\widehat{u}}$

- We only require that $B^+ \in L^{\widehat{u}}$, not necessarily $B \in L^{\widehat{u}}$
- By Jensen inequality, if $B^- \in L^1$, then condition 2 holds
- If u is bounded from above (as the exponential utility), then the condition 2 is satisfied by any claims.

Assumptions

In all subsequent results it is assumed that:

- $u : \mathbb{R} \to \mathbb{R}$ is concave, increasing (not constant on $\mathbb{R}$)
- $W \in L^{\widehat{u}}$ is suitable
- B is admissible.

Theorem

Suppose that W satisfies

$$U^W_{-B} := \sup_{H \in \mathcal{H}^W} E[u(-B + (H \cdot S)_T] < u(+\infty).$$

Then $\mathcal{M}^W$ is not empty and

$$\sup_{H \in \mathcal{H}^W} E[u(-B + (H \cdot S)_T)]$$

$$= \min_{\lambda > 0,\ Q \in \mathcal{M}^W} \left\{ \lambda \widehat{Q}(-B) + E\left[\Phi\left(\lambda \frac{dQ_r}{dP}\right)\right] + \lambda \| Q_s \| \right\},$$

If $W \in M^{\widehat{u}}$ and $B \in M^{\widehat{u}}$ then $\mathcal{M}^W$ can be replaced by $\mathbb{M}_\sigma \cap L^{\widehat{\Phi}}$ and no singular terms appear.

Further results

Under stronger regularity conditions on the utility u and on W there are results also on:

- the existence of the optimal solution to the primal utility maximization problem,
- on the representation of the optimal solution as a stochastic integral,
- on the supermartingale property of the optimal wealth process.

Indifference price

The real number $\pi(B)$ solution of the equation

$$\sup_{H\in\mathcal{H}^W} Eu(x+(H\cdot S)_T) = \sup_{H\in\mathcal{H}^W} Eu(x+\pi(B)-B+(H\cdot S)_T)$$

is called the seller indifference price of the claim B

¿From the previous results, we now obtain the dual representation of $\pi(B)$ and we show that the **indifference price is, except for the sign, a convex risk measures on** $L^{\widehat{u}}$

The domain of the indifference price functional

Define: $I_u(f) = E[u(f)]$ and

$$\mathcal{B} := \left\{ B \in L^{\widehat{u}} \mid \exists \epsilon > 0 : Eu(-(1+\epsilon)B^+) > -\infty \right\}$$

LEMMA:

$$\mathcal{B} = \{ B \in L^{\widehat{u}} \mid (-B) \in \operatorname{int}(\mathrm{Dom}(I_u)) \}$$

is an open convex set in $L^{\widehat{u}}$ containing $M^{\widehat{u}}$ and thus L^∞

Proposition

If the initial wealth $x \in \mathbb{R}$ satisfies $U_x^W = \sup_{H \in \mathcal{H}^W} E[u(x + (H \cdot S)_T] < u(+\infty)$, then the seller's indifference price

$$\pi : \mathcal{B} \to \mathbb{R}$$

π is well a defined, norm continuous, subdifferentiable, convex, monotone, translation invariance map on $\mathcal{B}$ and it admits the representation

$$\pi(B) = \max_{Q \in \mathcal{M}^W} \{Q(B) - \alpha(Q)\}$$

where the (minimal) penalty term $\alpha(Q)$ is given by

$$\alpha(Q) = x + \|Q_s\| + \inf_{\lambda > 0} \left\{ \frac{E[\Phi(\lambda \frac{dQ_r}{dP})] - U_x^W}{\lambda} \right\}.$$

Corollary

If $U_x^W < u(+\infty)$, the seller's indifference price π defines a convex risk measure $\rho(B) = \pi(-B)$ on $\mathcal{B}$, with the following representation:

$$\rho(B) = \pi(-B) = \max_{Q \in \mathcal{M}^W} \{Q(-B) - \alpha(Q)\}.$$

If both the loss control W and the claim B are in $M^{\widehat{u}}$, then no singular terms appear in the representation and this risk measure has the Fatou property. In terms of π, this means

$$B_n \uparrow B \Rightarrow \pi(B_n) \uparrow \pi(B)$$

Example: set $u(x) = -e^{-\gamma x}$, $\gamma > 0$

PROPOSITION:
Suppose that $B \in L^{\widehat{u}}$ satisfies $E[e^{\gamma(1+\epsilon)B^+}] < \infty$, for some $\varepsilon > 0$, and $W \in L^{\widehat{u}}$ is suitable. If $\mathcal{M}^W$ is not empty
then the indifference price is

$$\pi_\gamma(B) = \max_{Q \in \mathcal{M}^W} \left\{ Q(B) - \frac{1}{\gamma} \mathbb{H}(Q, P) \right\}$$

where the penalty term is given by

$$\mathbb{H}(Q, P) = H(Q_r, P) + \gamma \|Q_s\|_P - \min_{Q \in \mathcal{M}^W} \{H(Q_r, P) + \gamma \|Q_s\|_P\}$$

Continuing the Example: $u(x) = -e^{-\gamma x}, \gamma > 0$

If both the loss control W and the claim B are in $M^{\widehat{u}}$, then no singular terms appear in the representation and

$$\rho_\gamma(B) := \pi_\gamma(-B) = \max_{Q \in M_\sigma \cap L^{\widehat{\Phi}}} \{E_Q[-B] - \frac{1}{\gamma}\mathbb{H}(Q,P)\}$$

where

$$\mathbb{H}(Q,P) = H(Q,P) - \min_{Q \in M_\sigma \cap L^{\widehat{\Phi}}} \{H(Q,P)\}$$

is a convex risk measures on Orlicz space

New Assumption

Assumption

The utility function $u : \mathbb{R} \to \mathbb{R}$ is strictly increasing, strictly concave, continuously differentiable and

$$\lim_{x \downarrow -\infty} u'(x) = +\infty, \quad \lim_{x \uparrow \infty} u'(x) = 0 \qquad \textit{(Inada conditions)}$$

Moreover, for any probability $Q \ll P$, the conjugate function Φ satisfies

$$E\left[\Phi\left(\frac{dQ}{dP}\right)\right] < +\infty \textit{ iff } E\left[\Phi\left(\lambda\frac{dQ}{dP}\right)\right] < +\infty \textit{ for all } \lambda > 0$$

The new domain of the optimization problem

Definition

$$\mathcal{M}_\Phi^W := \{Q \in \mathcal{M}^W \mid E\left[\Phi\left(\frac{dQ^r}{dP}\right)\right] < +\infty\}$$

$$K_B^W := \{f \in L^0 \mid f \in L^1(Q^r),\ E_{Q^r}[f] \le Q^s(-B) + \|Q^s\|,\ \forall Q \in \mathcal{M}_\Phi^W\},$$

and the corresponding optimization problem

$$U_B^W := \sup_{f \in K_B^W} E[u(f - B)].$$

Existence of the optimal solution

Theorem

The maximum U_B^W is attained over K_B^W and the unique maximizer is

$$f_B = -\Phi'(\lambda_B \frac{dQ_B^r}{dP}) + B.$$

The relation between primal and dual optimizers is given by:

$$E_{Q_B^r}[f_B] = Q_B^s(-B) + \|Q_B^s\|.$$

The next proposition gives *a priori* bounds for this singular contribution $Q^s(-B)$ appearing in the duality relation:

Proposition

For any $B \in \mathcal{B}$, *let*

$L := \sup\{\beta > 0 \mid E[\widehat{u}(\beta B^+)] < +\infty\}$ *and* $l := \sup\{\alpha > 0 \mid E[\widehat{u}(\alpha B^-)] <$

Then, for any fixed $Q \in \mathcal{M}_\Phi^W$,

$$-\frac{1}{L}\|Q^s\| \leq Q^s(-B) \leq \frac{1}{l}\|Q^s\|$$

and in particular we recover again $Q^s(B) = 0$ *when* $B \in M^{\widehat{u}}$.

On the representation as stochastic integral

The result in the Theorem does not guarantee in full generality that the optimal random variable $f_B \in K_B^W$ can be represented as terminal value from an investment strategy in $L(S)$, that is, $f_B = \int_0^T H_t dS_t$.

The next proposition presents a partial result in this direction.

Proposition

Suppose that $B \in \mathcal{B}$, $Q_B^s = 0$ and $Q_B^r \sim P$. Then f_B can be represented as terminal wealth from a suitable strategy H.