Lectures for the Course on Foundations of Mathematical Finance

First Part: Convex Risk Measures

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Outline Lectures

- Convex Risk Measures on L^∞
- For a nice dual representation we need $\sigma(L^\infty, L^1)$ lower semicontinuity
- For Convex Risk Measures on more general spaces what replaces this continuity assumption?
- Which are these more general spaces and why are they needed?
- Orlicz spaces associated to a utility function
- How do we associate a Risk Measure to a utility function
- Utility maximization with unbounded price process (here Orlicz spaces are needed) and with random endowment

...Outline Lectures

- Indifference price as a convex risk measure on the Orlicz space associated to the utility function
- Weaker assumption than convexity: Quasiconvexity
- Quasiconvex (and convex) dynamic risk measures: their dual representation
- The conditional version of the Certainty Equivalent as a quasiconvex map
- The need of (dynamic and stochastic) generalized Orlicz space
- The dual representation of the Conditional Certainty Equivalent

Lectures based on the following papers

- First Part
  On the extension of the Namioka-Klee theorem and on the Fatou property for risk measures,
  Joint with **S. Biagini (2009),** In: Optimality and risk: modern trends in mathematical finance. The Kabanov Festschrift.
- Second Part
  1. *A unified framework for utility maximization problems: an Orlicz space approach*,
     Joint with **S. Biagini (2008)**, Annals of Applied Probability.
  2. *Indifference Price with general semimartingales*
     Joint with **S. Biagini and M. Grasselli (2009)**, Forthcoming on Mathematical Finance

...Lectures based on the following papers

- Third Part
  1. *Dual representation of Quasiconvex Conditional maps,*
     Joint with **M. Maggis (2009)**, Preprint
  2. *Conditional Certainty Equivalent,*
     Joint with **M. Maggis (2009)**, Preprint

Notations

- $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space.
- $L^0 = L^0(\Omega, \mathcal{F}, \mathbb{P})$ is the space of all $\mathbb{P}$.a.s. finite random variables.
- L^∞ is the subspace of all essentially bounded random variables.
- $L(\Omega)$ is a linear subspace of L^0.
- We suppose that
$$L^\infty \subseteq L(\Omega) \subseteq L^0.$$
- We adopt in $L(\Omega)$ the $\mathbb{P}$-a.s. usual order relation between random variables.

Monetary Risk Measure

Definition

A map $\rho : L(\Omega) \to \mathbb{R}$ is called a *monetary risk measure* on $L(\Omega)$ if it has the following properties:

(*CA*) Cash additivity :
$\forall\ x \in L(\Omega)$ and $\forall\ c \in \mathbb{R}\ \ \rho(x + c) = \rho(x) - c$.

(*M*) Monotonicity :
$\rho(x) \leq \rho(y)\ \forall\ x, y \in L(\Omega)$ such that $x \geq y$.

(*G*) Grounded property : $\rho(\mathbf{0}) = 0$.

Acceptance set

Definition

Let $\rho : L(\Omega) \to \mathbb{R}$ be a monetary risk measure. The set of acceptable positions is defined as

$$\mathcal{A}_\rho := \{x \in L(\Omega) : \rho(x) \leq 0\}.$$

Remark

$\rho : L(\Omega) \to \mathbb{R}$ satisfies cash additivity (CA) if and only if there exists a set $A \subseteq L(\Omega)$:

$$\rho(x) = \inf\{a \in \mathbb{R} : a + x \in A\}.$$

Coherent and Convex Risk Measures

Coherent and convex risk measures are monetary risk measures which satisfies some of the following properties:

- (CO) Convexity : For all $\lambda \in [0,1]$ and for all $x, y \in L(\Omega)$ we have that $\rho(\lambda x + (1-\lambda)y) \leq \lambda\rho(x) + (1-\lambda)\rho(y)$.
- (SA) Subadditivity : $\rho(x+y) \leq \rho(x) + \rho(y) \;\; \forall\; x, y \in L(\Omega)$.
- (PH) Positive homogeneity : $\rho(\lambda x) = \lambda\rho(x) \;\forall\; x \in L(\Omega)$ and $\forall\; \lambda \geq 0$.

Proposition

Let $\rho : L(\Omega) \to \mathbb{R}$ *and* $\mathcal{A}_\rho = \{x \in L(\Omega) : \rho(x) \leq 0\}$. *Then the following properties hold true:*

- (a) *If* ρ *satisfies* (CO) *then* $\mathcal{A}_\rho$ *is a convex set.*
- (b) *If* ρ *satisfies* (PH) *then* $\mathcal{A}_\rho$ *is a cone.*
- (c) *If* ρ *satisfies* (M) *then* $\mathcal{A}_\rho$ *is a monotone set, i.e.* $x \in \mathcal{A}_\rho, y \geq x \Rightarrow y \in \mathcal{A}_\rho$.

Viceversa, let $A \subseteq L(\Omega)$ *and* $\rho_A(x) := \inf\{a \in \mathbb{R} : a + x \in A\}$. *Then the following properties hold true:*

- (a′) *If* A *is a convex set then* ρ_A *satisfies* (CO).
- (b′) *If* A *is a cone then* ρ_A *satisfies* (PH).
- (c′) *If* A *is a monotone set then* ρ_A *satisfies* (M).

Dual System

- Let X and X' be two vector lattices and $< \cdot, \cdot >: X \times X' \to \mathbb{R}$ a bilinear form.
- This dual system $(X, X', < \cdot, \cdot >)$ will be denoted simply with (X, X').
- We consider the locally convex topological vector space (X, τ), where τ is a topology compatible with the dual system (X, X'), so that
- the topological dual space $(X, \tau)'$ coincides with X', i.e:

$$(X, \tau)' = X'$$

- Examples:
  - $(L^\infty, \sigma(L^\infty, L^1))' = L^1$; $< x, x' >= E[xx']$
  - $(L^\infty, \| \cdot \|_\infty)' = ba$; $< x, x' >= x'(x)$
  - $(L^p, \| \cdot \|_p)' = L^q$, $p \in [1, \infty)$; $< x, x' >= E[xx']$

Convex conjugate

The convex conjugate f^* and the convex bi-conjugate f^{**} of a convex function $f : X \to \mathbb{R} \cup \{\infty\}$ with $\mathrm{Dom}(f) := \{x \in X : f(x) < +\infty\} \neq \emptyset$ are given by

- $f^* : X' \to \mathbb{R} \cup \{\infty\}$

$$f^*(x') := \sup_{x \in X}\{x'(x) - f(x)\} = \sup_{x \in \mathrm{Dom}(f)}\{x'(x) - f(x)\}$$

- $f^{**} : X \to \mathbb{R} \cup \{\infty\}$

$$f^{**}(x) := \sup_{x' \in X'}\{x'(x) - f^*(x')\} = \sup_{x' \in \mathrm{Dom}(f^*)}\{x'(x) - f^*(x')\}$$

One of the main results about convex conjugates is the Fenchel-Moreau theorem.

Theorem (Fenchel-Moreau)

Let $f : X \to \mathbb{R} \cup \{\infty\}$ be a convex lower semi-continuous function with $Dom(f) \neq \emptyset$. Then

$$f(x) = f^{**}(x), \ \ \forall\, x \in X,$$

$$f(x) = \sup_{x' \in X'} \{x'(x) - f^*(x')\} \ \ \forall\, x \in X.$$

A simple application of this theorem gives the dual representation of risk measures. Set first:

$$\mathcal{Z} := \{x' \in X'_+ : x'(\mathbf{1}) = 1\}$$

Proposition

Let $f : X \to \mathbb{R} \cup \{\infty\}$ *be a convex lower semi-continuous function with* $Dom(f) \neq \emptyset$*. Then there exists a convex, lower semi-continuous function* $\alpha : X' \to \mathbb{R} \cup \{\infty\}$ *with* $\mathcal{D} := Dom(\alpha) \neq \emptyset$ *such that*

$$f(x) = \sup_{x' \in \mathcal{D}} \{x'(x) - \alpha(x')\} \;\; \forall\, x \in X.$$

(M) *If* f *is monotone (i.e.* $x \leq y \Longrightarrow f(x) \leq f(y)$*) then* $\mathcal{D} \subseteq X'_+$.

(CA) *If* $f(x + c) = f(x) + c \;\; \forall\, c \in \mathbb{R}$*, then* $\mathcal{D} \subseteq \{x' \in X' : x'(\mathbf{1}) = 1\}$.

(M + CA) *If* f *satisfies both (M) and (CA) then* $\mathcal{D} \subseteq \mathcal{Z}$.

(PH) *If* $f(\lambda x) = \lambda f(x) \;\; \forall\, \lambda \geq 0$ *then* $\alpha(x') = \begin{cases} 0 & x' \in \mathcal{D} \\ +\infty & x' \notin \mathcal{D} \end{cases}$

Representation of convex risk measures on L^∞

- A monetary risk measure on $L^\infty = L^\infty(\Omega, \mathcal{F}, P)$ is norm continuous
- $(L^\infty, \|\cdot\|)' = ba := ba(\Omega, \mathcal{F}, P)$,
- Therefore, applying the previous Proposition to a convex risk measure (a monotone decreasing, cash additive, convex map)

$$\rho : L^\infty(\Omega, \mathcal{F}, P) \to \mathbb{R}$$

we immediately deduce the following Proposition:

Representation of convex risk measures on L^∞

Proposition

A convex risk measure $\rho : L^\infty(\Omega, \mathcal{F}, P) \to \mathbb{R}$ *admits the representation*

$$\rho(x) = \sup_{\mu \in ba_+(1)} \{\mu(-x) - \alpha(\mu)\}$$

where

$$ba_+(1) = \{\mu \in ba \mid \mu \geq 0,\ \mu(1_\Omega) = 1\},$$

If in addition ρ *is* $\sigma(L^\infty, L^1)$ *-LSC then*

$$\rho(x) = \sup_{Q << P} \{E_Q[-x] - \alpha(Q)\}$$

Generalization

- We will need a representation result for convex maps defined on more general spaces: Examples of these spaces are the Orlicz spaces
- Therefore, we need to understand under which conditions we may deduce a representation of the type:

$$\rho(x) = \sup_{Q} \{E_Q[-x] - \alpha(Q)\}$$

- We will need a sort of LSC condition
- For this we need to introduce some terminology and results from Banach lattices theory.

How and why Orlicz spaces

Very simple observation: u is concave, so the steepest behavior is on the left tail.
Economically, this reflects the risk aversion of the agent: the losses are weighted in a more severe way than the gains.

We will turn the left tail of u into a Young function $\widehat{u}$.
Then, $\widehat{u}$ gives rise to an Orlicz space $L^{\widehat{u}}$, naturally associated to our problem, which allows for an **unified treatment of Utility Maximization**.

Orlicz function spaces on a probability space

A Young function Ψ is an even, convex function

$$\Psi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$$

with the properties:

1- $\Psi(0) = 0$

2- $\Psi(\infty) = +\infty$

3- $\Psi < +\infty$ in a neighborhood of 0.

The Orlicz space L^{Ψ} on $(\Omega, \mathcal{F}, P)$ is then

$$L^{\Psi} = \{x \in L^0 \mid \exists \alpha > 0 \, E[\Psi(\alpha x)] < +\infty\}$$

It is a Banach space with the gauge norm

$$\|x\|_{\Psi} = \inf \left\{ c > 0 \mid E\left[\Psi\left(\frac{x}{c}\right)\right] \leq 1 \right\}$$

The subspace $\mathbf{M}^\Psi$

Consider the closed subspace of L^Ψ

$$M^\Psi = \{x \in L^0(P) \mid E[\Psi(\alpha x)] < +\infty \ \forall \alpha > 0\}$$

- In general,

  $$M^\Psi \subsetneq L^\Psi.$$

  When Ψ is continuous on $\mathbb{R}$, we have $M^\Psi = \overline{L^\infty}^\Psi$
- But when Ψ satisfies the Δ_2 growth-condition (as il the L^p case) the two spaces coincide:

  $$M^\Psi = L^\Psi.$$

  and $L^\Psi = \{x \mid E[\Psi(x)] < +\infty\} = \overline{L^\infty}^\Psi$.

Generalities

A **Frechet lattice** (E, τ) is a completely metrizable locally solid Riesz space - not necessarily locally convex.

Examples: $L^p(\Omega, \mathcal{F}, P)$, $p \in [0, 1)$ with the usual metric and pointwise order.

A **Banach lattice** $(E, \|\|)$ is a Banach Riesz space with monotone norm, i.e.

$$|y| \leq |x| \Rightarrow \|y\| \leq \|x\|$$

Examples: $L^p(\Omega, \mathcal{F}, P)$, $p \in [1, \infty]$, as well as all the Orlicz spaces $L^\Psi(\Omega, \mathcal{F}, P)$.

The extension of Namioka-Klee Theorem

Namioka-Klee Theorem, 1957: Every linear and positive functional on a Frechet lattice is continuous.

Extended Namioka-Klee Theorem (ExNKT) [Biagini-F.06]: Every convex and monotone functional $\pi : E \to \mathbb{R} \cup \{+\infty\}$ on a Frechet lattice is continuous on the interior of $\mathrm{Dom}(\pi)$.

Linearity $\to$ convexity
Positivity $\to$ monotonicity

See also: **Ruszczynki - Shapiro**, Optimization of Convex Risk Function, Math. Op. Res. 31/3 2006.)

Fenchel and Extended Namioka Theorem imply the representation wrt the topological dual E'

From now on X denotes a Riesz Space (a vector Lattice) and (E, τ) denotes a locally convex Frechet lattice and E' denotes the topological dual of (E, τ).

Proposition If $\pi : E \to \mathbb{R}$ is convex and monotone (increasing) then

$$\pi(x) = \sup_{x' \in E'_+} \{\langle x', x\rangle - \pi^*(x')\}$$

Fenchel and Extended Namioka Theorem imply the representation wrt the topological dual E'

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Proposition If $\pi : E \to \mathbb{R}$ is convex and monotone (increasing) then

$$\pi(x) = \sup_{x' \in E'_+} \{\langle x', x \rangle - \pi^*(x')\}$$

- However, we are looking for a representation in terms of the order continuous dual E^c
  When $E = L^\infty$, $E' = ba = L^1 \oplus E^d$ (E^d are the **pure charges**) we want a representation in terms of L^1.

Order convergence

A net $\{x_\alpha\}_\alpha$ in X is order convergent to $x \in X$, written $x_\alpha \xrightarrow{o} x$, if there is a decreasing net $\{y_\alpha\}_\alpha$ in X satisfying $y_\alpha \downarrow 0$ and $|x_\alpha - x| \leq y_\alpha$ for each α, i.e.:

$$x_\alpha \xrightarrow{o} x \ \text{ iff } \ |x_\alpha - x| \leq y_\alpha \downarrow 0$$

Example: Let $X = L^p(\Omega, \mathcal{F}, P)$, $p \in [0, \infty]$. Then

$$x_\alpha \xrightarrow{o} x \quad \Leftrightarrow \quad x_\alpha \xrightarrow{P-a.s.} x \quad \text{and } \{x_\alpha\} \text{ is dominated in } L^p$$

Continuity wrt the order convergence

A functional $f : X \to \mathbb{R} \cup \{+\infty\}$ is order continuous if

$$x_\alpha \overset{o}{\to} x \Rightarrow f(x_\alpha) \to f(x)$$

$f : X \to \mathbb{R} \cup \{\infty\}$ is order lower semicontinuous if

$$x_\alpha \overset{o}{\to} x \Rightarrow f(x) \leq \liminf f(x_\alpha).$$

Some definitions about Riesz Spaces

Recall that:

- The space X is order separable if any subset A which admits a supremum in X contains a countable subset with the same supremum.
- The space X is order complete when each order bounded subset A has a supremum (least upper bound) and an infimum (largest lower bound).

Order separability allows to work with sequences instead of nets

What is the Fatou property?

Proposition: Let X be order complete and order separable Riesz space. If $\pi : X \to \mathbb{R} \cup \{+\infty\}$ is increasing, TFAE

1-π is order-l.s.c.: $x_\alpha \xrightarrow{o} x \Rightarrow \pi(x) \leq \liminf \pi(x_\alpha)$.

2- π is continuous from below: $x_\alpha \uparrow x \Rightarrow \pi(x_\alpha) \uparrow \pi(x)$.

3- π has the Fatou property: $x_n \xrightarrow{o} x \Rightarrow \pi(x) \leq \liminf \pi(x_n)$.

Moral: the Fatou property for monotone increasing functionals is nothing but order lower semicontinuity!

Note: All the Orlicz spaces $L^\Psi(\Omega, \mathcal{F}, P)$ (and hence all $L^p(\Omega, \mathcal{F}, P)$, $p \in [1, \infty]$) satisfy the requirements on X.

The order continuous dual

For Frechet lattices the dual space E' admits the decomposition $E' = E^c \oplus E^d$, where E^c is the band of all order continuous linear functionals and

$$E^d = \left\{ x' \in E' \mid |x'| \wedge |y'| = 0 \text{ for all } y' \in E^c \right\}$$

is the disjoint complement of E^c, also called the band of singular functionals.

Examples:

$E = L^\infty$, $E' = ba = L^1 \oplus E^d$ (E^d are the **pure charges**)
$E = L^\Psi$, $E' = L^{\Psi^*} \oplus E^d$, ($\Psi^*$ is the convex conjugate of Ψ)
$E = L^\Psi$, $E' = L^{\Psi^*} \oplus \{0\}$ if Ψ satisfies Δ_2,
$E = L^p$, $E' = L^q \oplus \{0\}$, $p \in [1, \infty)$,

The representation in terms of E^c

If we want a representation of π wrt to the order continuous functionals:

$$\pi(x) = \sup_{x' \in (E^c)_+} \left\{ \langle x', x \rangle - \pi^*(x') \right\},$$

then we must require that:

- π is order l.s.c.
- a mild "compatibility" condition (a Komlos subsequence property) between the topology and the order structure holds.

On Komlos subsequence type property

Definition: A topology τ on a Riesz space X has the C property if $x_\alpha \xrightarrow{\tau} x$ implies that there exist a subsequence $\{x_{\alpha_n}\}_n$ and convex combinations $y_n \in conv(x_{\alpha_n}, \cdots)$ s.t. $y_n \xrightarrow{o} x$. That is

$$x_\alpha \xrightarrow{\tau} x \Rightarrow y_n \xrightarrow{o} x \quad \text{for some} \quad y_n \in conv(x_{\alpha_n}, \ldots).$$

On the representation of order l.s.c. functionals

Proposition: Suppose that the topology $\sigma(E, E^c)$ has the C property and that $\pi : E \to \mathbb{R} \cup \{\infty\}$ is proper, convex and increasing. The following are equivalent:

1- π is $\sigma(E, E^c)-$**lower semicontinuous**

2- π admits the representation on $(E^c)_+$

$$\pi(x) = \sup_{x' \in (E^c)_+} \left\{ \langle x', x \rangle - \pi^*(x') \right\}$$

3- π is **order lower semicontinuous** (or it has Fatou prop.).

On sufficient conditions for the C property

Proposition: In any locally convex Frechet lattice E such that

$$(E, \tau) \hookrightarrow (L^1, \|\cdot\|_{L^1}),$$

with a linear lattice embedding, the topology

$$\sigma(E, E^c)$$

has the C property.

The case of Orlicz spaces

If $L^\Psi = L^\Psi(\Omega, \mathcal{F}, P)$ is an Orlicz space, then

$$(L^\Psi, \|\cdot\|_\Psi) \hookrightarrow (L^1, \|\cdot\|_{L^1})$$

So, the topology

$$\sigma(L^\Psi, L^{\Psi^*}) = \sigma(L^\Psi, (L^\Psi)^c)$$

satisfies the C property

Here Ψ^* is the convex conjugate of Ψ and L^{Ψ^*} is the Orlicz space associated to the Young function Ψ^*.

Risk measures on Orlicz spaces

Proposition: For a convex risk measure (monotone decreasing, convex, cash additive)

$$\rho : L^{\Psi} \to \mathbb{R} \cup \{+\infty\}$$

the following are equivalent:

1. ρ is $\sigma(L^{\Psi}, L^{\Psi^*})$ l.s.c.
2. ρ can be represented as

$$\rho(x) = \sup_{Q \in \mathcal{D}} \{E_Q[-x] - \alpha(Q)\}, \ \mathcal{D} = \{\frac{dQ}{dP} \in L^{\Psi^*}_+ \mid E[\frac{dQ}{dP}] = 1\}$$

3. ρ is order l.s.c. (or it has the Fatou property)

A counter-example

π is order-l.s.c. $\not\Rightarrow \pi$ is $\sigma(E, E^c)-$l.s.c.
when the weak topology hasn't the C property

- $E = C([0,1])$, with the supremum norm and pointwise order $y \geq x$ iff $y(t) \geq x(t)\ \forall t \in [0,1]$
- E is a Banach lattice
- $E^c = \{0\}$, E' are the Borel signed measures, $E' = \{0\} \oplus E^d$
- Therefore $\sigma(E, E^c) = \sigma(E, 0)$ doesn't satisfy the C property!!

Example continues

- Let $\pi : E \to \mathbb{R}$,
$$\pi(x) = \max_{t \in [0,1]} x(t)$$
- π is increasing and convex $\Rightarrow$ norm continuous
- Hence $\pi(x) = \sup_{x' \in (E')_+} \{\langle x', x \rangle - \pi^*(x')\}$
- but π does not admit a representation in terms of $E^c = \{0\}$, since it is not constant.
- Hence it is not $\sigma(E, E^c)-$l.s.c.
- However, we can easily prove that π is order l.s.c.

On continuity from below

- We now analyze the equivalent formulations of continuity from below for maps between Riesz spaces

$$f : X \to Y$$

(we replaced $\mathbb{R}$ with Y).
- This will be useful when we consider conditional risk measures

$$\rho : L(\Omega, \mathcal{F}_t, P) \to L(\Omega, \mathcal{F}_s, P)$$

- The equivalent formulations holds for monotone maps that are quasiconvex, not necessarily convex.

Definition

For a map $f : X \to Y$ consider the lower level set

$$\mathcal{A}_y := \{x \in X \mid f(x) \leq y\},\ y \in Y.$$

Properties

The map $f : X \to Y$ is said to be

- (MON) monotone increasing if $x_1 \leq x_2 \Rightarrow f(x_1) \leq f(x_2)$
- (QCO) quasiconvex if for every $y \in Y$ the set $\mathcal{A}_y$ is convex
- ($\tau-$LSC) $\tau-$lower semicontinuous if for every $y \in Y$ the set $\mathcal{A}_y$ is τ closed, where τ is a linear topology on X.
- ($o-$LSC) order lower semicontinuous if for every $x_\alpha, x \in X$

$$x_\alpha \stackrel{o}{\to} x \quad \Rightarrow \quad f(x) \leq \liminf f(x_\alpha)$$

- (CFB) continuous from below if for every $x_\alpha, x \in X$

$$x_\alpha \uparrow x \quad \Rightarrow \quad f(x_\alpha) \uparrow f(x).$$

Proposition (Equivalent formulations of CFB)

Suppose that X and Y are two Riesz spaces, that X is order complete and order separable and that $f : X \to Y$.
Then if the topology $\sigma(X, X^c)$ satisfies the $C-$property and if f is (MON) and (QCO):

$$(\sigma(X, X^c) - \text{LSC}) \Leftrightarrow (o - \text{LSC}) \Leftrightarrow (\text{CFB}).$$