Chapter 4

Structural Models of Credit Risk

Broadly speaking, credit risk concerns the possibility of financial losses due to changes in the credit quality of market participants. The most radical change in credit quality is a *default event*. Operationally, for medium to large cap firms, default is normally triggered by a failure of the firm to meet its debt servicing obligations, which usually quickly leads to bankruptcy proceedings, such as Chapter 11 in the U.S. Thus default is considered a rare and singular event after which the firm ceases to operate as a viable concern, and which results in *large* financial losses to *some* security holders. With some flexible thinking, this view of credit risk also extends to sovereign bonds issued by countries with a non-negligible risk of default, such as those of developing countries.

Under structural models, a default event is deemed to occur for a firm when its assets reach a sufficiently low level compared to its liabilities. These models require strong assumptions on the dynamics of the firm's asset, its debt and how its capital is structured. The main advantage of structural models is that they provide an intuitive picture, as well as an endogenous explanation for default. We will discuss other advantages and some of their disadvantages in what follows.

4.1 The Merton Model (1974)

The Merton model takes an overly simple debt structure, and assumes that the total value A_t of a firm's assets follows a geometric Brownian motion under the physical measure

$$dA_t = \mu A_t dt + \sigma A_t dW_t, \quad A_0 > 0, \tag{4.1}$$

where μ is the mean rate of return on the assets and σ is the asset volatility. We also need further assumptions: there are no bankruptcy charges, meaning the liquidation value equals the firm value; the debt and equity are frictionless tradeable assets.

Large and medium cap firms are funded by shares ("equity") and bonds ("debt"). The Merton model assumes that debt consists of a single outstanding bond with face value K and maturity T. At maturity, if the total value of the assets is greater than the debt, the latter is paid in full and the remainder is distributed among shareholders. However, if

$A_T < K$ then default is deemed to occur: the bondholders exercise a debt covenant giving them the right to liquidate the firm and receive the liquidation value (equal to the total firm value since there are no bankruptcy costs) in lieu of the debt. Shareholders receive nothing in this case, but by the principle of limited liability are not required to inject any additional funds to pay for the debt.

From these simple observations, we see that shareholders have a cash flow at T equal to

$$(A_T - K)^+,$$

and so equity can be viewed as a European call option on the firm's assets. On the other hand, the bondholder receives $\min(A_T, K)$. Moreover, the physical probability of default at time T, measured at time t, is

$$P_t[\tau = T] = P_t[A_T \le K] = N[-d_2^P]$$

where $d_2^P = (\sigma\sqrt{T-t})^{-1}(\log(A_t/K) + (\mu - \sigma^2/2)(T-t))$.

The value E_t at earlier times $t < T$ can be derived using the classic martingale argument (see exercise 24 for an alternative derivation). Assuming one can trade the firm value A_t, we note that $e^{-rt}A_t$ is a martingale under the risk-neutral measure Q with market price of risk $\phi = (\mu - r)/\sigma$ and Radon-Nikodym derivative

$$\frac{dQ}{dP} = \exp\left(\phi W_T - \frac{1}{2}\phi^2 T\right). \tag{4.2}$$

Then we find the standard Black-Scholes call option formula

$$\begin{aligned} E_t &= E^Q[e^{-r(T-t)}(A_T - K)^+] = \mathrm{BSCall}(A_t, K, r, \sigma, T-t) && (4.3) \\ &= A_t N[d_1] - e^{-r(T-t)} K N[d_2] && (4.4) \end{aligned}$$

where

$$d_1 = \frac{\log(A_t/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = \frac{\log(A_t/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \tag{4.5}$$

Bond holders, on the other hand, receive

$$\min(K, A_T) = A_T - (A_T - K)^+ = K - (K - A_T)^+.$$

Therefore the value D_t for the debt at earlier times $t < T$ can be obtained as the value of a zero-coupon bond minus a European put option. Of course the fundamental identity of accounting holds:

$$A_t = E_t + D_t,$$

and all three assets are discounted risk neutral martingales. A zero coupon defaultable bond with face value 1 and maturity T will have the price $\bar{P}_t(T) = D_t/K$, and has the yield spread

$$YS_t(T) = \frac{1}{T-t}\log\frac{Ke^{-r(T-t)}}{D_t} = -\frac{1}{T-t}\log\left(e^{r(T-t)}\frac{A_t}{K}(1 - N[d_1]) + N[d_2].\right) \tag{4.6}$$

Despite being derived from a debt structured with a single maturity T, this equation is often interpreted as giving a function of T: it is imagined that if an additional bond of small face value with a different maturity were issued by the firm, it would also be priced according to (4.6). The qualitative behaviour of this term structure is that credit spreads start at zero for $T = 0$, increase sharply to a maximum, and then decrease either to zero at large times if $r - \sigma^2/2 \leq 0$ or a positive value if $r - \sigma^2/2 > 0$. This is in accordance with the diffusive character of the model. For very short maturity times, the asset price diffusion will almost surely never cross the default barrier. The probability density of default then increases for longer maturities but starts to decrease again as the geometric Brownian motion drifts away from the barrier.

This behaviour is also observed in first passage and excursion models, except that spreads exhibit a faster decrease for longer maturities. It is at odds with empirical observations in two respects: (i) observed spreads remain positive even for small time horizons and (ii) tend to increase as the time horizon increases. The first feature follows from the fact that there is always a small probability of immediate default. The second is a consequence of greater uncertainty for longer time horizons. One of the main reasons to study reduced-form models is that, as we will see, they can easily avoid such discrepancies.

The previously obtained formula for the physical default probability (that is under the measure P) can be used to calculate risk neutral default probability provided we replace μ by r. Thus one finds that

$$Q[\tau > T] = N\left(N^{-1}(P[\tau > T]) - \phi\sqrt{T}\right).$$

and as long as $\phi > 0$ we see that market implied (i.e. risk neutral) survival probabilities are always less than historical ones.

In the event of default, the bondholder receives only a fraction A_T/K, called the *recovery fraction*, of the bond principal K: the fractional loss $(K - A_T)/K$ is called the *loss given default* or LGD. As you will see in an exercise, the probability distribution of LGD can be computed explicitly in the Merton model.

Note that equity value increases with the firm's volatility (since its payoff is convex in the underlier), so shareholders are generally inclined to press for riskier positions to be taken by their managers. The opposite is true for bondholders. So-called "agency problems" relate to the contradictory aims of shareholders, bondholders and other "stakeholders".

Structural models like Merton's model depend on the unobserved variable A_t. On the other hand, for publicly traded companies, the share price (and hence the total equity) is closely observed in the market. The usual "ad-hoc" approach to obtaining an estimate for the firm's asset values A_t and volatility σ in Merton's model uses the Black-Scholes formula for a call option, that is,

$$E_t = \mathrm{BSCall}(A_t, K, r, \sigma, T - t), \tag{4.7}$$

where K and T are determined by the firm's debt structure. One combines this with a second equation by equating the equity volatility to the coefficient of the Brownian term obtained by applying Itô's formula to (4.7), namely,

$$\sigma A_t \frac{\partial \text{BSCall}}{\partial A} = \sigma^E E_t. \tag{4.8}$$

As we will see in Section 4.3, a consistent method is to use Duan's maximum likelihood result [5] to estimate σ and μ directly from the equity time series E^i. Once an estimate for σ is obtained in this way, it can be inserted back into the pricing formula (4.7) in order to produce estimates for the firm values A^i.

The Merton model is only a starting point for studying credit risk, and is obviously far from realistic:

- The non-stationary structure of the debt that leads to the termination of operations on a fixed date, and default can only happen on that date. Geske [10] extended the Merton model to the case of bonds of different maturities.
- It is incorrect to assume that the firm value is tradeable. In fact, the firm value and its parameters is not even directly observed.
- Interest rates should certainly be taken to be stochastic: this is not a serious drawback, and its generalization was included in Merton's original paper.
- The short end of the yield spread curve in calibrated versions of the Merton model typically remains essentially zero for months, in strong contradiction with observations.

The so-called first passage models extend the Merton framework by allowing default to happen at intermediate times.

4.2 Black-Cox model

The simplest *first passage model* again takes a firm with asset value given by (4.1) and outstanding debt with face value K at maturity T. However, instead of admitting only the possibility of default at maturity time T, Black and Cox (1976) [3] postulated that default occurs at the first time that the firm's asset value drops below a certain time-dependent barrier $K(t)$. This can be explained by the right of bondholders to exercise a "safety covenant" that allows them to liquidate the firm if at any time its value drops below the specified threshold $K(t)$. Thus, the default time is given by

$$\tau = \inf\{t > 0 : A_t < K(t)\} \tag{4.9}$$

For the choice of the time dependent barrier, observe that if $K(t) > K$ then bondholders are always completely covered, which is certainly unrealistic. On the other hand, one should clearly have $K_T \leq K$ for a consistent definition of default. One natural, but certainly not the only, choice is to take an increasing time-dependent barrier $K(t) = K_0 e^{kt}, K_0 \leq K e^{-kT}$.

The first passage time to the default barrier can now be reduced to the first passage time for Brownian motion with drift. Observing that

$$\{A_t < K(t)\} = \{W_t + \sigma^{-1}(r - \sigma^2/2 - k)t \le \sigma^{-1}\log(K_0/A_0)\},$$

we obtain that the risk neutral probability of default occurring before time $t \le T$ is then given by

$$Q[0 \le \tau < t] = Q\left[\min_{s\le t}(A_s/K(s)) \le 1\}\right] = Q\left[\min_{s\le t} X_s \le \sigma^{-1}\log\left(\frac{K_0}{A_0}\right)\right] \qquad (4.10)$$

where $X_t = W_t + mt, m = \sigma^{-1}(r - \sigma^2/2 - k)$. This is a classic problem of probability we discuss in Appendix A, whose solution is given by

$$\begin{aligned} Q[\min_{s\le t} X_t \le d] &= 1 - \mathrm{FP}(-d; -m, t) \\ \mathrm{FP}(d; m, t) &:= N\left[\frac{d - mt}{\sqrt{t}}\right] - e^{2md} N\left[\frac{-d - mt}{\sqrt{t}}\right], \quad d \ge 0 \end{aligned} \qquad (4.11)$$

Thus we obtain the formula

$$Q[0 \le \tau < t] = 1 - \mathrm{FP}(-d; -m, t) \qquad (4.12)$$

with $m = \sigma^{-1}(r - \sigma^2/2 - k)$ and $d = \sigma^{-1}\log(K_0/A_0) < 0$.

The pay-off for equity holders at maturity is

$$(A_T - K)^+ \mathbf{1}_{\{\min_{s\le T} X_s > d\}} = (e^{kT} A_0 e^{\sigma X_T} - K)^+ \mathbf{1}_{\{\min_{s\le T} X_s > d\}}. \qquad (4.13)$$

This is equivalent to the payoff of a down-and-out call option, and can be priced by "Black-Scholes"-type closed form expressions found for example in (Merton 74) [24]. The equity in the Black-Cox model is smaller than the share value obtained in the Merton model, and is not monotone in the volatility.

In the event of default, the pay-off for debt holders is $A_\tau = K(\tau)$ at the time of default, and the fair "recovery value" can be computed by integrating $K(s)$, discounted, with respect to the risk-neutral PDF for the time of default. The value of the bond at time t prior to default is a sum $D_t = D_t^b + D_t^m$ of the recovery value and the value of the payment at maturity. The recovery value is thus

$$D_t^b = \int_t^T e^{r(t-s)} K(s)(-\partial_s \mathrm{FP}(-d_t; -m, s - t))ds \qquad (4.14)$$

where $d_t = \sigma^{-1}\log(K(t)/A_t)$. The remaining term can be written

$$D_t^m = E^Q[e^{-r(T-t)}[A_T - (A_T - K)^+]\mathbf{1}_{\{\tau > T\}}|\mathcal{F}_t]$$

which is a difference of barrier call options (one with zero strike). Computation of these integrals can be done explicitly, as you will be asked to do in an exercise.

One can go further with the Black-Cox model and consider what happens if an additional bond is issued with face value \$1 (considered to be negligible), and maturity $T_1 < T$. In the event $\tau \le T_1$ the bond would pay the "recovery fraction" $R(\tau) := K(\tau)/K$, while in the event $\tau \ge T_1$ the bond pays the principal at maturity.

4.3 Time-Changed Brownian Motion Models

The intractable nature of first passage problems for processes beyond Brownian motion has impeded efforts to extend the Black-Cox picture in the way that the Black-Scholes equity model has been generalized to stochastic volatility and jump-diffusion models. Hurd [14] has introduced one flexible way to extend to much more general processes. The mathematical ingredients are a Brownian motion $W_t, t \geq 0$ and an independent "time change" process $G_t, t \geq 0$. A time change is any non-decreasing process with $G_0 = 0$. Examples are Poisson processes, processes of the form $\int_0^t \lambda_s ds$ with $\lambda_s \geq 0$, and Lévy subordinators (non-decreasing processes with independent increments and infinitely divisible transition densities). Then one defines the *log-leverage process* to be $X_t = \log[A_t/K(t)] = x_0 + \sigma W_{G_t} + \beta\sigma^2 G_t$.

The standard way to define the time of default would be

$$\tau = \inf\{t \geq 0 | X_t \leq 0\}$$

but the first passage problem can no longer be solved explicitly. [14] instead proposes an alternative definition

$$\tau = \inf\{t \geq 0 | G_t \geq \tau_{BM}\}, \quad \tau_{BM} = \inf\{t \geq 0 | x_0 + \sigma W_t + \beta\sigma^2 t \leq 0\}$$

With this definition, one finds tractable formulas for all quantities of interest, starting with the survival probability:

$$P[\tau > t] = E[P[\tau > t|\mathcal{G}]] = E[\mathrm{FP}(x_0/\sigma; \beta\sigma, G_t)] \tag{4.15}$$

Note the use of iterated conditioning over the sigma-algebra $\mathcal{G}$ generated by the time change G.

Here are two classes of time-changes that can easily be studied.

1. G is taken to be a Lévy subordinator [4], which are defined to be independent increment, increasing pure-jump processes. A good example is the drifting gamma process, in which case X will be a so-called variance-gamma process.

2. $G_t = \int_0^t \lambda_s ds$ for some positive process λ. A good example is to take λ to be a CIR process (3.36).

4.4 KMV

KMV[1] is the name given to a particularly successful practical implementation of structural credit modeling. It is instructive to see what assumptions they make in order to produce commercially acceptable credit methods. The main difficulty, as in all structural models, is in assigning dynamics to the firm value, which is an unobserved process. We outline the main points, including the key point where KMV diverges from a strict structural model.

[1] The firm KMV is named after Kealhofer, McQuown and Vasicek, the founders of the company in 2002. It has since been sold to Moody's.

- Default trigger: The question is to determine a level for K from the structure of a firm's debt (which in practise consists of bond issues of different maturity, coupon rate, seniority and features such as convertibility). In a nutshell, KMV puts the value somewhere between the face value of short term debt, and the face value of the total debt, arguing that the firm will always have to service short term debt, but can be more flexible in servicing the long term debt. Typically, the trigger is given by the full short term debt plus half the long term debt.

- Value of firm: Rather than try to estimate the firm value directly from detailed balance sheet data (a procedure highly sensitive to assumptions and method) KMV infers A_t from the value of debt (which is taken from balance sheet) and equity.

- Equity: KMV takes the safest course, defining it to be the market capitalization (current share value times the number of shares outstanding). Since E_t is observed in the market, it is used to infer the value of A_t using the Black-Scholes call option formula:

$$E_t = \text{BSCall}(A_t, T - t, r, \sigma, K) \tag{4.16}$$

  The maturity date T is not so clear, but should represent the approximate time scale of the debt[2].

- Calibration: The appropriate way to estimate model parameters and the firm value process, given a time series $\{\hat{E}_{t_1}, \ldots, \hat{E}_{t_N}\}$ of market capitalization observed at times $t_i = i\Delta t, i = 1, \ldots, N$ is to use Duan's theorem [5]. One finds that

$$(\hat{\sigma}, \hat{\mu}) = \arg\max_{\sigma,\mu} \mathcal{L}_E(\hat{E}_{t_1}, \ldots, \hat{E}_{t_N}; \sigma, \mu)$$

  and

$$\hat{A}_{t_i} = F_i^{-1}(\hat{E}_{t_i}, \hat{\sigma})$$

  where $E_i = F_i(A, \sigma)$ denotes the function BSCall. Here the log-likelihood function is

$$\begin{aligned} \mathcal{L}_E(\hat{E}_{t_1}, \ldots, \hat{E}_{t_N}; \sigma, \mu) &= \mathcal{L}_A(F_1^{-1}(\hat{E}_{t_1}), \ldots, F_N^{-1}(\hat{E}_{t_N}); \sigma, \mu) \\ &\quad - \sum_{i=1}^{N} \log \left[\frac{\partial F_i}{\partial A} \right] (F_i^{-1}(E_{t_i}, \sigma)) \end{aligned}$$

  where $\mathcal{L}_A(A_{t_1}, \ldots, A_{t_N}; \sigma, \mu)$ is the explicit log-likelihood function of the jointly log-normal variables $A_{t_1}, \ldots, A_{t_N}$. Thus from a time series of equity data, the method leads to a time series $\hat{A}_{t_i}$ and an estimate $\hat{\sigma}, \hat{\mu}$. Of course, even if the model were true, if the estimate $\hat{\sigma}$ is far from the true σ, the inferred time series $\hat{A}$ would differ systematically from the "true" values of A.

[2]For example, one could take it to be the *duration* of the debt.

From the data on a given date, KMV computes the key credit score DD_t for the firm at that time, called *distance to default*. Roughly speaking it gives the amount by which $\log A_t$ exceeds $\log K$ measured in standard deviations σ of the one year PDF $\log A_{t+1}$. That is

$$\mathrm{DD}_t = \frac{\log(A_t/K)}{\sigma} \tag{4.17}$$

By a strict structural interpretation, EDF, the expected default frequency, meaning the probability of observing the firm to default within one year, ought to equal the normal probability $\mathrm{EDF}_t = N(\mathrm{DD}_t)$. KMV, however, breaks the model at this point, and instead relies on its large database of historical defaults to map DD to EDF by a proprietary function $\mathrm{EDF} = f(\mathrm{DD})$. $f(\mathrm{DD})$ is designed to give the actual fraction of all firms with the given DD that have been observed to default within one year.

Studies such as Duffie et al [6] indicate that the distance to default DD_t is a reasonable firm-specific dynamic (defined by current observations of the firm) quantity that correlates strongly with credit spreads and observed historical default frequency.

4.5 Optimal Capital Structure Models

Papers by Leland and Toft [21, 22] use the Black-Cox framework to explore the question of how a firm best capitalizes itself. They proposed that the owners of the firm will choose to issue debt in the way that maximizes their equity. Two important control parameters are the level of the default trigger K and the overall size of the debt. They consider two factors, namely $\tau \geq 0$ the fractional tax benefit paid on debt coupons to the debt issuer, and $\alpha \geq 0$, the bankruptcy costs as a fraction of firm value.

Assumption 4.

1. The firm's asset value process A_t follows geometric Brownian motion:

$$\frac{dA_t}{A_t} = (r - \delta)dt + \sigma dW_t^Q \tag{4.18}$$

for a constant risk-free rate r and "dividend rate" $\delta \geq 0$.

2. The debt is issued as a "perpetual bond" that pays constant coupon rate C per unit of time;

3. The debt contract specifies a level K such that the firm defaults at any time that $A_t \leq K$. That is, the time of bankruptcy is

$$\tau_B = \inf\{t | A_t \leq K\} \tag{4.19}$$

4. A tax rebate is paid at the rate τC on debt coupons, for some constant $\tau \in [0, 1)$;

5. At the time of default, the equity is zero, and the debt is valued at $(1-\alpha)K$ for some fraction $\alpha \in [0, 1)$. That is, bankruptcy charges of αK are paid at the time τ_B.

The picture to have is that at some time (take this to be $t = 0$), the owners of the debt-free firm with value A_0 decide to "raise capital" by issuing debt in the form of a perpetual bond that pays a constant coupon rate C and confers on the bondholders the right to put the firm in default when the firm value falls below K. By doing this, the owners hope to increase the value of their equity by some amount. The question then is: **Question:** What values (C^*, K^*) maximize the equity E at time $t = 0$?

Using risk neutral valuation leads to the value of the issued debt at time 0

$$D_0 = E^Q[e^{-r\tau_B}(1-\alpha)K\mathbf{1}_{\{0\le\tau_B<\infty\}}] + E^Q[\int_0^{\tau_B} Ce^{-rt}dt] := D(A_0; K, C) \tag{4.20}$$

The value of the firm after recapitalization will be

$$v_0 = A + E^Q[\int_0^{\tau_B} \tau Ce^{-rt}dt] - E^Q[e^{-r\tau_B}\alpha K\mathbf{1}_{\{0\le\tau_B<\infty\}}] := v(A_0; K, C) \tag{4.21}$$

Finally, one can show directly that the firm equity will be

$$E_0 = v_0 - D_0 = E^Q[\int_0^{\tau_B} (\delta A_t - (1-\tau)C)e^{-rt}dt] \tag{4.22}$$

It turns out that all the above valuation formulas can be expressed in terms of the Laplace transform $L(\alpha, b, \mu) = \Phi_{\tau_b}(\alpha)$ for $\tau_b = \inf\{t|W_t + \mu t \ge b\}$, which is given by (A.34) :

$$\Phi_{\tau_b}(\alpha) := E[e^{-\alpha\tau_b}] = e^{b(\mu-\sqrt{\mu^2+2\alpha})}$$

First we note that $A_t = \exp[\log A_0 + \sigma W_t + (r - \delta - \frac{1}{2}\sigma^2)t]$. Changing the sign of the Brownian motion in the usual way converts the lower crossing problem into an upper crossing problem, and one can see that the first term of (4.20) equals

$$(1-\alpha)KL(r, d_0, -m) = (1-\alpha)K\left(\frac{A_0}{K}\right)^{-\gamma}$$

where $d_0 = \frac{1}{\sigma}\log(A_0/K), m = \frac{1}{\sigma}(r - \delta - \frac{1}{2}\sigma^2)$ and $\gamma := \frac{1}{\sigma}(m + \sqrt{m^2 + 2r})$. The second term of (4.20) becomes

$$E^Q[\frac{C}{r}(1-e^{-r\tau_B})\mathbf{1}_{\{0\le\tau_B<\infty\}} + \frac{C}{r}\mathbf{1}_{\{\tau_B=\infty\}}]$$
$$= \frac{C}{r}[1 - L(r, d_0, -m)] = \frac{C}{r}\left[1 - \left(\frac{A_0}{K}\right)^{-\gamma}\right] \tag{4.23}$$

Similarly,

$$v_0 = A_0 + \frac{C\tau}{r}\left[1 - \left(\frac{A_0}{K}\right)^{-\gamma}\right] - \alpha K \left(\frac{A_0}{K}\right)^{-\gamma}$$

Finally, when $A_0 > K$ we have

$$E_0 = v_0 - D_0 = A_0 - \frac{(1-\tau)C}{r}\left[1 - \left(\frac{A_0}{K}\right)^{-\gamma}\right] - K \left(\frac{A_0}{K}\right)^{-\gamma}$$

while $E_0 = 0$ when $A_0 \leq K$.

One can now address the big question as follows

1. Fix C, and find the value $K^*(C) < \frac{(1-\tau)C}{r}$ so that the "smooth pasting condition"

$$\frac{dE}{dA}|_{A=K} = 0$$

holds. $K^*(C)$ is the optimal K for this value of C.

2. Plug this value of K into the formula for E, and maximize over C to find C^* and $K^* = K^*(C^*)$.

The solution in step 1 is easily seen to be $K^*(C) = \frac{\gamma(1-\tau)C}{(\gamma+1)r}$. The solution for C^* is

$$C^* = A_0 \frac{r(1+\gamma)}{\gamma(1-\tau)}\left(\frac{(1+\gamma)\tau + \alpha(1-\tau)\gamma}{\tau}\right)^{-1/\gamma} \tag{4.24}$$

4.6 Default Events and Bond Prices

We have learned from structural models, particularly the Black-Cox model, that two factors are critical in determining the value of defaultable bonds, namely the probability of default (PD) and the loss given default (LGD). In this section we explore some general properties of the probability of default, and see what these imply for defaultable bonds.

Recall that the *default time* τ is a *stopping time*, that is, a random variable $\tau : \Omega \to \mathbb{R}_+ \cup \{\infty\}$ such that $\{\tau \leq t\} \in \mathcal{F}_t$, for every $t \geq 0$. In other words, a random time τ is a stopping time if the cádlág[3] stochastic process

$$H_t(\omega) = \mathbf{1}_{\{\tau \leq t\}}(\omega) = \begin{cases} 1, & \text{if } \tau(\omega) \leq t \\ 0, & \text{otherwise} \end{cases} \tag{4.25}$$

is adapted to the filtration $\mathcal{F}_t$. For default times, H_t is known as the *default indicator process*, and $H_t^c = 1 - H_t$ is the *survival indicator process*.

[3]for "continueux á droite, limite á gauche", or "right continuous, left limit".

We say that a stopping time $\tau > 0$ is *predictable* if there is an *announcing sequence* of stopping times $\tau_1 \leq \tau_2 \leq \cdots$ such that

$$\lim_{n\to\infty} \tau_n = \tau, \qquad \text{P-a.s}$$

If you have studied stochastic analysis you will recognize this as the statement that the indicator process H_t is predictable. For the record, the opposite of a predictable stopping time is a *totally inaccessible* stopping time, that is, a stopping time τ such that

$$P[\tau = \hat{\tau} < \infty] = 0,$$

for any predictable stopping time $\hat{\tau}$. It can be shown that every stopping time can be expressed as the minimum of a predictable and a totally inaccessible stopping time.

Example 2. For any adapted process X_t, we may define stopping times $\tau_d = \inf\{t|X_t \geq d\}$. If X is an Itô diffusion (and therefore has continuous paths a.s.), this is a predictable stopping time: one takes any sequence $\{d_n\}_{n=1}^{\infty}$ that increases to d, and then $\{\tau_{d_n}\}$ is an announcing sequence. Conversely, if X_t is a pure jump process (for example a Poisson process), then τ_d is totally inaccessible.

4.6.1 Unconditional default probability

Given a default time τ, the *probability of survival* in t years is

$$P[\tau > t] = 1 - P[\tau \leq t] = 1 - E[\mathbf{1}_{\{\tau \leq t\}}]. \tag{4.26}$$

Several other related quantities can be derived from this basic probability. For instance,

$$P[s < \tau \leq t] = P[\tau > s] - P[\tau > t]$$

is the unconditional probability of default occurring in the time interval $[s, t]$.

Using Bayes's rule for conditional probability, one can deduce that the probability of survival in t years conditioned on survival up to $s \leq t$ years is

$$P[\tau > t|\tau > s] = \frac{P[\{\tau > t\} \cap \{\tau > s\}]}{P[\tau > s]} = \frac{P[\tau > t]}{P[\tau > s]}, \tag{4.27}$$

since $\{\tau > t\} \subset \{\tau > s\}$. From this we can define the *forward default probability* for the interval $[s, t]$ as

$$P[s \leq \tau \leq t|\tau > s] = 1 - P[\tau > t|\tau > s] = 1 - \frac{P[\tau > t]}{P[\tau > s]}. \tag{4.28}$$

Assuming that $P[\tau > t]$ is strictly positive and differentiable in t, we define the *forward default rate function* as

$$h(t) = -\frac{\partial \log P[\tau > t]}{\partial t}. \tag{4.29}$$

It then follows that

$$P[\tau > t | \tau > s] = e^{-\int_s^t h(u)du}. \tag{4.30}$$

The forward default rate measures the instantaneous rate of arrival for a default event at time t conditioned on survival up to t. Indeed, if $h(t)$ is continuous we find that for a short time interval $[t, t + \Delta t]$,

$$h(t)\Delta t \approx P[t \leq \tau \leq t + \Delta t | \tau > t].$$

4.6.2 Conditional default probability

The above probabilities are all derived from $P[\tau > t]$, that is, conditionally on the constant information set available at time 0. More generally, one can focus on $\mathrm{SP}_s(t) := P[\tau > t|\mathcal{F}_s]$, that is, the *survival probability* in t years conditioned on *all* the information available at time $s \leq t$. If we assume positivity and differentiability in t, then this can be written as

$$\mathrm{SP}_s(t) = H_s^c e^{-\int_s^t h_s(u)du}, \tag{4.31}$$

where

$$h_s(t) = -\frac{\partial \log \mathrm{SP}_s(t)}{\partial t}. \tag{4.32}$$

We define $h_s(t)$ to be the *forward default rate process* given all the information up to time s: it clearly has the initial values $h_0(t) = h(t)$.

The indicator process H_t defined in (4.25) is a submartingale. Moreover, since H can be shown to be of class $\mathcal{D}$, it follows from the Doob-Meyer decomposition[4] that there exists a unique nondecreasing predictable process Λ_t, called the *compensator*, such that $H_t - \Lambda_t$ is a uniformly integrable martingale. Since defaults happen only once, in general we know that

$$\Lambda_t = \Lambda_{\tau \wedge t}.$$

In some cases the compensator can be written as

$$\Lambda_t = \int_0^t \lambda_s ds \tag{4.33}$$

for a non-negative, progressively measurable process λ_t. In this case the process λ_t is called the *default intensity*. In other cases, such as Black-Cox models, the compensator does not admit a default intensity (intuitively, the instantaneous probability of default is either 0 or ∞).

Under suitable technical conditions it can be shown that

$$\lambda_t = h_t(t). \tag{4.34}$$

Therefore, while the forward default rate function $h(t)$ gives the instantaneous rate of default conditioned only on survival up to t, the default intensity λ_t measures the instantaneous rate of default conditioned on all the information available up to time t.

[4] Please see [26] for a fundamental discussion of such matters.

Finally, we observe that starting from a sufficiently regular family of survival probabilities one can obtain forward default rates by (4.32) and the associated intensity by (4.34). This is analogous to knowing a differentiable system of bond prices and then obtaining forward and spot interest rates from it. As we have seen, going in the opposite direction, that is from the spot interest r_t to bond prices, is not always straightforward, and the same is true for going from intensities to survival probabilities.

4.6.3 Implied Survival Probabilities and Credit Spreads

We now investigate how the prices of defaultable zero-coupon bonds can be used to infer *risk-neutral* default probabilities. As we will see later, PD and LGD become entangled in the bond pricing formula, and for that reason we will assume in this section that bonds pay nothing in the event of default ("zero-recovery"). We also assume in this section that interest rates r_t and the default time τ are *independent* under the risk-neutral measure Q.

Let $\bar{P}_t(T)\mathbf{1}_{\{\tau>t\}}$ be the price at time $t \leq T$ of a defaultable zero-coupon bond issued by a certain firm with maturity T and face value equal to one unit of currency. Then, since we assume bonds pay zero recovery, we know that

$$\bar{P}_t(T)\mathbf{1}_{\{\tau>t\}} = E_t^Q\left[e^{-\int_t^T r_s ds}\mathbf{1}_{\{\tau>T\}}\right] \tag{4.35}$$

and since r_t and τ are *independent* this becomes

$$\bar{P}_t(T)\mathbf{1}_{\{\tau>t\}} = P_t(T)Q[\tau > T|\mathcal{F}_t].$$

Therefore, as long as $\tau > t$ the risk-neutral survival probability is given by

$$Q[\tau > T|\mathcal{F}_t] = \frac{\bar{P}_t(T)}{P_t(T)} = \exp[-\int_t^T (\bar{f}_t(s) - f_t(s))ds] \tag{4.36}$$

Comparing with the definition of the yield spread $YS_t(T)$ and the forward default rate $h_t(T)$, we see that

$$YS_t(T) = \frac{1}{T-t}\int_t^T h_t(s)ds, \quad h_t(s) = \bar{f}_t(s) - f_t(s).$$

Thus the term structure of risk-neutral survival probabilities is completely determined by the term structure of both defaultable and default-free zero-coupon bonds. In what follows, even though the interpretation relies on the above two assumptions, (4.36) will be called the *implied survival probability*, emphasizing the fact that it is derived from market prices and associated to the risk-neutral measure Q.

4.7 Exercises

In the following exercises, we take the asset value process with parameters $\mu = r = 0.05$ and $\sigma = 0.20$.

Exercise 22. The *leverage ratio* or *debt/equity ratio* of a firm is defined to be the total debt of the firm divided by the equity $L_t = D_t/E_t$. A highly leveraged firm has a lot of debt which might indicate a danger of default, and certainly a susceptibility to rising interest rates.

1. Consider the Merton model with the above parameters. Use MATLAB or similar to plot the value of the debt D_0 as a function of debt maturity T for the following leverage levels: $10, 3, 1, 1/3, 1/10$. (Hint: W.L.O.G. you may set $K = 1$. Then you will need to use a root finding method (e.g. Newton-Raphson) to find the value of A_0 for a given level of leverage.)

2. Thinking of $D_0(T)$ as the price of a zero coupon defaultable bond, compute the *credit spread* (defaultable bond yield minus default free bond yield) as a function over T, again for the same leverage levels.

Exercise 23. (Alternative derivation of the Merton model) Suppose $dA_t = A_t[\mu dt + \sigma dW_t]$ under the physical measure, plus the other assumptions of the Merton model. Suppose further that debt and equity are tradeable assets that satisfy $A_t = D_t + E_t$ and follow processes $D_t = D(t, A_t), E_t = E(t, A_t)$ for differentiable functions. By considering a locally risk-free self-financing portfolio of bonds and equity (which by necessity will earn the risk-free rate of return), prove directly that both D, E satisfy the Black-Scholes equation

$$\partial_t f + \frac{1}{2}\sigma^2 A^2 \partial_A^2 f + rA\partial_A f - rf = 0$$

Exercise 24. In the Merton model with the number of shares N constant, the stock price is $S_t = E_t/N$. Find a formula for the stock volatility as a function of time to maturity and current equity, i.e. $\sigma_t^{(S)} = f(T - t, S_t)$. Find the stochastic differential equation for S_t under the risk-neutral measure. Is S_t Markovian? Compute the asymptotics of stock volatility when the leverage ratio tends to ∞. What does this say about the behaviour of the stock price?

Exercise 25. For any $s < T$, compute the conditional probability density function $p_\tau(t|\mathcal{F}_s), t > s$ for the time to default τ in the Black-Cox model. What is the behaviour of p_τ as $t \to s+$? Does this model have a default intensity?

Exercise 26. *Loss given default* LGD is defined to be the fraction of the bond principal that is not returned to the bondholder at default. Compute the probability distribution of LGD in the Merton model. Plot the pdf with $T = 5$ for the leverage levels 100, 30, 10, 3, 1, and compute the mean and standard deviation in each case.