Assignment 2
Foundations of Mathematical Finance - Winter 2010
Fields Institute

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1. Solve the utility optimization problem

$$u(x) = \sup_{H \in \mathcal{H}} E[U(x + (H \cdot S)_T)], \quad x \in \mathrm{dom}(U),$$

in a complete market with $\mathcal{M}^e(S) = \{Q\}$ and finite probability space $\Omega = \{\omega_1, \ldots, \omega_n\}$ for each of the following utility functions U:

1. $U(x) = -\frac{e^{-\gamma x}}{\gamma}, \quad \gamma > 0.$
2. $U(x) = \log(x).$
3. $U(x) = \frac{x^\alpha}{\alpha}, \quad \alpha \in (-\infty, 1)\backslash\{0\}.$

For each function, make sure to address the following points:

- Find the conjugate function $V(y) = \sup_x [U(x) - yx]$.
- Compute the dual value function $v(y) = E\left[V\left(y\frac{dQ}{dP}\right)\right]$.
- Find $\hat{y}(x)$ satisfying $v'(\hat{y})(x)) = -x$
- Obtain $\hat{X}_T(x)$ using the relation $U'(\hat{X}_T(x)) = \hat{y}(x)\frac{dQ}{dP}$.
- Justify the existence of a portfolio $\hat{H} \in \mathcal{H}$ such that $\hat{X}(x) = x + (\hat{H} \cdot S)_T$.