Assignment 1
Foundations of Mathematical Finance - Winter 2010
Fields Institute

25/01/2010

1. Let $(\Omega, \mathcal{F}, P)$ be a finite probability space. Show that $L^0_-(\Omega, \mathcal{F}, P)$ is a close polyhedral cone. Moreover, show that $C = K + L^\infty_-$ is a closed convex set, where K denotes the set of attainable claims at price 0.

2. Consider discounted assets $S = (S^1_t, \dots, S^d_t)_{t=0}^T$ and let $H \in \mathcal{H}$ be a self–financing strategy such that $(H \cdot S)_T \geq 0$ and $P[(H \cdot S)_T > 0] > 0$ (that is, H is an arbitrage in the multi–period market). Show that there exists $1 \leq t \leq T$ and a set $A \in \mathcal{F}_{t-1}$ with $P(A) > 0$ such that $\mathbf{1}_A H_t \Delta S_t \geq 0$ and $P[\mathbf{1}_A H_t \Delta S_t > 0] > 0$ (that is, H_t is an arbitrage in the single period market (S_{t-1}, S_t)).

3. Show that the set $I(f) := \{E_Q[f] \mid Q \in \mathcal{M}^e(S)\}$ is a bounded interval in $\mathbb{R}$.

4. Consider a one–period market model with prices at time $t = 0$ given by the constants $\hat{S}_0 \in \mathbb{R}^{d+1}$ and prices a time time $T = 1$ given by the $\mathbb{R}^{d+1}$–value random variable $\hat{S}_1(\omega)$ where $\omega \in \Omega = \{\omega_1, \dots, \omega_N\}$. Show that this model is arbitrage–free if and only if there exists a vector $\psi \in \mathbb{R}^N_{++}$ (that is, a vector with strictly positive components) such that

$$\hat{S}^j_0 = \sum_{n=1}^{N} S^i_1(\omega_n)\psi_n, \qquad j = 0, \dots, d$$

The vector ψ is called the *state–price* density for the model.