Interest Rate Theory

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Contents

Björk,T. *Arbitrage Theory in Continuous Time.*
3:rd ed. 2009. Oxford University Press.

1.

Mathematics Recap

Ch. 10-12

Contents

1.

Conditional Expectation

Conditional Expectation

If $\mathcal{F}$ is a sigma-algebra and X is a random variable which is $\mathcal{F}$-measurable, we write this as $X \in \mathcal{F}$.

If $X \in \mathcal{F}$ and if $\mathcal{G} \subseteq \mathcal{F}$ then we write $E[X|\mathcal{G}]$ for the conditional expectation of X given the information contained in $\mathcal{G}$. Sometimes we use the notation $E_{\mathcal{G}}[X]$.

The following proposition contains everything that we will need to know about conditional expectations within this course.

Main Results

Proposition 1: Assume that $X \in \mathcal{F}$, and that $\mathcal{G} \subseteq \mathcal{F}$. Then the following hold.

- The random variable $E[X|\mathcal{G}]$ is completely determined by the information in $\mathcal{G}$ so we have

$$E[X|\mathcal{G}] \in \mathcal{G}$$

- If we have $Y \in \mathcal{G}$ then Y is completely determined by $\mathcal{G}$ so we have

$$E[XY|\mathcal{G}] = YE[X|\mathcal{G}]$$

  In particular we have

$$E[Y|\mathcal{G}] = Y$$

- If $\mathcal{H} \subseteq \mathcal{G}$ then we have the "law of iterated expectations"

$$E[E[X|\mathcal{G}]|\mathcal{H}] = E[X|\mathcal{H}]$$

- In particular we have

$$E[X] = E[E[X|\mathcal{G}]]$$

2.

Changing Measures

Absolute Continuity

Definition: Given two probability measures P and Q on $\mathcal{F}$ we say that Q is **absolutely continuous** w.r.t. P on $\mathcal{F}$ if, for all $A \in \mathcal{F}$, we have

$$P(A) = 0 \quad \Rightarrow \quad Q(A) = 0$$

We write this as

$$Q << P.$$

If $Q << P$ and $P << Q$ then we say that P and Q are **equivalent** and write

$$Q \sim P$$

Equivalent measures

It is easy to see that P and Q are equivalent if and only if

$$P(A) = 0 \quad \Leftrightarrow \quad Q(A) = 0$$

or, equivalently,

$$P(A) = 1 \quad \Leftrightarrow \quad Q(A) = 1$$

Two equivalent measures thus agree on all certain events and on all impossible events, but can disagree on all other events.

Simple examples:

- All non degenerate Gaussian distributions on R are equivalent.

- If P is Gaussian on R and Q is exponential then $Q << P$ but not the other way around.

Absolute Continuity ct'd

Consider a given probability measure P and a random variable $L \geq 0$ with $E^P[L] = 1$. Now **define** Q by

$$Q(A) = \int_A L dP$$

then it is easy to see that Q is a probability measure and that $Q << P$.

A natural question is now if **all** measures $Q << P$ are obtained in this way. The answer is yes, and the precise (quite deep) result is as follows.

The Radon Nikodym Theorem

Consider two probability measures P and Q on $(\Omega, \mathcal{F})$, and assume that $Q << P$ on $\mathcal{F}$. Then there exists a unique random variable L with the following properties

1. $Q(A) = \int_A L dP, \quad \forall A \in \mathcal{F}$
2. $L \geq 0, \quad P - a.s.$
3. $E^P[L] = 1,$
4. $L \in \mathcal{F}$

The random variable L is denoted as

$$L = \frac{dQ}{dP}, \quad \text{on } \mathcal{F}$$

and it is called the **Radon-Nikodym derivative** of Q w.r.t. P on $\mathcal{F}$, or the **likelihood ratio** between Q and P on $\mathcal{F}$.

A simple example

The Radon-Nikodym derivative L is intuitively the local scale factor between P and Q. If the sample space Ω is finite so $\Omega = \{\omega_1, \dots, \omega_n\}$ then P is determined by the probabilities $p_1, \dots, p_n$ where

$$p_i = P(\omega_i) \quad i = 1, \dots, n$$

Now consider a measure Q with probabilities

$$q_i = Q(\omega_i) \quad i = 1, \dots, n$$

If $Q << P$ this simply says that

$$p_i = 0 \quad \Rightarrow \quad q_i = 0$$

and it is easy to see that the Radon-Nikodym derivative $L = dQ/dP$ is given by

$$L(\omega_i) = \frac{q_i}{p_i} \quad i = 1, \dots, n$$

If $p_i = 0$ then we also have $q_i = 0$ and we can define the ratio q_i/p_i arbitrarily.

If $p_1, \ldots, p_n$ as well as $q_1, \ldots, q_n$ are all positive, then we see that $Q \sim P$ and in fact

$$\frac{dP}{dQ} = \frac{1}{L} = \left(\frac{dQ}{dP} \right)^{-1}$$

as could be expected.

Computing expected values

A main use of Radon-Nikodym derivatives is for the computation of expected values.

Suppose therefore that $Q << P$ on $\mathcal{F}$ and that X is a random variable with $X \in \mathcal{F}$. With $L = dQ/dP$ on $\mathcal{F}$ then have the following result.

Proposition 3: With notation as above we have

$$E^Q[X] = E^P[L \cdot X]$$

The Abstract Bayes' Formula

We can also use Radon-Nikodym derivatives in order to compute conditional expectations. The result, known as the abstract **Bayes' Formula**, is as follows.

Theorem 4: Consider two measures P and Q with $Q << P$ on $\mathcal{F}$ and with

$$L^{\mathcal{F}} = \frac{dQ}{dP} \quad \text{on } \mathcal{F}$$

Assume that $\mathcal{G} \subseteq \mathcal{F}$ and let X be a random variable with $X \in \mathcal{F}$. Then the following holds

$$E^Q\left[X \middle| \mathcal{G}\right] = \frac{E^P\left[L^{\mathcal{F}} X \middle| \mathcal{G}\right]}{E^P\left[L^{\mathcal{F}} \middle| \mathcal{G}\right]}$$

Dependence of the σ-algebra

Suppose that we have $Q << P$ on $\mathcal{F}$ with

$$L^{\mathcal{F}} = \frac{dQ}{dP} \quad \text{on } \mathcal{F}$$

Now consider smaller σ-algebra $\mathcal{G} \subseteq \mathcal{F}$. Our problem is to find the R-N derivative

$$L^{\mathcal{G}} = \frac{dQ}{dP} \quad \text{on } \mathcal{G}$$

We recall that $L^{\mathcal{G}}$ is characterized by the following properties

1. $Q(A) = E^P\left[L^{\mathcal{G}} \cdot I_A\right] \quad \forall A \in \mathcal{G}$
2. $L^{\mathcal{G}} \geq 0$
3. $E^P\left[L^{\mathcal{G}}\right] = 1$
4. $L^{\mathcal{G}} \in \mathcal{G}$

A natural guess would perhaps be that $L^{\mathcal{G}} = L^{\mathcal{F}}$, so let us check if $L^{\mathcal{F}}$ satisfies points 1-4 above.

By assumption we have

$$Q(A) = E^P\left[L^{\mathcal{F}} \cdot I_A\right] \quad \forall A \in \mathcal{F}$$

Since $\mathcal{G} \subseteq \mathcal{F}$ we then have

$$Q(A) = E^P\left[L^{\mathcal{F}} \cdot I_A\right] \quad \forall A \in \mathcal{G}$$

so point 1 above is certainly satisfied by $L^{\mathcal{F}}$. It is also clear that $L^{\mathcal{F}}$ satisfies points 2 and 3. It thus seems that $L^{\mathcal{F}}$ is also a natural candidate for the R-N derivative $L^{\mathcal{G}}$, but the problem is that we do not in general have $L^{\mathcal{F}} \in \mathcal{G}$.

This problem can, however, be fixed. By iterated expectations we have, for all $A \in \mathcal{G}$,

$$E^P\left[L^{\mathcal{F}} \cdot I_A\right] = E^P\left[E^P\left[L^{\mathcal{F}} \cdot I_A \middle| \mathcal{G}\right]\right]$$

Since $A \in \mathcal{G}$ we have

$$E^P\left[L^{\mathcal{F}} \cdot I_A \middle| \mathcal{G}\right] = E^P\left[L^{\mathcal{F}} \middle| \mathcal{G}\right] I_A$$

Let us now define $L^{\mathcal{G}}$ by

$$L^{\mathcal{G}} = E^P\left[L^{\mathcal{F}} \middle| \mathcal{G}\right]$$

We then obviously have $L^{\mathcal{G}} \in \mathcal{G}$ and

$$Q(A) = E^P\left[L^{\mathcal{G}} \cdot I_A\right] \quad \forall A \in \mathcal{G}$$

It is easy to see that also points 2-3 are satisfied so we have proved the following result.

A formula for $L^{\mathcal{G}}$

Proposition 5: If $Q << P$ on $\mathcal{F}$ and $\mathcal{G} \subseteq \mathcal{F}$ then, with notation as above, we have

$$L^{\mathcal{G}} = E^P \left[L^{\mathcal{F}} \middle| \mathcal{G} \right]$$

The likelihood process on a filtered space

We now consider the case when we have a probability measure P on some space Ω and that instead of just one σ-algebra $\mathcal{F}$ we have a **filtration**, i.e. an increasing family of σ-algebras $\{\mathcal{F}_t\}_{t\geq 0}$.

The interpretation is as usual that $\mathcal{F}_t$ is the information available to us at time t, and that we have $\mathcal{F}_s \subseteq \mathcal{F}_t$ for $s \leq t$.

Now assume that we also have another measure Q, and that for some fixed T, we have $Q << P$ on $\mathcal{F}_T$. We define the random variable L_T by

$$L_T = \frac{dQ}{dP} \quad \text{on } \mathcal{F}_T$$

Since $Q << P$ on $\mathcal{F}_T$ we also have $Q << P$ on $\mathcal{F}_t$ for all $t \leq T$ and we define

$$L_t = \frac{dQ}{dP} \quad \text{on } \mathcal{F}_t \quad 0 \leq t \leq T$$

For every t we have $L_t \in \mathcal{F}_t$, so L is an adapted process, known as the **likelihood process**.

The L process is a P martingale

We recall that

$$L_t = \frac{dQ}{dP} \quad \text{on } \mathcal{F}_t \quad 0 \le t \le T$$

Since $\mathcal{F}_s \subseteq \mathcal{F}_t$ for $s \le t$ we can use Proposition 5 and deduce that

$$L_s = E^P \left[L_t | \mathcal{F}_s \right] \quad s \le t \le T$$

and we have thus proved the following result.

Proposition: Given the assumptions above, the likelihood process L is a P-martingale.

Where are we heading?

We are now going to perform measure transformations on Wiener spaces, where P will correspond to the objective measure and Q will be the risk neutral measure.

For this we need define the proper likelihood process L and, since L is a P-martingale, we have the following natural questions.

- What does a martingale look like in a Wiener driven framework?

- Suppose that we have a P-Wiener process W and then change measure from P to Q. What are the properties of W under the new measure Q?

These questions are handled by the Martingale Representation Theorem, and the Girsanov Theorem respectively.

3.

The Martingale Representation Theorem

Intuition

Suppose that we have a Wiener process W under the measure P. We recall that if h is adapted (and integrable enough) and if the process X is defined by

$$X_t = x_0 + \int_0^t h_s dW_s$$

then X is a a martingale. We now have the following natural question:

Question: Assume that X is an arbitrary martingale. Does it then follow that X has the form

$$X_t = x_0 + \int_0^t h_s dW_s$$

for some adapted process h?

In other words: Are **all** martingales stochastic integrals w.r.t. W?

Answer

It is immediately clear that all martingales can **not** be written as stochastic integrals w.r.t. W. Consider for example the process X defined by

$$X_t = \begin{cases} 0 & \text{for} \quad 0 \leq t < 1 \\ Z & \text{for} \quad t \geq 1 \end{cases}$$

where Z is an random variable, independent of W, with $E[Z] = 0$.

X is then a martingale (why?) but it is clear (how?) that it cannot be written as

$$X_t = x_0 + \int_0^t h_s dW_s$$

for any process h.

Intuition

The intuitive reason why we cannot write

$$X_t = x_0 + \int_0^t h_s dW_s$$

in the example above is of course that the random variable Z "has nothing to do with" the Wiener process W. In order to exclude examples like this, we thus need an assumption which guarantees that our probability space only contains the Wiener process W and nothing else.

This idea is formalized by assuming that the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ **is the one generated by the Wiener process** W.

The Martingale Representation Theorem

Theorem. Let W be a P-Wiener process and assume that the filtation is the **internal** one i.e.

$$\mathcal{F}_t = \mathcal{F}_t^W = \sigma\left\{W_s;\ 0 \leq s \leq t\right\}$$

Then, for every $(P, \mathcal{F}_t)$-martingale X, there exists a real number x and an adapted process h such that

$$X_t = x + \int_0^t h_s dW_s,$$

i.e.

$$dX_t = h_t dW_t.$$

Proof: Hard. This is very deep result.

Note

For a given martingale X, the Representation Theorem above guarantees the existence of a process h such that

$$X_t = x + \int_0^t h_s dW_s,$$

The Theorem does **not**, however, tell us how to find or construct the process h.

4.

The Girsanov Theorem

Setup

Let W be a P-Wiener process and fix a time horizon T. Suppose that we want to change measure from P to Q on $\mathcal{F}_T$. For this we need a P-martingale L with $L_0 = 1$ to use as a likelihood process, and a natural way of constructing this is to choose a process g and then define L by

$$\begin{cases} dL_t &= g_t dW_t \\ L_0 &= 1 \end{cases}$$

This definition does not guarantee that $L \geq 0$, so we make a small adjustment. We choose a process φ and define L by

$$\begin{cases} dL_t &= L_t \varphi_t dW_t \\ L_0 &= 1 \end{cases}$$

The process L will again be a martingale and we easily obtain

$$L_t = e^{\int_0^t \varphi_s dW_s - \frac{1}{2}\int_0^t \varphi_s^2 ds}$$

Thus we are guaranteed that $L \geq 0$. We now change measure form P to Q by setting

$$dQ = L_t dP, \quad \text{on } \mathcal{F}_t,\ 0 \leq t \leq T$$

The main problem is to find out what the properties of W are, under the new measure Q. This problem is resolved by the **Girsanov Theorem**.

The Girsanov Theorem

Let W be a P-Wiener process. Fix a time horizon T.

Theorem: Choose an adapted process φ, and define the process L by

$$\begin{cases} dL_t &= L_t \varphi_t dW_t \\ L_0 &= 1 \end{cases}$$

Assume that $E^P[L_T] = 1$, and define a new mesure Q on $\mathcal{F}_T$ by

$$dQ = L_t dP, \quad \text{on } \mathcal{F}_t,\ 0 \le t \le T$$

Then $Q << P$ and the process W^Q, defined by

$$W_t^Q = W_t - \int_0^t \varphi_s ds$$

is Q-Wiener. We can also write this as

$$dW_t = \varphi_t dt + dW_t^Q$$

Changing the drift in an SDE

The single most common use of the Girsanov Theorem is as follows.

Suppose that we have a process X with P dynamics

$$dX_t = \mu_t dt + \sigma_t dW_t$$

where μ and σ are adapted and W is P-Wiener.

We now do a Girsanov Transformation as above, and the question is what the Q-dynamics look like.

From the Girsanov Theorem we have

$$dW_t = \varphi_t dt + dW_t^Q$$

and substituting this into the P-dynamics we obtain the Q dynamics as

$$dX_t = \{\mu_t + \sigma_t \varphi_t\} dt + \sigma_t dW_t^Q$$

Moral: The drift changes but the diffusion is unaffected.

The Converse of the Girsanov Theorem

Let W be a P-Wiener process. Fix a time horizon T.

Theorem. Assume that:

- $Q << P$ on $\mathcal{F}_T$, with likelihood process

$$L_t = \frac{dQ}{dP}, \quad \text{on } \mathcal{F}_t \ 0, \leq t \leq T$$

- The filtation is the **internal** one .i.e.

$$\mathcal{F}_t = \sigma \left\{ W_s; \ 0 \leq s \leq t \right\}$$

Then there exists a process φ such that

$$\begin{cases} dL_t &= L_t \varphi_t dW_t \\ L_0 &= 1 \end{cases}$$

2.

The Martingale Approach

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Financial Markets

Price Processes:

$$S_t = \left[S_t^0, ..., S_t^N\right]$$

Example: (Black-Scholes, $S^0 := B,\ S^1 := S$)

$$\begin{aligned} dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\ dB_t &= r B_t dt. \end{aligned}$$

Portfolio:

$$h_t = \left[h_t^0, ..., h_t^N\right]$$

$h_t^i =$ number of units of asset i at time t.

Value Process:

$$V_t^h = \sum_{i=0}^{N} h_t^i S_t^i = h_t S_t$$

Self Financing Portfolios

Definition: (intuitive)
A portfolio is **self-financing** if there is no exogenous infusion or withdrawal of money. "The purchase of a new asset must be financed by the sale of an old one."

Definition: (mathematical)
A portfolio is **self-financing** if the value process satisfies

$$dV_t = \sum_{i=0}^{N} h_t^i dS_t^i$$

Major insight:
If the price process S is a **martingale**, and if h is **self-financing**, then V is a **martingale**.

NB! This simple observation is in fact the basis of the following theory.

Arbitrage

The portfolio u is an **arbitrage** portfolio if

- The portfolio strategy is self financing.
- $V_0 = 0$.
- $V_T \geq 0, \; P - a.s.$
- $P\left(V_T > 0\right) > 0$

Main Question: When is the market free of arbitrage?

First Attempt

Proposition: If $S_t^0, \cdots, S_t^N$ are P-martingales, then the market is free of arbitrage.

Proof:
Assume that V is an arbitrage strategy. Since

$$dV_t = \sum_{i=0}^{N} h_t^i dS_t^i,$$

V is a P-martingale, so

$$V_0 = E^P[V_T] > 0.$$

This contradicts $V_0 = 0$.

True, but useless.

Example: (Black-Scholes)

$$\begin{aligned} dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\ dB_t &= r B_t dt. \end{aligned}$$

(We would have to assume that $\alpha = r = 0$)

We now try to improve on this result.

Choose S_0 as numeraire

Definition:
The **normalized price vector** Z is given by

$$Z_t = \frac{S_t}{S_t^0} = \left[1, Z_t^1, ..., Z_t^N\right]$$

The **normalized value process** V^Z is given by

$$V_t^Z = \sum_0^N h_t^i Z_t^i.$$

Idea:
The arbitrage and self financing concepts should be independent of the accounting unit.

Invariance of numeraire

Proposition: One can show (see the book) that

- S-arbitrage $\Longleftrightarrow$ Z-arbitrage.
- S-self-financing $\Longleftrightarrow$ Z-self-financing.

Insight:

- If h self-financing then

$$dV_t^Z = \sum_1^N h_t^i dZ_t^i$$

- Thus, if the **normalized** price process Z is a P-martingale, then V^Z is a martingale.

Second Attempt

Proposition: If $Z_t^0, \cdots, Z_t^N$ are P-martingales, then the market is free of arbitrage.

True, but still fairly useless.

Example: (Black-Scholes)

$$\begin{aligned} dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\ dB_t &= r B_t dt. \end{aligned}$$

$$\begin{aligned} dZ_t^1 &= (\alpha - r) Z_t^1 dt + \sigma Z_t^1 dW_t, \\ dZ_t^0 &= 0 dt. \end{aligned}$$

We would have to assume "risk-neutrality", i.e. that $\alpha = r$.

Arbitrage

Recall that h is an arbitrage if

- h is self financing
- $V_0 = 0$.
- $V_T \geq 0, \ P - a.s.$
- $P(V_T > 0) > 0$

Major insight

This concept is invariant under an **equivalent change of measure!**

Martingale Measures

Definition: A probability measure Q is called an **equivalent martingale measure** (EMM) if and only if it has the following properties.

- Q and P are equivalent, i.e.

$$Q \sim P$$

- The normalized price processes

$$Z_t^i = \frac{S_t^i}{S_t^0}, \quad i = 0, \ldots, N$$

  are **Q-martingales**.

Wan now state the main result of arbitrage theory.

First Fundamental Theorem

Theorem: The market is arbitrage free

iff

there exists an equivalent martingale measure.

Comments

- It is very easy to prove that existence of EMM imples no arbitrage (see below).
- The other imnplication is technically very hard.
- For discrete time and finite sample space Ω the hard part follows easily from the separation theorem for convex sets.
- For discrete time and more general sample space we need the Hahn-Banach Theorem.
- For continuous time the proof becomes technically very hard, mainly due to topological problems. See the textbook.

Proof that EMM implies no arbitrage

This is basically done above. Assume that there exists an EMM denoted by Q. Assume that $P(V_T \geq 0) = 1$ and $P(V_T > 0) > 0$. Then, since $P \sim Q$ we also have $Q(V_T \geq 0) = 1$ and $Q(V_T > 0) > 0$.

Recall:

$$dV_t^Z = \sum_1^N h_t^i dZ_t^i$$

Q is a martingale measure

$$\Downarrow$$

V^Z is a Q-martingale

$$\Downarrow$$

$$V_0 = V_0^Z = E^Q\left[V_T^Z\right] > 0$$

$$\Downarrow$$

No arbitrage

Choice of Numeraire

The **numeraire** price S_t^0 can be chosen arbitrarily. The most common choice is however that we choose S^0 as the **bank account**, i.e.

$$S_t^0 = B_t$$

where

$$dB_t = r_t B_t dt$$

Here r is the (possibly stochastic) short rate and we have

$$B_t = e^{\int_0^t r_s ds}$$

Example: The Black-Scholes Model

$$\begin{aligned} dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\ dB_t &= r B_t dt. \end{aligned}$$

Look for martingale measure. We set $Z = S/B$.

$$dZ_t = Z_t(\alpha - r)dt + Z_t \sigma dW_t,$$

Girsanov transformation on $[0, T]$:

$$\begin{cases} dL_t = L_t \varphi_t dW_t, \\ L_0 = 1. \end{cases}$$

$$dQ = L_T dP, \quad \text{on } \mathcal{F}_T$$

Girsanov:

$$dW_t = \varphi_t dt + dW_t^Q,$$

where W^Q is a Q-Wiener process.

The Q-dynamics for Z are given by

$$dZ_t = Z_t \left[\alpha - r + \sigma\varphi_t\right] dt + Z_t \sigma dW_t^Q.$$

Unique martingale measure Q, with Girsanov kernel given by

$$\varphi_t = \frac{r - \alpha}{\sigma}.$$

Q-dynamics of S:

$$dS_t = rS_t dt + \sigma S_t dW_t^Q.$$

Conclusion: The Black-Scholes model is free of arbitrage.

Pricing

We consider a market $B_t, S_t^1, \ldots, S_t^N$.

Definition:
A **contingent claim** with **delivery time** T, is a random variable

$$X \in \mathcal{F}_T.$$

"At $t = T$ the amount X is paid to the holder of the claim".

Example: (European Call Option)

$$X = \max\left[S_T - K, 0\right]$$

Let X be a contingent T-claim.

Problem: How do we find an arbitrage free price process $\Pi_t\left[X\right]$ for X?

Solution

The extended market

$$B_t, S_t^1, \ldots, S_t^N, \Pi_t[X]$$

must be arbitrage free, so there must exist a martingale measure Q for $(B_t, S_t, \Pi_t[X])$. In particular

$$\frac{\Pi_t[X]}{B_t}$$

must be a Q-martingale, i.e.

$$\frac{\Pi_t[X]}{B_t} = E^Q\left[\left.\frac{\Pi_T[X]}{B_T}\right| \mathcal{F}_t\right]$$

Since we obviously (why?) have

$$\Pi_T[X] = X$$

we have proved the main pricing formula.

Risk Neutral Valuation

Theorem: For a T-claim X, the arbitrage free price is given by the formula

$$\Pi_t [X] = E^Q \left[e^{-\int_t^T r_s ds} \times X \middle| \mathcal{F}_t \right]$$

Example: The Black-Scholes Model

Q-dynamics:

$$dS_t = rS_t dt + \sigma S_t dW_t^Q.$$

Simple claim:

$$X = \Phi(S_T),$$

$$\Pi_t [X] = e^{-r(T-t)} E^Q \left[\Phi(S_T) \middle| \mathcal{F}_t \right]$$

Kolmogorov $\Rightarrow$

$$\Pi_t [X] = F(t, S_t)$$

where $F(t, s)$ solves the Black-Scholes equation:

$$\begin{cases} \frac{\partial F}{\partial t} + rs\frac{\partial F}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 F}{\partial s^2} - rF &= 0, \\ F(T, s) &= \Phi(s). \end{cases}$$

Problem

Recall the valuation formula

$$\Pi_t[X] = E^Q\left[e^{-\int_t^T r_s ds} \times X \middle| \mathcal{F}_t\right]$$

What if there are several different martingale measures Q?

This is connected with the **completeness** of the market.

Hedging

Def: A portfolio is a **hedge** against X ("replicates X") if

- h is self financing
- $V_T = X, \quad P - a.s.$

Def: The market is **complete** if every X can be hedged.

Pricing Formula:
If h replicates X, then a natural way of pricing X is

$$\Pi_t [X] = V_t^h$$

When can we hedge?

Existence of hedge

$$\Updownarrow$$

Existence of stochastic integral representation

Fix T-claim X.

If h is a hedge for X then

- $V_T^Z = \frac{X}{B_T}$

- h is self financing, i.e.

$$dV_t^Z = \sum_1^K h_t^i dZ_t^i$$

Thus V^Z is a Q-**martingale**.

$$V_t^Z = E^Q \left[\left. \frac{X}{B_T} \right| \mathcal{F}_t \right]$$

Lemma:
Fix T-claim X. Define martingale M by

$$M_t = E^Q\left[\left.\frac{X}{B_t}\right| \mathcal{F}_t\right]$$

Suppose that there exist predictable processes $h^1, \cdots, h^N$ such that

$$M_t = x + \sum_{i=1}^{N} \int_0^t h_s^i dZ_s^i,$$

Then X can be replicated.

Proof

We guess that

$$M_t = V_t^Z = h_t^B \cdot 1 + \sum_{i=1}^{N} h_t^i Z_t^i$$

Define: h^B by

$$h_t^B = M_t - \sum_{i=1}^{N} h_t^i Z_t^i.$$

We have $M_t = V_t^Z$, and we get

$$dV_t^Z = dM_t = \sum_{i=1}^{N} h_t^i dZt^i,$$

so the portfolio is self financing. Furthermore:

$$V_T^Z = M_T = E^Q \left[\left. \frac{X}{B_T} \right| \mathcal{F}_T \right] = \frac{X}{B_T}.$$

Second Fundamental Theorem

The second most important result in arbitrage theory is the following.

Theorem:

The market is complete

iff

the martingale measure Q is unique.

Proof: It is obvious (why?) that if the market is complete, then Q must be unique. The other implication is very hard to prove. It basically relies on duality arguments from functional analysis.

Black-Scholes Model

Q-dynamics

$$\begin{aligned} dS_t &= rS_t dt + \sigma S_t dW_t^Q, \\ dZ_t &= Z_t \sigma dW_t^Q \end{aligned}$$

$$M_t = E^Q \left[e^{-rT} X \middle| \mathcal{F}_t \right],$$

Representation theorem for Wiener processes

$\Downarrow$

there exists g such that

$$M_t = M(0) + \int_0^t g_s dW_s^Q.$$

Thus

$$M_t = M_0 + \int_0^t h_s^1 dZ_s,$$

with $h_t^1 = \frac{g_t}{\sigma Z_t}$.

Result:
X can be replicated using the portfolio defined by

$$\begin{aligned} h_t^1 &= g_t/\sigma Z_t, \\ h_t^B &= M_t - h_t^1 Z_t. \end{aligned}$$

Moral: The Black Scholes model is complete.

Special Case: Simple Claims

Assume X is of the form $X = \Phi(S_T)$

$$M_t = E^Q \left[e^{-rT} \Phi(S_T) \middle| \mathcal{F}_t \right],$$

Kolmogorov backward equation $\Rightarrow M_t = f(t, S_t)$

$$\begin{cases} \frac{\partial f}{\partial t} + rs\frac{\partial f}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 f}{\partial s^2} & = & 0, \\ f(T, s) & = & e^{-rT}\Phi(s). \end{cases}$$

Itô $\Rightarrow$

$$dM_t = \sigma S_t \frac{\partial f}{\partial s} dW_t^Q,$$

so

$$g_t = \sigma S_t \cdot \frac{\partial f}{\partial s},$$

Replicating portfolio h:

$$\begin{aligned} h_t^B &= f - S_t \frac{\partial f}{\partial s}, \\ h_t^1 &= B_t \frac{\partial f}{\partial s}. \end{aligned}$$

Interpretation: $f(t, S_t) = V_t^Z$.

Define $F(t,s)$ by

$$F(t,s) = e^{rt} f(t,s)$$

so $F(t,S_t) = V_t$. Then

$$\begin{cases} h_t^B = \frac{F(t,S_t) - S_t \frac{\partial F}{\partial s}(t,S_t)}{B_t}, \\ h_t^1 = \frac{\partial F}{\partial s}(t,S_t) \end{cases}$$

where F solves the **Black-Scholes equation**

$$\begin{cases} \frac{\partial F}{\partial t} + rs\frac{\partial F}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 F}{\partial s^2} - rF = 0, \\ F(T,s) = \Phi(s). \end{cases}$$

Main Results

- The market is arbitrage free $\Leftrightarrow$ There exists a martingale measure Q

- The market is complete $\Leftrightarrow$ Q is unique.

- Every X must be priced by the formula

$$\Pi_t\left[X\right] = E^Q\left[e^{-\int_t^T r_s ds} \times X \middle| \mathcal{F}_t \right]$$

  for some choice of Q.

- In a non-complete market, different choices of Q will produce different prices for X.

- For a hedgeable claim X, all choices of Q will produce the same price for X:

$$\Pi_t\left[X\right] = V_t = E^Q\left[e^{-\int_t^T r_s ds} \times X \middle| \mathcal{F}_t \right]$$

Completeness vs No Arbitrage
Rule of Thumb

Question:
When is a model arbitrage free and/or complete?

Answer:
Count the number of risky assets, and the number of random sources.

$$R = \text{number of random sources}$$
$$N = \text{number of risky assets}$$

Intuition:
If N is large, compared to R, you have lots of possibilities of forming clever portfolios. Thus lots of chances of making arbitrage profits. Also many chances of replicating a given claim.

Rule of thumb

Generically, the following hold.

- The market is arbitrage free if and only if

$$N \leq R$$

- The market is complete if and only if

$$N \geq R$$

Example:
The Black-Scholes model.

$$\begin{aligned} dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\ dB_t &= r B_t dt. \end{aligned}$$

For B-S we have $N = R = 1$. Thus the Black-Scholes model is arbitrage free and complete.

Stochastic Discount Factors

Given a model under P. For every EMM Q we define the corresponding **Stochastic Discount Factor**, or **SDF**, by

$$D_t = e^{-\int_0^t r_s ds} L_t,$$

where

$$L_t = \frac{dQ}{dP}, \quad \text{on } \mathcal{F}_t$$

There is thus a one-to-one correspondence between EMMs and SDFs.

The risk neutral valuation formula for a T-claim X can now be expressed under P instead of under Q.

Proposition: With notation as above we have

$$\Pi_t [X] = \frac{1}{D_t} E^P [D_T X | \mathcal{F}_t]$$

Proof: Bayes' formula.

Martingale Property of $S \cdot D$

Proposition: If S is an arbitrary price process, then the process

$$S_t D_t$$

is a P-martingale.

Proof: Bayes' formula.

3.

Incomplete Markets

Ch. 15

Derivatives on Non Financial Underlying

Recall: The Black-Scholes theory assumes that the market for the underlying asset has (among other things) the following properties.

- The underlying is a liquidly traded asset.
- Shortselling allowed.
- Portfolios can be carried forward in time.

There exists a large market for derivatives, where the underlying does not satisfy these assumptions.

Examples:

- Weather derivatives.
- Derivatives on electric energy.
- CAT-bonds.

Typical Contracts

Weather derivatives:

"Heating degree days". Payoff at maturity T is given by

$$\mathcal{Z} = \max\{X_T - 30, 0\}$$

where X_T is the (mean) temperature at some place.

Electricity option:

The right (but not the obligation) to buy, at time T, at a predetermined price K, a constant flow of energy over a predetermined time interval.

CAT bond:

A bond for which the payment of coupons and nominal value is contingent on some (well specified) natural disaster to take place.

Problems

Weather derivatives:
The temperature is not the price of a traded asset.

Electricity derivatives:
Electric energy cannot easily be stored.

CAT-bonds:
Natural disasters are not traded assets.

We will treat all these problems within a **factor model**.

Typical Factor Model Setup

Given:

- An underlying factor process X, which is **not** the price process of a traded asset, with dynamics under the objective probability measure P as

$$dX_t = \mu(t, X_t)\, dt + \sigma(t, X_t)\, dW_t.$$

- A risk free asset with dynamics

$$dB_t = rB_t dt,$$

Problem:

Find arbitrage free price $\Pi_t[\mathcal{Z}]$ of a derivative of the form

$$\mathcal{Z} = \Phi(X_T)$$

Concrete Examples

Assume that X_t is the temperature at time t at the village of Peniche (Portugal).

Heating degree days:

$$\Phi(X_T) = 100 \cdot \max\{X_T - 30, 0\}$$

Holiday Insurance:

$$\Phi(X_T) = \begin{cases} 1000, & \text{if } X_T < 20 \\ 0, & \text{if } X_T \geq 20 \end{cases}$$

Question

Is the price $\Pi_t[\Phi]$ uniquely determined by the P-dynamics of X, and the requirement of an arbitrage free derivatives market?

NO!!

WHY?

Stock Price Model $\sim$ Factor Model

Black-Scholes:

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t, \\ dB_t &= r B_t dt. \end{aligned}$$

Factor Model:

$$\begin{aligned} dX_t &= \mu(t, X_t)dt + \sigma(t, X_t)dW_t, \\ dB_t &= r B_t dt. \end{aligned}$$

What is the difference?

Answer

- X is **not the price of a traded asset!**
- We can not form a portfolio based on X.

1. **Rule of thumb:**

$$\begin{aligned} N &= 0, \quad \text{(no risky asset)} \\ R &= 1, \quad \text{(one source of randomness, } W\text{)} \end{aligned}$$

   We have $N < R$. The exogenously given market, consisting only of B, is incomplete.

2. **Replicating portfolios:**
   We can only invest money in the bank, and then sit down passively and wait.

We do **not** have **enough underlying assets** in order to price X-derivatives.

- There is **not** a unique price for a **particular** derivative.

- In order to avoid arbitrage, **different** derivatives have to satisfy **internal consistency** relations.

- If we take **one** "benchmark" derivative as given, then all other derivatives can be priced **in terms of** the market price of the benchmark.

We consider two given claims $\Phi(X_T)$ and $\Gamma(X_T)$. We assume they are traded with prices

$$\begin{aligned} \Pi_t[\Phi] &= f(t, X_t) \\ \Pi_t[\Gamma] &= g(t, X_t) \end{aligned}$$

Program:

- Form portfolio based on Φ and Γ. Use Itô on f and g to get portfolio dynamics.

$$dV = V\left\{u^f \frac{df}{f} + u^g \frac{dg}{g}\right\}$$

- Choose portfolio weights such that the $dW-$ term vanishes. Then we have

$$dV = V \cdot kdt,$$

("synthetic bank" with k as the short rate)

- Absence of arbitrage implies

$$k = r$$

- Read off the relation $k = r$!

From Itô:

$$df = f\mu_f dt + f\sigma_f dW,$$

where

$$\begin{cases} \mu_f &= \frac{f_t + \mu f_x + \frac{1}{2}\sigma^2 f_{xx}}{f}, \\ \sigma_f &= \frac{\sigma f_x}{f}. \end{cases}$$

Portfolio dynamics

$$dV = V\left\{u^f \frac{df}{f} + u^g \frac{dg}{g}\right\}.$$

Reshuffling terms gives us

$$dV = V \cdot \left\{u^f \mu_f + u^g \mu_g\right\} dt + V \cdot \left\{u^f \sigma_f + u^g \sigma_g\right\} dW.$$

Let the portfolio weights solve the system

$$\begin{cases} u^f + u^g &= 1, \\ u^f \sigma_f + u^g \sigma_g &= 0. \end{cases}$$

$$
\begin{aligned}
u^f &= -\frac{\sigma_g}{\sigma_f - \sigma_g}, \\
u^g &= \frac{\sigma_f}{\sigma_f - \sigma_g},
\end{aligned}
$$

Portfolio dynamics

$$dV = V \cdot \left\{ u^f \mu_f + u^g \mu_g \right\} dt.$$

i.e.

$$dV = V \cdot \left\{ \frac{\mu_g \sigma_f - \mu_f \sigma_g}{\sigma_f - \sigma_g} \right\} dt.$$

Absence of arbitrage requires

$$\frac{\mu_g \sigma_f - \mu_f \sigma_g}{\sigma_f - \sigma_g} = r$$

which can be written as

$$\frac{\mu_g - r}{\sigma_g} = \frac{\mu_f - r}{\sigma_f}.$$

$$\frac{\mu_g - r}{\sigma_g} = \frac{\mu_f - r}{\sigma_f}.$$

Note!

The quotient does **not** depend upon the particular choice of contract.

Result

Assume that the market for X-derivatives is free of arbitrage. Then there exists a universal process λ, such that

$$\frac{\mu_f(t) - r}{\sigma_f(t)} = \lambda(t, X_t),$$

holds for all t and for every choice of contract f.

NB: The same λ for all choices of f.

$$\begin{aligned} \lambda &= \text{Risk premium per unit of volatility} \\ &= \text{“Market Price of Risk” (cf. CAPM).} \\ &= \text{Sharpe Ratio} \end{aligned}$$

Slogan:
“On an arbitrage free market all X-derivatives have the same market price of risk.”

The relation

$$\frac{\mu_f - r}{\sigma_f} = \lambda$$

is actually a PDE!

Pricing Equation

$$\begin{cases} f_t + \{\mu - \lambda\sigma\} f_x + \frac{1}{2}\sigma^2 f_{xx} - rf & = & 0 \\ f(T,x) & = & \Phi(x), \end{cases}$$

P-**dynamics:**

$$dX = \mu(t,X)dt + \sigma(t,X)dW.$$

Can we solve the PDE?

No!!

Why??

Answer

Recall the PDE

$$\begin{cases} f_t + \{\mu - \lambda\sigma\} f_x + \frac{1}{2}\sigma^2 f_{xx} - rf & = & 0 \\ f(T,x) & = & \Phi(x), \end{cases}$$

- In order to solve the PDE **we need to know λ.**
- λ is not given exogenously.
- λ is not determined endogenously.

Question:

Who determines λ?

Answer:

THE MARKET!

Interpreting λ

Recall that the f dynamics are

$$df = f\mu_f dt + f\sigma_f dW_t$$

and λ is defined as

$$\frac{\mu_f(t) - r}{\sigma_f(t)} = \lambda(t, X_t),$$

- λ **measures the aggregate risk aversion** in the market.
- If λ is big then the market is highly risk averse.
- If λ is zero then the market is **risk netural**.
- If you make an assumption about λ, then you implicitly make an assumption about the aggregate risk aversion of the market.

Moral

- Since the market is incomplete the requirement of an arbitrage free market will **not** lead to unique prices for X-derivatives.

- Prices on derivatives are determined by two main factors.

  1. **Partly** by the requirement of an arbitrage free derivative market. **All** pricing functions satisfies the **same** PDE.
  2. **Partly** by supply and demand on the market. These are in turn determined by attitude towards risk, liquidity consideration and other factors. All these are aggregated into the particular λ used (implicitly) by the market.

Risk Neutral Valuation

We recall the PDE

$$\left\{ \begin{array}{rcl} f_t + \{\mu - \lambda\sigma\} f_x + \frac{1}{2}\sigma^2 f_{xx} - rf & = & 0 \\ f(T,x) & = & \Phi(x), \end{array} \right.$$

Using Feynman-Kac we obtain a risk neutral valuation formula.

Risk Neutral Valuation

$$f(t,x) = e^{-r(T-t)} E^Q_{t,x}\left[\Phi(X_T)\right]$$

Q**-dynamics:**

$$dX_t = \{\mu - \lambda\sigma\}\, dt + \sigma dW_t^Q$$

- Price = expected value of future payments
- The expectation should **not** be taken under the "objective" probabilities P, but under the "risk adjusted" probabilities Q.

Interpretation of the risk adjusted probabilities

- The risk adjusted probabilities can be interpreted as probabilities in a (fictuous) risk neutral world.
- When we **compute prices**, we can calculate **as if** we live in a risk neutral world.
- This does **not** mean that we live in, or think that we live in, a risk neutral world.
- The formulas above hold regardless of the attitude towards risk of the investor, as long as he/she prefers more to less.

Diversification argument about λ

- If the risk factor is **idiosyncratic** and **diversifiable**, then one can argue that the factor should not be priced by the market. Compare with APT.

- Mathematically this means that $\lambda = 0$, i.e. $P = Q$, i.e. **the risk neutral distribution coincides with the objective distribution**.

- We thus have the **"actuarial pricing formula"**

$$f(t,x) = e^{-r(T-t)} E^{P}_{t,x}\left[\Phi(X_T)\right]$$

where we use the objective probabiliy measure P.

Modeling Issues

Temperature:

A standard model is given by

$$dX_t = \{m(t) - bX_t\}\, dt + \sigma dW_t,$$

where m is the mean temperature capturing seasonal variations. This often works reasonably well.

Electricity:

A (naive) model for the spot electricity price is

$$dS_t = S_t \{m(t) - a \ln S_t\}\, dt + \sigma S_t dW_t$$

This implies lognormal prices (why?). Electricty prices are however very far from lognormal, because of "spikes" in the prices. Complicated.

CAT bonds:

Here we have to use the theory of point processes and the theory of extremal statistics to model natural disasters. Complicated.

Martingale Analysis

Model: Under P we have

$$\begin{aligned} dX_t &= \mu(t, X_t)\, dt + \sigma(t, X_t)\, dW_t, \\ dB_t &= rB_t dt, \end{aligned}$$

We look for martingale measures. Since B is the only traded asset we need to find $Q \sim P$ such that

$$\frac{B_t}{B_t} = 1$$

is a Q martingale.

Result: In this model, **every** $Q \sim P$ is a martingale measure.

Girsanov

$$dL_t = L_t \varphi_t dW_t$$

P-dynamics

$$dX_t = \mu(t, X_t)\, dt + \sigma(t, X_t)\, dW_t,$$

$$dL_t = L_t \varphi_t dW_t$$

$dQ = L_t dP$ on $\mathcal{F}_t$

Girsanov:

$$dW_t = \varphi_t dt + dW_t^Q$$

Martingale pricing:

$$F(t, x) = e^{-r(T-t)} E^Q \left[Z \middle| \mathcal{F}_t \right]$$

Q-dynamics of X:

$$dX_t = \{\mu(t, X_t) + \sigma(t, X_t)\, \varphi_t\}\, dt + \sigma(t, X_t)\, dW_t^Q,$$

Result: We have $\lambda_t = -\varphi_t$, i.e,. the Girsanov kernel φ equals minus the market price of risk.

Several Risk Factors

We recall the dynamics of the f-derivative

$$df = f\mu_f dt + f\sigma_f dW_t$$

and the Market Price of Risk

$$\frac{\mu_f - r}{\sigma_f} = \lambda, \quad \text{i.e.} \quad \mu_f - r = \lambda\sigma_f.$$

In a multifactor model of the type

$$dX_t = \mu(t, X_t)\, dt + \sum_{i=1}^{n} \sigma_i(t, X_t)\, dW_t^i,$$

it follows from Girsanov that for every risk factor W^i there will exist a market price of risk $\lambda_i = -\varphi_i$ such that

$$\mu_f - r = \sum_{i=1}^{n} \lambda_i \sigma_i$$

Compare with CAPM.

4.

Bonds and Interest Rates. Short Rate Models

Ch. 22-23

Definitions

Bonds:
T-bond = zero coupon bond, paying $\$1$ at the date of maturity T.

$$
\begin{aligned}
p(t,T) &= \text{price, at } t\text{, of a } T\text{-bond.} \\
p(T,T) &= 1.
\end{aligned}
$$

Main Problem

- Investigate the **term structure**, i.e. how prices of bonds with different dates of maturity are related to each other.
- Compute arbitrage free prices of interest rate derivatives (bond options, swaps, caps, floors etc.)

Risk Free Interest Rates

At time t:

- Sell one S-bond
- Buy exactly $p(t,S)/p(t,T)$ $T-$bonds
- Net investment at t: $\$0$.

At time S:

- Pay $\$1$

At time T:

- Collect $\$p(t,S)/p(t,T) \cdot 1$

Net Effect

- The contract is made at t.
- An investment of 1 at time S has yielded $p(t,S)/p(t,T)$ at time T.
- The equivalent constant rates, R, are given as the solutions to

Continuous rate:

$$e^{R\cdot(T-S)}\cdot 1 = \frac{p(t,S)}{p(t,T)}$$

Simple rate:

$$[1 + R\cdot(T-S)]\cdot 1 = \frac{p(t,S)}{p(t,T)}$$

Continuous Interest Rates

1. The **forward rate for the period** $[S, T]$**, contracted at** t is defined by

$$R(t; S, T) = -\frac{\log p(t, T) - \log p(t, S)}{T - S}.$$

2. The **spot rate**, $R(S, T)$, for the period $[S, T]$ is defined by

$$R(S, T) = R(S; S, T).$$

3. The **instantaneous forward rate at** T**, conracted at** t is defined by

$$f(t, T) = -\frac{\partial \log p(t, T)}{\partial T} = \lim_{S \to T} R(t; S, T).$$

4. The **instantaneous short rate at** t is defined by

$$r(t) = f(t, t).$$

Simple Rates (LIBOR)

1. The **simple forward rate L(t;S,T)for the period** $[S, T]$**, contracted at** t is defined by

$$L(t; S, T) = \frac{1}{T - S} \cdot \frac{p(t, S) - p(t, T)}{p(t, T)}$$

2. The **simple spot rate,** $L(S, T)$, for the period $[S, T]$ is defined by

$$L(S, T) = \frac{1}{T - S} \cdot \frac{1 - p(S, T)}{p(S, T)}$$

Practical Formula (LIBOR)

The **simple spot rate,** $L(T, T+\delta)$, for the period $[T, T+\delta]$ is given by

$$p(T, T+\delta) = \frac{1}{1 + \delta L(T, T+\delta)}$$

i.e.

$$L = \frac{1}{\delta} \cdot \frac{1-p}{p}$$

Bond prices $\sim$ forward rates

$$p(t,T) = p(t,s) \cdot \exp\left\{-\int_s^T f(t,u)du\right\},$$

In particular we have

$$p(t,T) = \exp\left\{-\int_t^T f(t,s)ds\right\}.$$

Interest Rate Options

Problem:
We want to price, at t, a European Call, with exercise date S, and strike price K, on an underlying T-bond. ($t < S < T$).

Naive approach: Use Black-Scholes's formula.

$$F(t,p) = pN\left[d_1\right] - e^{-r(S-t)}KN\left[d_2\right].$$

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{S-t}}\left\{\ln\left(\frac{p}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(S-t)\right\}, \\ d_2 &= d_1 - \sigma\sqrt{S-t}. \end{aligned}$$

where

$$p = p(t,T)$$

Is this allowed?

- p shall be the price of a traded asset. OK!
- The volatility of p must be constant. Here we have a problem because of **pull-to-par**, i.e. the fact that $p(T, T) = 1$. Bond volatilities will tend to zero as the bond approaches the time of maturity.
- The short rate must be **constant** and **deterministic**. Here the approach collapses completely, since the whole point of studying bond prices lies in the fact that interest rates are stochastic.

There is some hope in the case when the remaining time to exercise the option is small in relation to the remaining time to maturity of the underlying bond (why?).

Deeply felt need

A consistent **arbitrage free** model for the bond market

Stochastic interest rates

We assume that the short rate r is a stochastic process.

Money in the bank will then grow according to:

$$\begin{cases} dB(t) & = & r(t)B(t)dt, \\ B(0) & = & 1. \end{cases}$$

i.e.

$$B(t) = e^{\int_0^t r(s)ds}$$

Models for the short rate

Model: (In reality)

P:

$$
\begin{aligned}
dr &= \mu(t,r)dt + \sigma(t,r)dW, \\
dB &= r(t)Bdt.
\end{aligned}
$$

Question: Are bond prices uniquely determined by the P-dynamics of r, and the requirement of an arbitrage free bond market?

NO!!

WHY?

Stock Models $\sim$ Interest Rates

Black-Scholes:

$$
\begin{aligned}
dS &= \alpha S dt + \sigma S dw, \\
dB &= r B dt.
\end{aligned}
$$

Interest Rates:

$$
\begin{aligned}
dr &= \mu(t, r) dt + \sigma(t, r) dW, \\
dB &= r_t B dt.
\end{aligned}
$$

Question: What is the difference?

Answer: The short rate r is **not the price of a traded asset!**

1. **Meta-Theorem:**

$$\begin{aligned} N &= 0, \quad \text{(no risky asset)} \\ R &= 1, \quad \text{(one source of randomness, } W\text{)} \end{aligned}$$

   We have $M < R$. The exongenously given market, consisting only of B, is incomplete.

2. **Replicating portfolios:**
   We can only invest money in the bank, and then sit down passively and wait.

We do **not** have **enough underlying assets** in order to price bonds.

- There is **not** a unique price for a **particular** $T-$bond.
- In order to avoid arbitrage, bonds of **different maturities** have to satisfy internal **consistency** relations.
- If we take **one** "benchmark" T_0-bond as given, then all other bonds can be priced **in terms of** the market price of the benchmark bond.

Assumption:

$$
\begin{aligned}
p(t,T) &= F(t,r_t;T) \\
p(t,T) &= F^T(t,r_t), \\
F^T(T,r) &= 1.
\end{aligned}
$$

Program

- Form portfolio based on $T-$ and $S-$bonds. Use Itô on $F^T(t, r(t))$ to get bond- and portfolio dynamics.

$$dV = V \left\{ u_T \frac{dF^T}{F^T} + u_S \frac{dF^S}{F^S} \right\}$$

- Choose portfolio weights such that the $dW-$ term vanishes. Then we have

$$dV = V \cdot kdt,$$

("synthetic bank" with k as the short rate)

- Absence of arbitrage $\Rightarrow k = r$.

- Read off the relation $k = r$!

From Itô:

$$dF^T = F^T \alpha_T dt + F^T \sigma_T d\tilde{W},$$

where

$$\begin{cases} \alpha_T &= \frac{F_t^T + \mu F_r^T + \frac{1}{2}\sigma^2 F_{rr}^T}{F^T}, \\ \sigma_T &= \frac{\sigma F_r^T}{F^T}. \end{cases}$$

Portfolio dynamics

$$dV = V \left\{ u^T \frac{dF^T}{F^T} + u^S \frac{dF^S}{F^S} \right\}.$$

Reshuffling terms gives us

$$dV = V \cdot \left\{ u^T \alpha_T + u^S \alpha_S \right\} dt + V \cdot \left\{ u^T \sigma_T + u^S \sigma_S \right\} dW.$$

Let the portfolio weights solve the system

$$\begin{cases} u^T + u^S &= 1, \\ u^T \sigma_T + u^S \sigma_S &= 0. \end{cases}$$

$$\begin{cases} u^T & = & -\frac{\sigma_S}{\sigma_T - \sigma_S}, \\ u^S & = & \frac{\sigma_T}{\sigma_T - \sigma_S}, \end{cases}$$

Portfolio dynamics

$$dV = V \cdot \left\{ u^T \alpha_T + u^S \alpha_S \right\} dt.$$

i.e.

$$dV = V \cdot \left\{ \frac{\alpha_S \sigma_T - \alpha_T \sigma_S}{\sigma_T - \sigma_S} \right\} dt.$$

Absence of arbitrage requires

$$\frac{\alpha_S \sigma_T - \alpha_T \sigma_S}{\sigma_T - \sigma_S} = r$$

which can be written as

$$\frac{\alpha_S(t) - r(t)}{\sigma_S(t)} = \frac{\alpha_T(t) - r(t)}{\sigma_T(t)}.$$

$$\frac{\alpha_S(t) - r(t)}{\sigma_S(t)} = \frac{\alpha_T(t) - r(t)}{\sigma_T(t)}.$$

Note!
The quotient does **not** depend upon the particular choice of maturity date.

Result

Assume that the bond market is free of arbitrage. Then there exists a universal process λ, such that

$$\frac{\alpha_T(t) - r(t)}{\sigma_T(t)} = \lambda(t),$$

holds for all t and for every choice of maturity T.

NB: The same λ for all choices of T.

$$\begin{aligned} \lambda &= \text{Risk premium per unit of volatility} \\ &= \text{“Market Price of Risk” (cf. CAPM).} \end{aligned}$$

Slogan:
“On an arbitrage free market all bonds have the same market price of risk.”

The relation

$$\frac{\alpha_T - r}{\sigma_T} = \lambda$$

is actually a PDE!

The Term Structure Equation

$$
\begin{cases} F_t^T + \{\mu - \lambda\sigma\} F_r^T + \frac{1}{2}\sigma^2 F_{rr}^T - rF^T & = & 0, \\ F^T(T, r) & = & 1. \end{cases}
$$

P**-dynamics:**

$$
dr = \mu(t, r)dt + \sigma(t, r)dW.
$$

$$
\lambda = \frac{\alpha_T - r}{\sigma_T}, \quad \text{for all } T
$$

In order to solve the TSE we need to know λ.

General Term Structure Equation

Contingent claim:

$$X = \Phi(r(T))$$

Result:
The price is given by

$$\Pi[t; X] = F(t, r(t))$$

where F solves

$$\begin{aligned} F_t + \{\mu - \lambda\sigma\} F_r + \frac{1}{2}\sigma^2 F_{rr} - rF &= 0, \\ F(T, r) &= \Phi(r). \end{aligned}$$

In order to solve the TSE we need to know λ.

Who determines λ?

THE MARKET!

Moral

- Since the market is incomplete the requirement of an arbitrage free bond market will not lead to unique bond prices.

- Prices on bonds and other interest rate derivatives are determined by two main factors.

  1. **Partly** by the requirement of an arbitrage free bond market (the pricing functions satisfies the TSE).
  2. **Partly** by supply and demand on the market. These are in turn determined by attitude towards risk, liquidity consideration and other factors. All these are aggregated into the particular λ used (implicitly) by the market.

Risk Neutral Valuation

Using Feynmac–Kač we obtain

$$F(t, r; T) = E^Q_{t,r}\left[e^{-\int_t^T r(s)ds} \times 1\right].$$

Q-dynamics:

$$dr = \{\mu - \lambda\sigma\}dt + \sigma dW$$

Risk Neutral Valuation

$$\Pi[t;X] = E_{t,r}^{Q}\left[e^{-\int_t^T r(s)ds} \times X\right]$$

Q-**dynamics:**

$$dr = \{\mu - \lambda\sigma\}dt + \sigma dW$$

- Price = expected value of future payments
- The expectation should **not** be taken under the "objective" probabilities P, but under the "risk adjusted" probabilities Q.

Interpetation of the risk adjusted probabilities

- The risk adjusted probabilities can be interpreted as probabilities in a (fictuous) risk neutral world.
- When we **compute prices**, we can calculate **as if** we live in a risk neutral world.
- This does **not** mean that we live in, or think that we live in, a risk neutral world.
- The formulas above hold regardless of the attitude towards risk of the investor, as long as he/she prefers more to less.

Martingale Analysis

Model: Under P we have

$$
\begin{aligned}
dr_t &= \mu(t, r_t)\, dt + \sigma(t, r_t)\, dW_t, \\
dB_t &= rB_t dt,
\end{aligned}
$$

We look for martingale measures. Since B is the only traded asset we need to find $Q \sim P$ such that

$$\frac{B_t}{B_t} = 1$$

is a Q martingale.

Result: In a short rate model, **every** $Q \sim P$ is a martingale measure.

Girsanov

$$dL_t = L_t \varphi_t dW_t$$

P-dynamics

$$dr_t = \mu(t, r_t) dt + \sigma(t, r_t) dW_t,$$

$$dL_t = L_t \varphi_t dW_t$$

$dQ = L_t dP$ on $\mathcal{F}_t$

Girsanov:

$$dW_t = \varphi_t dt + dW_t^Q$$

Martingale pricing:

$$\Pi[t; Z] = E^Q \left[e^{-\int_0^t r_s ds} Z \middle| \mathcal{F}_t \right]$$

Q-dynamics of r:

$$dr_t = \{\mu(t, r_t) + \sigma(t, r_t) \varphi_t\} dt + \sigma(t, r_t) dW_t^Q,$$

Result: We have $\lambda_t = -\varphi_t$, i.e,. the Girsanov kernel φ equals minus the market price of risk.

5.

Martingale Models for the Short Rate

Ch. 24

Martingale Modelling

Recall:

$$\Pi_t\left[X\right] = E^Q\left[e^{-\int_0^t r_s ds} \cdot X \middle| \mathcal{F}_t\right]$$

- All prices are determined by the Q-dynamics of r.
- Model dr directly under Q!

Problem: Parameter estimation!

Pricing under risk adjusted probabilities

Q-dynamics:

$$dr = \mu(t,r)dt + \sigma(t,r)dW$$

where W denotes a Q-Wiener process.

$$\begin{aligned}\Pi_t[X] &= E^Q\left[e^{-\int_t^T r_s ds} \times X \middle| \mathcal{F}_t\right] \\ p(t,T) &= E^Q\left[e^{-\int_t^T r_s ds} \times 1 \middle| \mathcal{F}_t\right]\end{aligned}$$

The case $\mathbf{X} = \Phi(r_T)$:

price given by

$$\Pi_t[X] = F(t, r_t)$$

$$\begin{cases} F_t + \mu F_r + \frac{1}{2}\sigma^2 F_{rr} - rF = 0, \\ F(T,r) = \Phi(r(T)). \end{cases}$$

1. Vasiček

$$dr = (b - ar)\,dt + \sigma dW,$$

2. Cox-Ingersoll-Ross

$$dr = (b - ar)\,dt + \sigma\sqrt{r}dW,$$

3. Dothan

$$dr = ardt + \sigma r dW,$$

4. Black-Derman-Toy

$$dr = \Phi(t)rdt + \sigma(t)rdW,$$

5. Ho-Lee

$$dr = \Phi(t)dt + \sigma dW,$$

6. Hull-White (extended Vasiček)

$$dr = \{\Phi(t) - ar\}\,dt + \sigma dW,$$

Bond Options

European call on a T-bond with strike price K and delivery date S.

$$
\begin{aligned}
X &= \max\left[p(S,T) - K,\ 0\right] \\
X &= \max\left[F^T(S, r_S) - K,\ 0\right]
\end{aligned}
$$

We have

$$\Pi_t\left[X\right] = F(t, r_t)$$

where F solves the PDE

$$
\begin{cases}
F_t + \mu F_r + \frac{1}{2}\sigma^2 F_{rr} - rF = 0, \\
F(S,r) = \Phi(r).
\end{cases}
$$

and where Φ is defined by

$$\Phi(r) = \max\left[F^T(S,r) - K,\ 0\right]$$

To solve the pricing PDE for F we need to calculate

$$\Phi(r) = \max\left[F^T(S,r) - K,\ 0\right]$$

and for this we need to compute the theoretical bond price function F^T. We thus also need to solve the PDE

$$\begin{cases} F_t^T + \mu F_r^T + \frac{1}{2}\sigma^2 F_{rr}^T - rF^T &= 0, \\ F^T(T,r) &= 1. \end{cases}$$

Lots of equations!

Need analytic solutions.

Affine Term Structures

We have an **Affine Term Structure** if

$$F(t, r; T) = e^{A(t,T) - B(t,T)r},$$

where A and B are deterministic functions.

Moral: If you want to obtain analytical formulas, then you must have an ATS.

Problem: How do we specify μ and σ in order to have an ATS?

Main Result for ATS

Proposition: Assume that μ and σ are of the form

$$\begin{aligned} \mu(t,r) &= \alpha(t)r + \beta(t), \\ \sigma^2(t,r) &= \gamma(t)r + \delta(t). \end{aligned}$$

Then the model admits an affine term structure

$$F(t,r;T) = e^{A(t,T)-B(t,T)r},$$

where A and B satisfy the system

$$\begin{cases} B_t(t,T) = -\alpha(t)B(t,T) + \frac{1}{2}\gamma(t)B^2(t,T) - 1, \\ B(T;T) = 0. \end{cases}$$

$$\begin{cases} A_t(t,T) = \beta(t)B(t,T) - \frac{1}{2}\delta(t)B^2(t,T), \\ A(T;T) = 0. \end{cases}$$

Parameter Estimation

Suppose that we have chosen a specific model, e.g. H-W . How do we estimate the parameters $a,\ b,\ \sigma$?

Naive answer:
Use standard methods from statistical theory.

WRONG!!

- The parameters are Q-parameters.
- Our observations are **not** under Q, but under P.
- Standard statistical techniques can **not** be used.
- We need to know the market price of risk (λ).
- Who determines λ?
- **The Market!**
- We must get **price information from the market** in order to estimate parameters.

Inversion of the Yield Curve

Q-dynamics with parameter list α:

$$dr = \mu(t, r; \alpha)dt + \sigma(t, r; \alpha)dW$$

$$\Downarrow$$

Theoretical term structure

$$p(0, T; \alpha); \quad T \geq 0$$

Observed term structure

$$p^{\star}(0, T); \quad T \geq 0.$$

Requirement:
A model such that the **theoretical** prices of today coincide with the **observed** prices of today. We want to choose tha parameter vector α such that

$$p(0,T;\alpha) \approx p^{\star}(0,T); \ \forall T \geq 0$$

Number of equations $= \infty$ (one for each T).
Number of unknowns = number of parameters.

Need:
Infinite parameter list.

The time dependent function Φ in Hull-White is precisely such an infinite parameter list (one parameter for every t).

Result

Hull-White can be calibrated exactly to any initial term strucutre. The calibrated model has the form

$$p(t,T) = \frac{p^\star(0,T)}{p^\star(0,t)} \times e^{C(t,r_t)}$$

where C is given by

$$B(t,T)f^\star(0,t) - \frac{\sigma^2}{2a^2}B^2(t,T)\left(1 - e^{-2aT}\right) - B(t,T)r_t$$

There are analytical formulas for interest rate options.

Short rate models

Pro:

- Easy to model r.
- Analytical formulas for bond prices and bond options.

Con:

- Inverting the yield curve can be hard work.
- Hard to model a flexible volatility structure for forward rates.
- With a one factor model, all points on the yield curve are perfectly correlated.

6.

Forward Rate Models

Ch. 25

Heath-Jarrow-Morton

Idea:

- Model the dynamics for the **entire yield curve**.
- The yield curve itself (rather than the short rate r) is the explanatory variable.
- Model forward rates. Use observed yield curve as boundary value.

Dynamics:

$$\begin{aligned} df(t,T) &= \alpha(t,T)dt + \sigma(t,T)dW(t), \\ f(0,T) &= f^{\star}(0,T). \end{aligned}$$

One SDE for every fixed maturity time T.

Existence of martingale measure

$$f(t,T) = \frac{\partial \log p(t,T)}{\partial T}$$

$$p(t,T) = \exp\left\{-\int_t^T f(t,s)ds\right\}$$

Thus:

Specifying forward rates.

$$\Longleftrightarrow$$

Specifying bond prices.

Thus:

No arbitrage
$\Downarrow$
restrictions on α and σ.

Strategy

- Start with P-dynamics for the forward rates

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)d\bar{W}(t)$$

where $\bar{W}$ is P-Wiener.

- Compute the corresponding bond price dynamics.

- Do a Girsanov transformation $P \to Q$.

- The Q-dynamics must then have the form

$$dp(t,T) = r_t p(t,T)dt + p(t,T)v(t,T)dW(t)$$

where W is Q-Wiener.

Practical Toolbox

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)d\bar{W}$$

$$\Downarrow$$

$$\begin{aligned} dp(t,T) &= p(t,T)\left\{ r(t) + A(t,T) + \frac{1}{2}\|S(t,T)\|^2 \right\} dt \\ &+ p(t,T)S(t,T)d\bar{W} \end{aligned}$$

$$\begin{cases} A(t,T) = -\int_t^T \alpha(t,s)ds, \\ S(t,T) = -\int_t^T \sigma(t,s)ds \end{cases}$$

Girsanov

$$\begin{cases} dL(t) & = & L(t)g^*(t)d\bar{W}(t), \\ L(0) & = & 1. \end{cases}$$

From Girsanov we have

$$d\bar{W}_t = gdt + dW_t$$

where W is Q-Wiener.

From the toolbox, the Q-dynamics are obtained as

$$\begin{aligned} dp(t,T) &= p(t,T)r(t)dt \\ &+ \left\{A(t,T) + \frac{1}{2}||S(t,T)||^2 + S(t,T)g(t)\right\}dt \\ &+ p(t,T)S(t,T)dW(t), \end{aligned}$$

Proposition:

There exists a martingale measure

$$\Updownarrow$$

There exists process $g(t) = [g_1(t), \cdots g_d(t)]^*$ s.t.

$$A(t,T) + \frac{1}{2}||S(t,T)||^2 + S(t,T)g(t) = 0, \quad \forall t, T$$

Taking T-derivatives we obtain the alternative formula

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma^*(t,s)ds - \sigma(t,T)g(t), \quad \forall t, T$$

Moral for Modeling

- Specify arbitrary volatilities $\sigma(t,T)$.
- Fix d "benchmark" maturities $T_1, \cdots, T_d$. For these maturities, specify drift terms $\alpha(t,T_1), \cdots \alpha(t,T_1)$.
- The Girsanov kernel is uniquely determined (for each fixed t) by

$$
\begin{aligned}
\sum_{i=1}^{d} \sigma_i(t,T_j) g_i(t) &= \sum_{i=1}^{d} \sigma_i(t,T_j) \int_0^T \sigma_i(t,s) ds \\
&- \alpha(t,T_j), \quad j = 1, \cdots d.
\end{aligned}
$$

- Thus Q is uniquely determined.
- All other drift terms will be uniquely defined by

$$
\alpha(t,T) = \sigma(t,T) \int_t^T \sigma(t,s) ds - \sigma(t,T) g(t), \quad \forall t, T
$$

Martingale Modelling

Q-dynamics:

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t)$$

- Specifying forward rates $\Longleftrightarrow$ specifying bond prices.
- Under Q all bond prices have r as the local rate of return.
- Thus, martingale modeling $\Longrightarrow$ restrictions on α and σ.

Which?

Martingale modeling

Recall:

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)ds - \sigma(t,T)g(t), \quad \forall t,T$$

Martingale modeling $\Longleftrightarrow g = 0$

Theorem: (HJM drift Condition) The bond market is arbitrage free if and only if

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)ds.$$

Moral: Volatility can be specified freely. The forward rate drift term is then uniquely determined.

Musiela parametrization

Parameterize forward rates by the time **to** maturity x, rather than time **of** maturity T.

Def:

$$r(t, x) = f(t, t + x).$$

Q-dynamics:

$$dr(t, x) = \mu(t, x)dt + \sigma(t, x)dW.$$

What are the relations between μ and σ under Q?

Compare with HJM!

$$df(t, T) = \alpha(t, T)dt + \sigma_0(t, T)dW.$$

where we use σ_0 to denote the HJM volatility.

$$
\begin{aligned}
dr(t,x) &= d\left[f(t,t+x)\right] \\
&= df(t,t+x) + f_T(t,t+x)dt \\
&= \left\{\alpha(t,t+x) + r_x(t,x)\right\}dt + \sigma_0(t,t+x)dW
\end{aligned}
$$

$$
\begin{aligned}
\mu(t,x) &= \alpha(t,t+x) + r_x(t,x) \\
\sigma(t,x) &= \sigma_0(t,t+x).
\end{aligned}
$$

HJM-condition:

$$
\alpha(t,T) = \sigma_0(t,T)\int_t^T \sigma_0(t,s)ds.
$$

Substitute!

The Musiela Equation

$$dr(t,x) = \left\{ \frac{\partial}{\partial x} r(t,x) + \sigma(t,x) \int_0^x \sigma(t,y)dy \right\} dt$$

$$+ \quad \sigma(t,x)dW$$

When σ is **deterministic** this is a **linear** equation in infinite dimensional space. Connections to control theory.

Forward Rate Models

Pro:

- Easy to model flexible volatility structure for forward rates.

- Easy to include multiple factors.

Con:

- The short rate will typically not be a Markov process.

- Computational problems.

7.

Change of Numeraire

Ch. 26

Change of Numeraire

Valuation formula:

$$\Pi_t[X] = E^Q\left[e^{-\int_t^T r_s ds} \times X \middle| \mathcal{F}_t\right]$$

Hard to compute. Double integral.

Note: If X and r are **independent** then

$$\begin{aligned}\Pi_t[X] &= E^Q\left[e^{-\int_t^T r_s ds} \middle| \mathcal{F}_t\right] \cdot E^Q\left[X \middle| \mathcal{F}_t\right] \\ &= p(t,T) \cdot E^Q\left[X \middle| \mathcal{F}_t\right].\end{aligned}$$

Nice! We do not have to compute $p(t,T)$. It can be observed directly on the market!
Single integral!

Sad Fact: X and r are (almost) never independent!

Idea

Use T-bond (for a fixed T) as numeraire. Define the **$\mathbf{T}$-forward measure** Q^T by the requirement that

$$\frac{\Pi(t)}{p(t,T)}$$

is a Q^T-martingale for every price process $\Pi(t)$.

Then

$$\frac{\Pi_t[X]}{p(t,T)} = E^T\left[\left.\frac{\Pi_T[X]}{p(T,T)}\right| \mathcal{F}_t\right]$$

$$\Pi_T[X] = X, \qquad p(T,T) = 1.$$

$$\Pi_t[X] = p(t,T)E^T[X|\mathcal{F}_t]$$

Do such measures exist?.

“The forward measure takes care of the stochastics over the interval $[t, T]$.”

Enormous computational advantages.

Useful for interest rate derivatives, currency derivatives and derivatives defined by several underlying assets.

General change of numeraire.

Idea: Use a fixed asset price process S_t as numeraire. Define the measure Q^S by the requirement that

$$\frac{\Pi(t)}{S_t}$$

is a Q^S-martingale for every arbitrage free price process $\Pi(t)$.

Constructing $\mathbf{Q}^S$

Fix a T-claim X. From general theory:

$$\Pi_0[X] = E^Q\left[\frac{X}{B_T}\right]$$

Assume that Q^S exists and denote

$$L_t = \frac{dQ^S}{dQ}, \quad \text{on } \mathcal{F}_t$$

Then

$$\begin{aligned} \frac{\Pi_0[X]}{S_0} &= E^S\left[\frac{\Pi_T[X]}{S_T}\right] = E^S\left[\frac{X}{S_T}\right] \\ &= E^Q\left[L_T \frac{X}{S_T}\right] \end{aligned}$$

Thus we have

$$\Pi_0[X] = E^Q\left[L_T \frac{X \cdot S_0}{S_T}\right],$$

For all $X \in \mathcal{F}_T$ we thus have

$$E^Q\left[\frac{X}{B_T}\right] = E^Q\left[L_T \frac{X \cdot S_0}{S_T}\right]$$

Natural candidate:

$$L_t = \frac{dQ_t^S}{dQ_t} = \frac{S_t}{S_0 B_t}$$

Proposition:

$$\begin{gathered}\Pi(t)/B_t \text{ is a } Q\text{-martingale.} \\ \Downarrow \\ \Pi(t)/S_t \text{ is a } Q^\star\text{-martingale.}\end{gathered}$$

Proof.

$$E^S\left[\left.\frac{\Pi(t)}{S_t}\right|\mathcal{F}_s\right]=\frac{E^Q\left[\left.L_t\frac{\Pi(t)}{S_t}\right|\mathcal{F}_s\right]}{L_s}$$

$$=\quad\frac{E^Q\left[\left.\frac{\Pi(t)}{B_tS_0}\right|\mathcal{F}_s\right]}{L_s}=\frac{\Pi(s)}{B(s)S_0L_s}$$

$$=\quad\frac{\Pi(s)}{S(s)}.\blacksquare$$

Result

$$\Pi_t[X] = S_t E^S \left[\left. \frac{X}{S_t} \right| \mathcal{F}_t \right]$$

We can observe S_t directly on the market.

Example: $X = S_t \cdot Y$

$$\Pi_t[X] = S_t E^S \left[Y \middle| \mathcal{F}_t \right]$$

Several underlying

$$X = \Phi\left[S_0(T), S_1(T)\right]$$

Assume Φ is linearly homogeous. Transform to Q^0.

$$\begin{aligned}\Pi_t\left[X\right] &= S_0(t)E^0\left[\left.\frac{\Phi\left[S_0(T), S_1(T)\right]}{S_0(T)}\right| \mathcal{F}_t\right] \\ &= S_0(t)E^0\left[\left.\varphi\left(Z_T\right)\right| \mathcal{F}_t\right]\end{aligned}$$

$$\varphi\left(z\right) = \Phi\left[1, z\right], \quad Z_t = \frac{S_1(t)}{S_0(t)}$$

Exchange option

$$X = \max\left[S_1(T) - S_0(T),\ 0\right]$$

$$\Pi_t\left[X\right] = S_0(t)E^0\left[\max\left[Z(T) - 1,\ 0\right]\middle|\,\mathcal{F}_t\right]$$

European Call on Z with strike price K. Zero interest rate.

Piece of cake!

Identifying the Girsanov Transformation

Assume Q-dynamics of S known as

$$dS_t = r_t S_t dt + S_t v_t dW_t$$

$$L_t = \frac{S_t}{S_0 B_t}$$

From this we immediately have

$$dL_t = L_t v_t dW_t.$$

and we can summarize.

Theorem: The Girsanov kernel is given by the numeraire volatility v_t, i.e.

$$dL_t = L_t v_t dW_t.$$

Forward Measures

Use price of T-bond as numeraire.

$$L_t^T = \frac{p(t,T)}{p(0,T)B_t}$$

$$\begin{aligned} dp(t,T) &= r_t p(t,T)dt + p(t,T)v(t,T)dW_t, \\ dL_t^T &= L_t^T v(t,T)dW_t \end{aligned}$$

Result:

$$\Pi_t[X] = p(t,T)E^T[X|\mathcal{F}_t]$$

Common Conjecture: "The forward rate is an unbiased estimator of the future spot rate:"

Lemma:

$$f(t,T) = E^T[r_t|\mathcal{F}_t]$$

A new look on option pricing

European call on asset S with strike price K and maturity T.

$$X = \max\left[S_T - K,\ 0\right]$$

Write X as

$$X = (S_T - K) \cdot I\left\{S_T \geq K\right\} = S_T I\left\{S_T \geq K\right\} - K I\left\{S_T \geq K\right\}$$

Use Q^S on the first term and Q^T on the second.

$$\Pi_0\left[X\right] = S_0 \cdot Q^S\left[S_T \geq K\right] - K \cdot p(0,T) \cdot Q^T\left[S_T \geq K\right]$$

Analytical Results

Assumption: Assume that $Z_{S,T}$, defined by

$$Z_{S,T}(t) = \frac{S_t}{p(t,T)},$$

has dynamics

$$dZ_{S,T}(t) = Z_{S,T}(t)m_T^S(t)dt + Z_{S,T}(t)\sigma_{S,T}(t)dW,$$

where $\sigma_{S,T}(t)$ is **deterministic**.

We have to compute

$$Q^T\left[S_T \geq K\right]$$

and

$$Q^S\left[S_T \geq K\right]$$

$$
\begin{aligned}
Q^T\left(S_T \geq K\right) &= Q^T\left(\frac{S_T}{p(T,T)} \geq K\right) \\
&= Q^T\left(Z_{S,T}(T) \geq K\right)
\end{aligned}
$$

By definition $Z_{S,T}$ is a Q^T-martingale, so Q^T-dynamics are given by

$$
dZ_{S,T}(t) = Z_{S,T}(t)\sigma_{S,T}(t)dW^T,
$$

with the solution

$$
Z_{S,T}(T) =
$$

$$
\frac{S_0}{p(0,T)} \times \exp\left\{-\frac{1}{2}\int_0^T \sigma_{S,T}^2(t)dt + \int_0^T \sigma_{S,T}(t)dW^T\right\}
$$

Lognormal distribution!

The integral

$$\int_0^T \sigma_{S,T}(t)dW^T$$

is Gaussian, with zero mean and variance

$$\Sigma^2_{S,T}(T) = \int_0^T \|\sigma_{S,T}(t)\|^2 dt$$

Thus

$$Q^T\left(S_t \geq K\right) = N[d_2],$$

$$d_2 = \frac{\ln\left(\frac{S_0}{Kp(0,T)}\right) - \frac{1}{2}\Sigma^2_{S,T}(T)}{\sqrt{\Sigma^2_{S,T}(T)}}$$

$$
\begin{aligned}
Q^S\left(S_t \geq K\right) &= Q^S\left(\frac{p(T,T)}{S_t} \leq \frac{1}{K}\right) \\
&= Q^S\left(Y_{S,T}(T) \leq \frac{1}{K}\right),
\end{aligned}
$$

$$
Y_{S,T}(t) = \frac{p(t,T)}{S_t} = \frac{1}{Z_{S,T}(t)}.
$$

$Y_{S,T}$ is a Q^S-martingale, so Q^S-dynamics are

$$
dY_{S,T}(t) = Y_{S,T}(t)\delta_{S,T}(t)dW^S.
$$

$$
Y_{S,T} = Z_{S,T}^{-1}
$$

$$
\Downarrow
$$

$$
\delta_{S,T}(t) = -\sigma_{S,T}(t)
$$

$$Y_{S,T}(T) =$$

$$\frac{p(0,T)}{S_0} \exp\left\{ -\frac{1}{2} \int_0^T \sigma^2_{S,T}(t)dt - \int_0^T \sigma_{S,T}(t)dW^S \right\},$$

$$Q^S\left(S_t \geq K\right) = N[d_1],$$

$$d_1 = d_2 + \sqrt{\Sigma^2_{S,T}(T)}$$

Proposition: Price of call is given by

$$\Pi_0[X] = S_0 N[d_2] - K \cdot p(0,T) N[d_1]$$

$$d_2 = \frac{\ln\left(\frac{S_0}{Kp(0,T)}\right) - \frac{1}{2}\Sigma^2_{S,T}(T)}{\sqrt{\Sigma^2_{S,T}(T)}}$$

$$d_1 = d_2 + \sqrt{\Sigma^2_{S,T}(T)}$$

$$\Sigma^2_{S,T}(T) = \int_0^T \|\sigma_{S,T}(t)\|^2 dt$$

Hull-White

Q-dynamics:

$$dr = \{\Phi(t) - ar\} dt + \sigma dW.$$

Affine term structure:

$$p(t,T) = e^{A(t,T) - B(t,T)r_t},$$

$$B(t,T) = \frac{1}{a}\left\{1 - e^{-a(T-t)}\right\}.$$

Check if Z has deterministic volatility

$$Z_t = \frac{S_t}{p(t,T_1)}, \qquad S_t = p(t,T_2)$$

$$Z_t = \frac{p(t,T_2)}{p(t,T_1)},$$

$$Z_t = \exp\{\Delta A(t) - \Delta B_t r_t\},$$

$$Z(t) = \exp\left\{\Delta A(t) - \Delta B_t r_t\right\},$$

$$\begin{aligned}\Delta A(t) &= A(t,T_2) - A(t,T_1), \\ \Delta B_t &= B(t,T_2) - B(t,T_1),\end{aligned}$$

$$dZ(t) = Z(t)\left\{\cdots\right\}dt + Z(t)\cdot\sigma_z(t)dW,$$

$$\sigma_z(t) = -\sigma\Delta B_t = \frac{\sigma}{a}e^{at}\left[e^{-aT_1} - e^{-aT_2}\right]$$

Deterministic volatility!

8

LIBOR Market Models

Ch. 27

Problems with infinitesimal rates

- Infinitesimal rates can never be observed in real life.
- Calibration to cap- or swaption data is difficult.

Disturbing facts from real life:

- The market uses Black-76 to quote caps and swaptions. Behind Blavk-76 are the assumptions
  - The short rate is constant.
  - The LIBOR rates are lognormally distributed.
- Logically inconsistent!
- Despite this, the market happily continues to use Black-76 for quoting purposes.

Project

- Construct a **logically consistent model** which (to some extent) justifies market practice.
- Construct an **arbitrage free** model with the property that caps, floors and/or swaptions are priced with a Black-76 type formula.

Main models

- **LIBOR market models** (Miltersen-Sandmann-Sondermann, Brace-Gatarek-Musiela)
- **Swap market models** (Jamshidian).

- Instead of modeling instantaneous rates, we model discrete **market rates**, such as
  - LIBOR rates (LIBOR market models)
  - Forward swap rates (swap market models).
- Under a suitable numeraire the market rates can be modeled lognormally.
- The market models with thus produce pricing formulas of the type Black-76.
- By construction the market models are very easy to calibrate to market data, i.e. to:
  - Caps and floors (LIBOR market model)
  - Swaptions (swap market model)
- Exotic derivatives has to be priced numerically.

Caps

Resettlement dates:

$$T_0 < T_1 < \ldots < T_n,$$

Tenor:

$$\alpha = T_{i+1} - T_i, \quad i = 0, \ldots, n-1.$$

Typically $\alpha = 1/4$, i.e. quarterly resettlement.

LIBOR forward rate for $[T_{i-1}, T_i]$:

$$L_i(t) = \frac{1}{\alpha} \cdot \frac{p_{i-1}(t) - p_i(t)}{p_i(t)}, \quad i = 1, \ldots, N.$$

where we use the notation

$$p_i(t) = p(t, T_i)$$

Definition:
A **cap** with **cap rate** R and **resettlement dates** $T_0, \dots, T_n$ is a contract which at each T_i give the holder the amount

$$X_i = \alpha \cdot \max\left[L_i(T_{i-1}) - R, 0\right], \quad i = 1, \dots, N$$

The cap is thus a portfolio of **caplets** $X_1, \dots, X_n$.

Black-76

The **Black-76** formula for the caplet

$$X_i = \alpha_i \cdot \max\left[L(T_{i-1}, T_i) - R, 0\right], \qquad (1)$$

is given by

$$\mathbf{Capl_i^B}(t) = \alpha \cdot p_i(t)\left\{L_i(t)N[d_1] - RN[d_2]\right\}$$

where

$$\begin{aligned} d_1 &= \frac{1}{\sigma_i\sqrt{T_i - t}}\left[\ln\left(\frac{L_i(t)}{R}\right) + \frac{1}{2}\sigma_i^2(T - t)\right], \\ d_2 &= d_1 - \sigma_i\sqrt{T_i - t}. \end{aligned}$$

- Black-76 presupposes that each LIBOR rate is lognormal.
- The constants $\sigma_1, \ldots, \sigma_N$ are known as the **Black volatilities**

Market price quotes

Market prices are quoted in terms of **Implied Black volatilities:** The can be quoted in two different ways.

- **Flat volatilities**
- **Spot volatilities** (also known as **forward volatilities**)

Market Price Data

For each $i = 1, \ldots, N$:

$$\mathbf{Cap_i^m(t)} \quad = \quad \text{market price of cap with resettlement dates } T_0, T_1, \ldots, T_i$$

Implied market prices of caplets:

$$\mathbf{Capl_i^m}(t) = \mathbf{Cap_i^m}(t) - \mathbf{Cap_{i-1}^m}(t),$$

with the convention $\mathbf{Cap_0^m}(t) = 0$

Defining Implied Black Volatility

Given market price data as above, the implied Black volatilities are defined as follows.

- The implied **flat volatilities** $\bar{\sigma}_1, \ldots, \bar{\sigma}_N$ are defined as the solutions of the equations

$$\mathbf{Cap_i^m}(t) = \sum_{k=1}^{i} \mathbf{Capl_k^B}(t; \bar{\sigma}_i), i = 1, \ldots, N. \quad (2)$$

- The implied **forward** or **spot** volatilities $\bar{\sigma}_1, \ldots, \bar{\sigma}_N$ are defined as solutions of the equations

$$\mathbf{Capl_i^m}(t) = \mathbf{Capl_i^B}(t; \bar{\sigma}_i), i = 1, \ldots, N. \quad (3)$$

The sequence $\bar{\sigma}_1, \ldots, \bar{\sigma}_N$ is called the volatility **term structure**.

Theoretical Price of a Caplet

By risk neutral valuation:

$$\mathbf{Capl_i}(t) = \alpha E_t^Q \left[e^{-\int_0^{T_i} r(s)ds} \cdot \max\left[L_i(T_{i-1}) - R, 0\right] \right],$$

Better to use T_i forward measure

$$\mathbf{Capl_i}(t) = \alpha p_i(t) E^{T_i} \left[\max\left[L_i(T_{i-1}) - R, 0\right] \middle| \mathcal{F}_t \right],$$

The crucial point is the distribution of L_i under Q^i where $Q^i = Q^{T_i}$

Important Fact: L_i is a **martingale** under Q^i

$$L_i(t) = \frac{1}{\alpha} \cdot \frac{p_{i-1}(t) - p_i(t)}{p_i(t)}$$

Idea: Model L_i as GBM under Q^i:

$$dL_i = \sigma_i L_i dW^i$$

LIBOR Market Model Definition

Define, for each i, the dynamics of L_i under Q^i as

$$dL_i(t) = L_i(t)\sigma_i(t)dW^i(t), \quad i = 1, \ldots, N,$$

where $\sigma_1(t), \ldots, \sigma_N(t)$ are **deterministic** and W^i is Q^i-Wiener.

The initial term structure $L_1(0), \ldots, L_N(0)$ is observed on the market.

Pricing Caps in the LIBOR Model

$$L_i(T) = L_i(t) \cdot e^{\int_t^T \sigma_i(s)dW^i(s) - \frac{1}{2}\int_t^T \|\sigma_i(s)\|^2 ds}.$$

Lognormal!

Theorem: Caplet prices are given by

$$\mathbf{Capl_i}(t) = \alpha_i \cdot p_i(t) \left\{ L_i(t)N[d_1] - RN[d_2] \right\},$$

where

$$d_1 = \frac{1}{\Sigma_i(t, T_{i-1})} \left[\ln\left(\frac{L_i(t)}{R} \right) + \frac{1}{2}\Sigma_i^2(t, T_{i-1}) \right],$$

$$d_2 = d_1 - \Sigma_i(t, T_{i-1}),$$

$$\Sigma_i^2(t, T) = \int_t^T \|\sigma_i(s)\|^2 ds.$$

Moral: Each caplet price is given by a Black-76 formula with Σ_i as the Black volatility.

Practical Handling of the LIBOR Model

We are standing at time $t = 0$.

- Collect implied caplet volatilities

$$\bar{\sigma}_1, \ldots, \bar{\sigma}_N$$

from the market.

- Choose model volatilities

$$\sigma_1(\cdot), \ldots, \sigma_N(\cdot)$$

such that

$$\bar{\sigma}_i = \frac{1}{T_i} \int_0^{T_{i-1}} \sigma_i^2(s) ds, \quad i = 1, \ldots, N.$$

- Now the model is calibrated.
- Use numerical methods to compute prices of exotics.

Terminal Measure dynamics

Define the Likelihood process η_i^j as

$$\eta_i^j(t) = \frac{dQ^j}{dQ^i}, \quad \text{on } \mathcal{F}_t$$

Can show that

$$d\eta_i^{i-1}(t) = \eta_i^{i-1}(t)\frac{\alpha_i L_i(t)}{1+\alpha_i L_i(t)}\sigma_i(t)dW^i(t).$$

Girsanov gives us

$$dW^i(t) = \frac{\alpha_i L_i(t)}{1+\alpha_i L_i(t)}\sigma_i^\star(t)dt + dW^{i-1}(t).$$

Proposition The Q^N dynamics of the LIBOR rates are

$$\begin{aligned} dL_i(t) &= -L_i(t)\left(\sum_{k=i+1}^{N}\frac{\alpha_k L_k(t)}{1+\alpha_k L_k(t)}\sigma_k(t)\sigma_i^\star(t)\right)dt \\ &+ L_i(t)\sigma_i(t)dW^N(t), \end{aligned}$$

Computational aspects

- The terminal measure dynamics of the system of LIBOR rates are quite messy.
- Various approximations for the drift term have been suggested.
- Numerical work.