Finite dimensional realizations of HJM models

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Definitions:

$\mathbf{p}_t(x)$: Price, at t of zero coupon bond maturing at $t+x$,

$\mathbf{r}_t(x)$: Forward rate, contracted at t, maturing at $t+x$

R_t : Short rate.

$$r_t(x) = -\frac{\partial \log p_t(x)}{\partial x}$$

$$p_t(x) = e^{-\int_0^x r_t(s)ds}$$

$$R_t = r_t(0).$$

Heath-Jarrow-Morton-Musiela

Idea: Model the dynamics for the **entire forward rate curve**.

The yield curve itself (rather than the short rate R) is the explanatory variable.

Model forward rates. Use observed forward rate curve as initial condition.

Q-dynamics:

$$\begin{aligned} dr_t(x) &= \alpha_t(x)dt + \sigma_t(x)dW_t, \\ r_0(x) &= r_0^\star(x), \quad \forall x \end{aligned}$$

W: d-dimensional Wiener process

One SDE for every fixed x.

Theorem: (HJMM drift Condition) The following relations must hold, under a martingale measure Q.

$$\alpha_t(x) = \frac{\partial}{\partial x} r_t(x) + \sigma_t(x) \int_0^x \sigma_t(s) ds.$$

Moral: Volatility can be specified freely. The forward rate drift term is then uniquely determined.

The Interest Rate Model

$$r_t = r_t(\cdot), \quad \sigma_t(x) = \sigma(r_t, x)$$

Heath-Jarrow-Morton-Musiela equation:

$$dr_t = \mu_0(r_t)dt + \sigma(r_t)dW_t$$

$$\mu_0(r_t, x) = \frac{\partial}{\partial x} r_t(x) + \sigma(r_t, x) \int_0^x \sigma(r_t, s) ds$$

The HJMM equation is an **infinite dimensional SDE** evolving in the space $\mathcal{H}$ of forward rate curves.

Sometimes you are lucky!

Example:

$$\sigma(r, x) = \sigma e^{-ax}$$

In this case the HJMM equation has a **finite dimensional state space realization**. We have in fact:

$$r_t(x) = B(t, x)Z_t - A(t, x)$$

where Z solves the one-dimensional SDE

$$dZ_t = \{\Phi(t) - aZ_t\}\, dt + \sigma dW_t$$

Furthermore the state process Z can be identified with the short rate $R = r(0)$.
(A, B and Φ are deterministic functions)

A Hilbert Space

Definition:
For each $(\alpha, \beta) \in R^2$, the space $\mathcal{H}_{\alpha,\beta}$ is defined by

$$\mathcal{H}_{\alpha,\beta} = \{f \in C^{\infty}[0, \infty); \ \|f\| < \infty\}$$

where

$$\|f\|^2 = \sum_{n=0}^{\infty} \beta^{-n} \int_0^{\infty} \left[f^{(n)}(x)\right]^2 e^{-\alpha x} dx$$

where

$$f^{(n)}(x) = \frac{d^n f}{dt^n}(x).$$

We equip $\mathcal{H}$ with the inner product

$$(f, g) = \sum_{n=0}^{\infty} \beta^{-n} \int_0^{\infty} f^{(n)}(x) g^{(n)}(x) e^{-\alpha x} dx$$

Properties of $\mathcal{H}$

Proposition:
The following hold.

- The linear operator
$$\mathbf{F} = \frac{\partial}{\partial x}$$
is bounded on $\mathcal{H}$

- $\mathcal{H}$ is complete, i.e. it is a Hilbert space.

- The elements in $\mathcal{H}$ are real analytic functions on R (not only on R_+).

NB: Filipovic and Teichmann!

Stratonovich Integrals

Definition The **Stratonovich integral**

$$\int_0^t X_s \circ dY_s$$

is defined as

$$\int_0^t X_s \circ dY_s = \int_0^t X_s dY_s + \frac{1}{2}\langle X, Y\rangle_t$$

$$\langle X, Y\rangle_t = \int_0^t dX_s dY_s,$$

Proposition: For any smooth F we have

$$dF(t, Y_t) = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial y} \circ dY_t$$

Stratonovich Form of HJMM

$$dr_t = \mu(r_t)dt + \sigma(r_t) \circ dW_t$$

where

$$\mu(r_t) = \mu_0(r_t) - \frac{1}{2}\frac{d\langle\sigma, W\rangle}{dt}$$

Main Point:
Using the Stratonovich differential we have no Itô second order term. Thus we can treat the SDE above as the ODE

$$\frac{dr_t}{dt} = \mu(r_t) + \sigma(r_t) \cdot v_t$$

where $v_t =$ "white noise".

Natural Questions

- What do the forward rate curves look like?
- What is the support set of the HJMM equation?
- When is a given model (e.g. Hull-White) consistent with a given family (e.g. Nelson-Siegel) of forward rate curves?
- When is the short rate Markov?
- When is a finite set of benchmark forward rates Markov?
- When does the interest rate model admit a realization in terms of a finite dimensional factor model?
- If there exists an FDR how can you construct a concrete realization?

Finite Dimensional Realizations

Main Problem:
When does a given interest rate model possess a finite dimensional realisation, i.e. when can we write r as

$$\begin{aligned} z_t &= \eta(z_t)dt + \delta(z_t) \circ dW(t), \\ r_t(x) &= G(z_t, x), \end{aligned}$$

where z is a **finite-dimensional** diffusion, and

$$G : R^d \times R_+ \to R$$

or alternatively

$$G : R^d \to \mathcal{H}$$

$\mathcal{H} =$ the space of forward rate curves

Examples:

$$\begin{aligned}
\sigma(r,x) &= e^{-ax}, \\
\sigma(r,x) &= xe^{-ax}, \\
\sigma(r,x) &= e^{-x^2}, \\
\sigma(r,x) &= \log\left(\frac{1}{1+x^2}\right), \\
\sigma(r,x) &= \int_0^\infty e^{-s} r(s) ds \cdot x^2 e^{-ax}.
\end{aligned}$$

Which of these admit a finite dimensional realisation?

Earlier literature

- **Cheyette** (1996)
- **Bhar & Chiarella** (1997)
- **Chiarella & Kwon** (1998)
- **Inui & Kijima** (1998)
- **Ritchken & Sankarasubramanian** (1995)
- **Carverhill** (1994)
- **Eberlein & Raible** (1999)
- **Jeffrey** (1995)

All these papers present **sufficient** conditions for existence of an FDR.

Present paper

We would like to obtain:

- Necessary and sufficient conditions.

- A better understanding of the deep structure of the FDR problem.

- A general theory of FDR for arbitrary infinite dimensional SDEs.

We attack the general problem by viewing it as a **geometrical** problem.

Invariant Manifolds

Def:
Consider an interest rate model

$$dr_t = \mu(r_t)dt + \sigma(r_t) \circ dW_t$$

on the space $\mathcal{H}$ of forward rate curves. A manifold (surface) $\mathcal{G} \subseteq \mathcal{H}$ is an **invariant manifold** if

$$r_0 \in \mathcal{G} \Rightarrow r_t \in \mathcal{G}$$

P-a.s. for all $t > 0$

Main Insight

There exists a finite dimensional realization.

iff

There exists a finite dimensional invariant manifold.

Characterizing Invariant Manifolds

Proposition: (Björk-Christensen)
Consider an interest rate model on Stratonovich form

$$dr_t = \mu(r_t)dt + \sigma(r_t) \circ dW_t$$

A manifold $\mathcal{G}$ is invariant under r if and only if

$$\begin{aligned} \mu(r) &\in T_{\mathcal{G}}(r), \\ \sigma(r) &\in T_{\mathcal{G}}(r), \end{aligned}$$

at all points of $\mathcal{G}$. Here $T_{\mathcal{G}}(r)$ is the tangent space of $\mathcal{G}$ at the point $r \in \mathcal{G}$.

Main Problem

Given:

- An interest rate model on Stratonovich form

$$dr_t = \mu(r_t)dt + \sigma(r_t) \circ dW_t$$

- An inital forward rate curve r_0:

$$x \longmapsto r_0(x)$$

Question:
When does there exist a finite dimensional manifold $\mathcal{G}$, such that

$$r_0 \in \mathcal{G}$$

and

$$\begin{aligned} \mu(r) &\in T_{\mathcal{G}}(r), \\ \sigma(r) &\in T_{\mathcal{G}}(r), \end{aligned}$$

A manifold satisfying these conditions is called a **tangential manifold**.

Abstract Problem

On the Hilbert space $\mathcal{H}$, we are given two vector fields $f_1(r)$ and $f_2(r)$. We are also given a point $r_0 \in \mathcal{H}$.

Problem:
When does there exist a finite dimensional manifold $\mathcal{G} \subseteq \mathcal{H}$ such that

- We have the inclusion

$$r_0 \in \mathcal{G}$$

- For all points $r \in \mathcal{G}$ we have the relations

$$\begin{aligned} f_1(r) &\in T_{\mathcal{G}}(r), \\ f_2(r) &\in T_{\mathcal{G}}(r) \end{aligned}$$

We call such a $\mathcal{G}$ an **tangential manifold**.

Easier Problem

On the space $\mathcal{H}$, we are given **one** vector field $f_1(r)$. We are also given a point $r_0 \in \mathcal{H}$.

Problem:
When does there exist a finite dimensional manifold $\mathcal{G} \subseteq \mathcal{H}$ such that

- We have the inclusion

$$r_0 \in \mathcal{G}$$

- We have the relation

$$f_1(r) \;\; \in \;\; T_{\mathcal{G}}(r)$$

Answer to Easy Problem:

ALWAYS!

Proof:
Solve the ODE

$$\frac{dr_t}{dt} = f_1(r_t)$$

with initial point r_0. Denote the solution at time t by

$$e^{f_1 t} r_0$$

Then the integral curve $\left\{e^{f_1 t} r_0; t \in R\right\}$ solves the problem, i.e.

$$\mathcal{G} = \left\{e^{f_1 t} r_0; t \in R\right\}$$

Furthermore, the mapping

$$G : R \to \mathcal{G}$$

where

$$G(t) = e^{f_1 t} r_0$$

parametrizes $\mathcal{G}$. We have

$$\mathcal{G} = Im[G]$$

Thus we even have a one dimensional coordinate system

$$\varphi : \mathcal{G} \to R$$

for $\mathcal{G}$, given by

$$\varphi = G^{-1}$$

Back to original problem:
We are given two vector fields $f_1(r)$ and $f_2(r)$ and a point $r_0 \in \mathcal{H}$.

Naive Conjecture:
There exists a two-dimensional tangential manifold, which is parametrized by the mapping

$$G : R^2 \to X$$

where

$$G(s,t) = e^{f_2 s} e^{f_1 t} r_0$$

Generally False!

Argument:
If there exists a 2-dimensional manifold, then it should also be parametrized by

$$H(s,t) = e^{f_1 s} e^{f_2 t} r_0$$

Moral:
We need some commutativity.

Lie Brackets

Given two vector fields $f_1(r)$ and $f_2(r)$, their **Lie bracket** $[f_1, f_2]$ is a vector field defined by

$$[f_1, f_2] = (Df_2)f_1 - (Df_1)f_2$$

where D is the Frechet derivative (Jacobian).

Fact:

$$e^{f_1 h} e^{f_2 h} r_0 \; - e^{f_2 h} e^{f_1 h} r_0 \approx [f_1, f_2] h^2$$

Fact:
If $\mathcal{G}$ is tangential to f_1 and f_2, then it is also tangential to $[f_1, f_2]$.

Definition:
Given vector fields $f_1(r), \ldots, f_n(r)$, the **Lie algebra**

$$\{f_1(r), \ldots, f_n(r)\}_{LA}$$

is the smallest linear space of vector fields, containing $f_1(r), \ldots, f_n(r)$, which is closed under the Lie bracket.

Conjecture:
$f_1(r), \ldots, f_n(r)$ generates a finite dimensional tangential manifold **iff**

$$dim\,\{f_1(r), \ldots, f_n(r)\}_{LA} < \infty$$

Frobenius' Theorem:
Given n independent vector fields $f_1, \ldots, f_n$. There will exist an n-dimensional tangential manifold **iff**

$$span\,\{f_1, \ldots, f_n\}$$

is closed under the Lie-bracket.

Corollary:
Given n vector fields $f_1, \ldots, f_n$. Then there exists exists a finite dimensional tangential manifold **iff** the Lie-algebra

$$\{f_1, \ldots, f_n\}_{LA}$$

generated by $f_1, \ldots, f_n$ has finite dimension at each point. The dimension of the manifold equals the dimension of the Lie-algebra.

Proposition:

Suppose that the vector fields $f_1, \ldots, f_n$ are independent and closed under the Lie bracket. Fix a point $r_0 \in X$. Then the tangential manifold is parametrized by

$$G : R^n \to \mathcal{G}$$

where

$$G(t_1, \ldots, t_n) = e^{f_n t_n} \ldots e^{f_2 t_2} e^{f_1 t_1} r_0$$

Main result

- Given any fixed initial forward rate curve r_0, there exists a finite dimensional invariant manifold $\mathcal{G}$ with $r_0 \in \mathcal{G}$ if and only if the Lie-algebra

  $$\mathcal{L} = \{\mu, \sigma\}_{LA}$$

  is finite dimensional.

- Given any fixed initial forward rate curve r_0, there exists a finite dimensional realization if and only if the Lie-algebra

  $$\mathcal{L} = \{\mu, \sigma\}_{LA}$$

  is finite dimensional. The dimension of the realization equals $dim\,\{\mu, \sigma\}_{LA}$.

Deterministic Volatility

$$\sigma(r, x) = \sigma(x)$$

Consider a **deterministic** volatility function $\sigma(x)$. Then the Ito and Stratonovich formulations are the same:

$$dr = \{\mathbf{F}r + S\}\, dt + \sigma dW$$

where

$$\mathbf{F} = \frac{\partial}{\partial x}, \quad S(x) = \sigma(x) \int_0^x \sigma(s) ds.$$

The Lie algebra $\mathcal{L}$ is generated by the two vector fields

$$\mu(r) = \mathbf{F}r + S, \quad \sigma(r) = \sigma$$

Proposition:
There exists an FDR **iff** σ is "quasi exponential", i.e. of the form

$$\sigma(x) = \sum_{i=1}^{n} p_i(x) e^{\alpha_i x}$$

where p_i is a polynomial.

Constant Direction Volatility

$$\sigma(r, x) = \varphi(r)\lambda(x)$$

Theorem
Assume that $\varphi''(r)(\lambda, \lambda) \neq 0$. Then the model admits a finite dimensional realization if and only if λ is quasi-exponential. The scalar field $\varphi(r)$ can be arbitrary.

Note: The degenerate case $\varphi(r)''(\lambda, \lambda) \equiv 0$ corresponds to CIR.

Short Rate Realizations

Question:
When is a given forward rate model realized by a short rate model?

$$\begin{aligned} r(t,x) &= G(t,R_t,x) \\ dR_t &= a(t,R_t)dt + b(t,R_t) \circ dW \end{aligned}$$

Answer:
There must exist a 2-dimensional realization. (With the short rate R and running time t as states).

Proposition: The model is a short rate model only if

$$dim\,\{\mu,\sigma\}_{LA} \leq 2$$

Theorem: The model is a generic short rate model if and only if

$$[\mu,\sigma]\,//\sigma$$

“All short rate models are affine“

Theorem: (Jeffrey) Assume that the forward rate volatitliy is of the form

$$\sigma(R_t, x)$$

Then the model is a generic short rate model if and only if σ is of the form

$$\begin{aligned}
\sigma(R,x) &= c && \text{(Ho-Lee)} \\
\sigma(R,x) &= ce^{-ax} && \text{(Hull-White)} \\
\sigma(R,x) &= \lambda(x)\sqrt{aR+b} && \text{(CIR)}
\end{aligned}$$

(λ solves a certain Ricatti equation)

Slogan:
Ho-Lee, Hull-White and CIR are the **only generic** short rate models.

Constructing an FDR

Problem:
Suppose that there actually **exists** an FDR, i.e. that

$$dim \{\mu, \sigma\}_{LA} < \infty.$$

How do you **construct** a realization?

Good news: There exists a general and easy theory for this, including a concrete algorithm. See Björk & Landen (2001).

Example: Deterministic Direction Volatility

Model:

$$\sigma_i(r, x) = \varphi(r)\lambda(x).$$

Minimal Realization:

$$\begin{cases} dZ_0 &= dt, \\ dZ_0^1 &= [c_0 Z_n^1 + \gamma\varphi^2(G(Z))]dt + \varphi(G(Z))dW_t, \\ dZ_i^1 &= (c_i Z_n^1 + Z_{i-1}^1)dt, \qquad i = 1, \ldots, n, \\ dZ_0^2 &= [d_0 Z_q^2 + \varphi^2(G(Z))]dt, \\ dZ_j^2 &= (d_j Z_q^2 + Z_{j-1}^2)dt, \qquad j = 1, \ldots, q. \end{cases}$$

Stochastic Volatility

Forward rate equation:

$$
\begin{aligned}
dr_t &= \mu_0(r_t, y_t)dt + \sigma(r_t, y_t)dW_t, \\
dy_t &= a(y_t)dt + b(y_t) \circ dV_t
\end{aligned}
$$

Here W and V are independent Wiener and y is a finite dimensional diffusion living on R^k.

$$
\mu_0 = \frac{\partial}{\partial x} r_t(x) + \sigma(r_t, y_t, x) \int_0^x \sigma(r_t, y_t, s) ds
$$

Problem: When does there exist an FDR?

Good news: This can be solved completely using the Lie algebra approach. See Björk-Landen-Svensson (2002).

Point Process Extensions

Including a driving **point process** leads to hard problems. More precisely

- The equivalence between existence of an FDR and existence of an invariant manifold still holds.

- The characterization of an invariant manifold as a tangential manifold is no longer true.

- This is because a point process act **globally** whereas a Wiener process act **locally**, thereby allowing differential calculus.

- Including a driving point process requires, for a general theory, completely different arguments. The picture is very unclear.

Point Processes: Special Cases

- Chiarella & Nikitopoulos Sklibosios (2003)
  Sufficient Conditions

- Tappe (2007)
  Necessary Conditions using Lie algebra techinques.

- Elhouar (2008)
  Wiener driven models with point process driven volatilities using Lie algebra techinques.

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