On λ-Quantile Dependent Convex Risk Measures

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Outline:

- The class of λ-quantile dependent convex measures of risk:
  - Definition;
  - The λ-quantile Fatou property;
  - The robust representation.
- The λ-quantile law invariant measures of risk:
  - Definition;
  - The robust representation;
  - The λ-quantile uniform preference;
  - Example: the λ-quantile dependent Weighted VaR.

Measures of risk: capital requirements that can be added to a financial position to make it acceptable.

Mathematically, $\rho : \mathcal{X} \to \mathbb{R} \cup \{\infty\}$. (e.g., $\mathcal{X} = L^p, 1 \leq p \leq \infty$.)

Axioms of convex measures of risk: $\forall X, Y \in \mathcal{X}$,

- Monotonicity: If $X \leq Y$, then $\rho(X) \geq \rho(Y)$.
- Cash invariance: $\rho(X + m) = \rho(X) - m$, $\forall m \in \mathbb{R}$.
- Convexity: $\rho(\alpha X + (1-\alpha)Y) \leq \alpha\rho(X) + (1-\alpha)\rho(Y)$, $\alpha \in [0, 1]$.

ρ is **coherent** if in addition it satisfies

- Positive homogeneity: $\rho(\lambda X) = \lambda\rho(X), \quad \forall \lambda \geq 0, \forall X \in \mathcal{X}$.

Robust Representation of Convex Measures of Risk:

ADEH (1999), Delbaen (2002), Föllmer and Schied (2004), Biagini and Frettelli (2009), Kaina and Rüschendorf (2009):

Notations:

$L^p := L^p(\Omega, \mathcal{F}, \mathbf{P})$, $1 \leq p \leq \infty$;

$L^q := L^q(\Omega, \mathcal{F}, \mathbf{P})$ the conjugate space of L^p;

$\rho : L^p \to \mathbb{R} \cup \{\infty\}$ a proper convex measure of risk;

$\mathcal{Q}_p := \{\mathbf{Q}$ probability measures : $\mathbf{Q} \ll \mathbf{P}$, $\frac{d\mathbf{Q}}{d\mathbf{P}} \in L^q\}$.

$$\rho(X) = \sup_{\mathbf{Q} \in \mathcal{Q}_p} \left(\mathbb{E}_{\mathbf{Q}}[-X] - \rho^*(\mathbf{Q})\right) \Longleftrightarrow \text{ lower semicontinuity of } \rho \ ,$$

$$\rho^*(\mathbf{Q}) = \sup_{X \in L^p} \left(\mathbb{E}_{\mathbf{Q}}[-X] - \rho(X)\right), \text{the Fenchel-Legendre transformation of } \rho.$$

Key: lower semicontinuity of $\rho \Leftrightarrow$ Fatou property.

Literatures on Fatou property:

- *Delbaen (2002)*: For coherent $\rho : L^\infty \to \mathbb{R}$,

Fatou(D) $\quad \forall (X_n) \subset L^\infty$ s.t. $|X_n| \leq C$,

$$X_n \to X \in L^\infty \ \mathbf{P}\text{-a.s.} \Longrightarrow \rho(X) \leq \liminf_{n\to\infty} \rho(X_n).$$

- *Föllmer and Schied (2004)*: Convex risk measures $\rho : L^\infty \to \mathbb{R}$.

- *Biagini and Frettelli (2009)*: For proper convex $\rho : L^p \to \mathbb{R} \cup \{\infty\}$,

Fatou(BF) $\quad \forall (X_n) \subset L^p$ s.t. $\mathbf{P}$-a.s. $|X_n| \leq Y \in L^p$,

$$X_n \to X \in L^p \ \mathbf{P}\text{-a.s.} \Longrightarrow \rho(X) \leq \liminf_{n\to\infty} \rho(X_n).$$

Examples of Risk Measures Depending on the Left Tails:

- Value-at-Risk:

$$VaR_\lambda(X) := -q_X^+(\lambda) = q_{-X}^-(1-\lambda).$$

- Conditional VaR:

$$CVaR_\lambda(X) := \frac{1}{\lambda}\int_0^\lambda VaR_\gamma(X)d\gamma = -\frac{1}{\lambda}\int_0^\lambda q_X^+(t)dt.$$

λ-Quantile Dependent Convex Risk Measures:

Definition 1 *For given $\lambda \in (0,1)$, a convex measure of risk*

$$\rho : L^p \to \mathbb{R} \cup \{\infty\}, \quad 1 \le p \le \infty,$$

*is λ-**quantile dependent**, if*

$$X \mathbb{I}_{\{X \le q_X^+(\lambda)\}} = Y \mathbb{I}_{\{Y \le q_Y^+(\lambda)\}} \quad \mathbf{P}\text{-}a.s. \quad \textit{implies} \quad \rho(X) = \rho(Y).$$

Namely, ρ depends on the the random variables up to their λ-quantiles.

Related Fatou property and robust representation?

The λ-quantile Fatou Property:

Fatou(D) $\quad \forall (X_n) \subset L^\infty$ s.t. $|X_n| \leq C$,

$$X_n \to X \in L^\infty \ \mathbf{P}\text{-a.s.} \Longrightarrow \rho(X) \leq \liminf_{n\to\infty} \rho(X_n).$$

Fatou(BF) $\quad \forall (X_n) \subset L^p$ s.t. $\mathbf{P}$-a.s. $|X_n| \leq Y \in L^p$,

$$X_n \to X \in L^p \ \mathbf{P}\text{-a.s.} \Longrightarrow \rho(X) \leq \liminf_{n\to\infty} \rho(X_n).$$

λ-quantile Fatou $\quad \forall (X_n) \subset L^p$ s.t. $q^+_{X_n}(\lambda) \leq c_\lambda$, $c_\lambda \in \mathbb{R}$,

$$X_n \to X \in L^p \ \mathbf{P}\text{-a.s.} \Longrightarrow \rho(X) \leq \liminf_{n\to\infty} \rho(X_n).$$

Relations Between the Three Fatou Properties:

- All require the uniform boundedness of (X_n).
  - **Fatou(D)**: $|X_n| \leq C$ for some $C \in \mathbb{R}$.
  - **Fatou(BF)**: $|X_n| \leq Y$ for some $Y \in L^p$.
  - **λ-quantile Fatou**: $q^+_{X_n}(\lambda) \leq c_\lambda$ for some $c_\lambda \in \mathbb{R}$.
- For $(X_n) \subset L^p$,

$$|X_n| \leq C \Longrightarrow |X_n| \leq Y \Longrightarrow q^+_{X_n}(\lambda) \leq c_\lambda.$$

- ρ has λ-quantile Fatou property$\Rightarrow$ ρ has Fatou(BF) $\Rightarrow$ ρ has Fatou(D).

Is the λ-quantile Fatou property equivalent to the lower semi-continuity as well as to the robust representation of the convex measure of risk?

Yes, if we restrict ourselves to the smaller class of λ-quantile dependent convex risk measures!

Robust Representation of λ-Quantile Dependent Convex Risk Measures:

Assume $\rho : L^p \to \mathbb{R} \cup \{\infty\}$ is *proper*, *convex* and λ-*quantile dependent*.

Theorem 2

1. $\rho(X) = \sup_{\mathbf{Q} \in \mathcal{Q}_p} (\mathbb{E}_{\mathbf{Q}}[-X] - \rho^*(\mathbf{Q})), \quad \rho^* = \sup_{X \in L^p} (\mathbb{E}_{\mathbf{Q}}[-X] - \rho(X)).$

$$\Updownarrow$$

2. *ρ is $\sigma(L^p, L^q)$ lower semicontinuous.*

$$\Updownarrow$$

3. *ρ is continuous from above.*

$$\Updownarrow$$

4. *ρ has the λ-quantile Fatou property.*

The λ-Quantile Law Invariant Convex Risk Measures:

Assume $\lambda \in (0,1)$ is given, and

$$\rho : L^p \to \mathbb{R} \cup \{\infty\}, \quad 1 \leq p \leq \infty,$$

is a convex measure of risk.

ρ is **λ-quantile dependent**, if

$$X \mathbb{I}_{\{X \leq q_X^+(\lambda)\}} = Y \mathbb{I}_{\{Y \leq q_Y^+(\lambda)\}} \quad \mathbf{P}\text{-a.s.} \quad \text{implies} \quad \rho(X) = \rho(Y).$$

ρ is **λ-quantile law invariant**, if

$$X \mathbb{I}_{\{X \leq q_X^+(\lambda)\}} \overset{\mathcal{D}}{\sim} Y \mathbb{I}_{\{Y \leq q_Y^+(\lambda)\}} \quad \text{implies} \quad \rho(X) = \rho(Y).$$

Robust Representation of The λ-Quantile Law Invariant Convex Risk Measures:

For law invariant convex risk measures: *Kusuoka (2001), Föllmer and Schied (2004), Frettelli and Rosazza Gianin (2005), Jouini, Schachermayer and Touzi (2006).*

Theorem 3 *Suppose* $(\Omega, \mathcal{F}, \mathbf{P})$ *is atomless,* $\rho : L^p \to \mathbb{R} \cup \{\infty\}$ *is proper, convex and* λ*-quantile law invariant.*

$$\rho \text{ has the } \lambda\text{-quantile Fatou property .}$$

$$\Updownarrow$$

$$\rho(X) = \sup_{\mathbf{Q} \in \mathcal{Q}_p} \left(\int_0^\lambda q_X(t) q_{-\frac{d\mathbf{Q}}{d\mathbf{P}}}(t) dt + q_X^+(\lambda) \int_\lambda^1 q_{-\frac{d\mathbf{Q}}{d\mathbf{P}}}(t) dt - \rho^*(\mathbf{Q}) \right),$$

$$\rho^*(\mathbf{Q}) = \sup_{X \in L^p} \left(\int_0^\lambda q_X(t) q_{-\frac{d\mathbf{Q}}{d\mathbf{P}}}(t) dt + q_X^+(\lambda) \int_\lambda^1 q_{-\frac{d\mathbf{Q}}{d\mathbf{P}}}(t) dt - \rho(X) \right).$$

An Example: The λ-quantile Dependent Weighted VaR

Definition 4 *(λ-quantile dependent WVaR)*

$\rho_{\mu,\lambda} : L^p \to \mathbb{R} \cup \{\infty\}$, $1 \leq p \leq \infty$,

$$\rho_{\mu,\lambda}(X) := \int_{(0,\lambda]} CVaR_\gamma(X)\mu(d\gamma) = -\int_0^\lambda q_X(t)\phi(t)dt = \int_0^\lambda q_X(t)q_{\nu_\phi}(t)dt.$$

μ*: probability measure on* $(0, \lambda]$,

$\phi(t) := \int_{(t,\lambda]} \frac{1}{s}\mu(ds)$,

ν_ϕ*: probability distribution measure s.t.* $q_{\nu_\phi}(t) := -\phi(t)$, $\forall t \in (0, \lambda]$.

In particular, $\mu(d\gamma) = \mathbb{I}_{\{\lambda\}}(d\gamma) \Rightarrow \phi(t) = \frac{1}{\lambda}\mathbb{I}_{(0,\lambda]}(t)$

$$\Rightarrow \rho_{\mu,\lambda}(X) = CVaR_\lambda(X) = -\frac{1}{\lambda}\int_0^\lambda q_X(t)dt.$$

Robust Representation of λ-quantile Dependent $WVaR$:

$$\rho_{\mu,\lambda}(X) = \max_{\mathbf{Q}\in\mathcal{Q}_\lambda} \mathbb{E}_{\mathbf{Q}}[-X],$$

with $\mathcal{Q}_\lambda := \{\mathbf{Q} \in \mathcal{Q}_p : \nu_{-\frac{d\mathbf{Q}}{d\mathbf{P}}} \underset{uni(\lambda)}{\succcurlyeq} \nu_\phi\}$.

Key: $\nu_{-\frac{d\mathbf{Q}}{d\mathbf{P}}} \underset{uni(\lambda)}{\succcurlyeq} \nu_\phi$, λ-quantile uniformly preference of $\nu_{-\frac{d\mathbf{Q}}{d\mathbf{P}}}$ over ν_ϕ.

Note: $\rho_{\mu,\lambda}(X) = \int_0^\lambda q_X(t) q_{\nu_\phi}(t) dt$

- is proper, coherent and λ-quantile law invariant.
- is continuous from above $\Leftrightarrow$ $\rho_{\mu,\lambda}$ has the λ-quantile Fatou property.

λ-Quantile Uniform Preference of Two Measures μ and ν:

(Second Order Stochastic Dominance)

Definition 5 $\mu \underset{uni(\lambda)}{\succcurlyeq} \nu$ *if*

for any "λ-quantile utility function" u defined by

$$u(x) = u_0(x)\mathbb{I}_{\{x \le q_\nu(\lambda)\}} + u_0(q_\nu(\lambda))\mathbb{I}_{\{x > q_\nu(\lambda)\}},$$

with u_0 a utility function, we have

$$\int_0^\lambda u\, d\mu \ge \int_0^\lambda u d\nu.$$

Proposition 6

$$\mu \underset{uni(\lambda)}{\succcurlyeq} \nu \Longleftrightarrow \int_0^t q_\mu(s)ds \ge \int_0^t q_\nu(s)ds,\ \forall t \in [0, \lambda]$$

$$\Longleftrightarrow \int_0^\lambda h(t)q_\mu(t)dt \ge \int_0^\lambda h(t)q_\nu(t)dt,\ \forall\ \textit{decreasing}\ h : [0, \lambda] \to \mathbb{R}^+.$$

Theorem 7 *(Robust Representation of the λ-quantile dependent $WVaR : \rho_{\mu,\lambda}$)*

$$\rho_{\mu,\lambda}(X) = \max_{\mathbf{Q}\in\mathcal{Q}_\lambda} \mathbb{E}_{\mathbf{Q}}[-X],$$

with $\mathcal{Q}_\lambda = \{\mathbf{Q} \in \mathcal{Q}_p, \quad \nu_{-\frac{d\mathbf{Q}}{d\mathbf{P}}} \underset{uni(\lambda)}{\succcurlyeq} \nu_\phi\}$,

ν_ϕ*: probability distribution measure s.t.* $q_{\nu_\phi}(t) := -\phi(t)$, $\forall t \in (0, \lambda]$,

$\phi(t) := \int_{(t,\lambda]} \frac{1}{s}\mu(s)$.

The maximum is obtained by choosing $\mathbf{Q}_X \in \mathcal{Q}_\lambda$ *such that* $\frac{d\mathbf{Q}_X}{d\mathbf{P}} = f(X)$*, where the decreasing function* f *is given by:*

$$f(x) = \begin{cases} \phi(F_X(x)) & \text{if } x \text{ is a continuity point of } F_X, \\ \frac{1}{F_X(x)-F_X(x-)} \int_{F_X(x-)}^{F_X(x)} \phi(t)dt & \text{if } x \text{ is a discrete point of } F_X, \end{cases}$$

for $F_X(x) \le \lambda$ *and* $f(x) = 0$*, otherwise.*

Special Case $CVaR_\lambda$:

$CVaR_\lambda(X) = \rho_{\mu,\lambda}(X), \quad \mu(d\gamma) = \mathbb{I}_{\{\lambda\}}(d\gamma), \quad q_{\nu_\phi}(t) = -\frac{1}{\lambda}\mathbb{I}_{(0,\lambda]}(t).$

From Theorem 7,

$$CVaR_\lambda(X) = \max_{\mathbf{Q}\in\mathcal{Q}_\lambda} \mathbb{E}_{\mathbf{Q}}[-X],$$

$$\mathcal{Q}_\lambda = \left\{ \mathbf{Q} \in \mathcal{Q}_p, \quad \nu_{-\frac{d\mathbf{Q}}{d\mathbf{P}}} \underset{uni(\lambda)}{\succcurlyeq} \nu_\phi \right\},$$

$$\nu_{-\frac{d\mathbf{Q}}{d\mathbf{P}}} \underset{uni(\lambda)}{\succcurlyeq} \nu_\phi \iff \int_0^t q_{\nu_{-\frac{d\mathbf{Q}}{d\mathbf{P}}}}(s)ds \geq \int_0^t q_{\nu_\phi}(s)ds = -\frac{t}{\lambda}$$

The set $\mathcal{Q}_\lambda$ coincides with the set

$$\left\{ \mathbf{Q}\,\text{probability measure} \;: \mathbf{Q} \ll \mathbf{P}, \quad \frac{d\mathbf{Q}}{d\mathbf{P}} \leq \frac{1}{\lambda} \quad \mathbf{P} - a.s. \right\}.$$

THANK YOU!