How do Variance Swaps Shape the Smile?

A Summary of Arbitrage Restrictions and Smile Asymptotics

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6th World Congress of the Bachelier Finance Society.
Toronto 2010

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Outline

Questions concerning when traded options are priced correctly have been asked many times in the field of financial mathematics; Black & Scholes 1973, Merton 1973, Hobson 1998, Davis & Hobson 2005, Cox and Obłój 2009 to name a few.

In

- Davis, Obłój & Raval. Arbitrage Bounds for Weighted variance Swap Prices. 2010. `arXiv:1001.2678v1`.

we extend this line of study to include swaps on realised variance. Today we show how the result can be used to generate restrictions on the prices of vanilla options. We then invert the role of the variance swap to deduce asymptotic bounds for the implied volatility smile.

Variance Swaps

A variance swap with maturity T has payoff proportional to

$$\langle \ln S \rangle_T - k_{vs}$$

and satisfies

$$\mathcal{P}(\langle \ln S \rangle_T - k_{vs}) = 0.$$

Lemma (Hedging and Pricing Realised Variance)

If $(S_t)_{t \in [0,T]}$ is a continuous semi-martingale then

$$\langle \ln S \rangle_T = -2 \ln(S_T / S_0) - 2 \int_0^T \frac{\mathrm{d}S_u}{S_u} \text{ a.s }. \tag{1}$$

Furthermore, if $(S_t)_{t \in [0,T]}$ is a *martingale* then it holds

$$\mathbb{E}\left[\langle \ln S \rangle_T\right] = 2\mathbb{E}[-\ln(S_T / S_0)] \tag{2}$$

and in particular $\mathbb{E}\left[\langle \ln S \rangle_T\right] < \infty \iff \mathbb{E}[|\ln(S_T)|] < \infty$.

Starting Point

Frictionless Markets

Consider a market over the time horizon $[0, T]$, with $T > 0$, trading a financial asset S. For $t \in [0, T]$, let S_t denote the asset price at time t, and suppose $S_0 = 1$. We also assume

Assumption A

- *The asset S at any time $t \in [0, T]$, and the quoted options at time 0 only, can be traded long or short in arbitrary amounts with no transaction costs.*
- *There are no interest rates and the asset pays no dividends.*
- *Suppose a finite number, $n \in \mathbb{N}$, of European put options on S are traded at time 0, maturing at time T for strikes $0 < k_1 < \ldots < k_n < \infty$. Let r_i denote the price of the put struck at k_i.*

(in the paper we have deterministic interest rates and dividends paid at a deterministic dividend yield)

Pricing Operator

Let $\mathcal{P}$ denote the prices of traded securities at time 0 and is specified as follows:

$$\mathcal{P}(1^t) = 1, \quad \mathcal{P}(S_t 1^t) = S_0 = 1, \quad \mathcal{P}(k_i - S_T)^+ = r_i, \; i = 1, \ldots, n, \tag{3}$$

and $\mathcal{P}$ acts linearly on the combinations of the above,

Let $\mathbb{X}_V$ denote the set of traded vanilla options on S in which the options are identified by the payoffs, i.e. $\mathbb{X}_V = \{(k_i - S_T)^+; \; i = 1 \ldots n\}$.

Definition

A *static portfolio* is a deterministic triple (π, ϕ, ψ), where:

- $\pi = (\pi_1, \ldots, \pi_n) \in \mathbb{R}^n$, in which π_i denotes the number of units of the put option with strike k_i.
- $\phi \in \mathbb{R}$ denotes the number of units of the asset S.
- Finally, $\psi \in \mathbb{R}$ denotes the number of units of the risk-free.

Consequently $\mathcal{P}X_T^{(\pi,\phi,\psi)} = X_0^{(\pi,\phi,\psi)}$.

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Arbitrage and Models

- A *model*, $\mathcal{M}$, for the asset price is a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{Q})$ with a positive *semimartingale* $(S_t)_{t\in[0,T]}$ with $S_0 = 1$ almost surely. The filtration satisfies the usual hypotheses and $\mathcal{F}_0$ is trivial. ($\mathbb{M}$ = set of all models)
- A model is called a $(\mathcal{P}, \mathbb{X})$-*market model* if (S_t) is an $(\mathcal{F}_t, \mathbb{Q})$-*martingale* and $\mathcal{P}X = \mathbb{E}^{\mathbb{Q}}[X]$ for all market quoted options $X \in \mathbb{X}_V$. ($\mathbb{M}^V$ = set of all vanilla market models)
- A *model-independent arbitrage* is a static portfolio (π, ϕ, ψ) with $\mathcal{P}X_T^{(\pi,\phi,\psi)} < 0$ and $X_T^{(\pi,\phi,\psi)} \geq 0$.
- The market prices $(\mathcal{P}, \mathbb{X})$ admit a *weak-arbitrage* (WA) if in any model $\mathcal{M} \in \mathbb{M}$, there exists an admissible strategy (π, ϕ, ψ) satisfying: $X_T^{(\pi,\phi,\psi)} \geq 0$ almost surely, $\mathbb{Q}[X_T^{(\pi,\phi,\psi)} > 0] > 0$ and $\mathcal{P}X_T^{(\pi,\phi,\psi)} \leq 0$.

Let

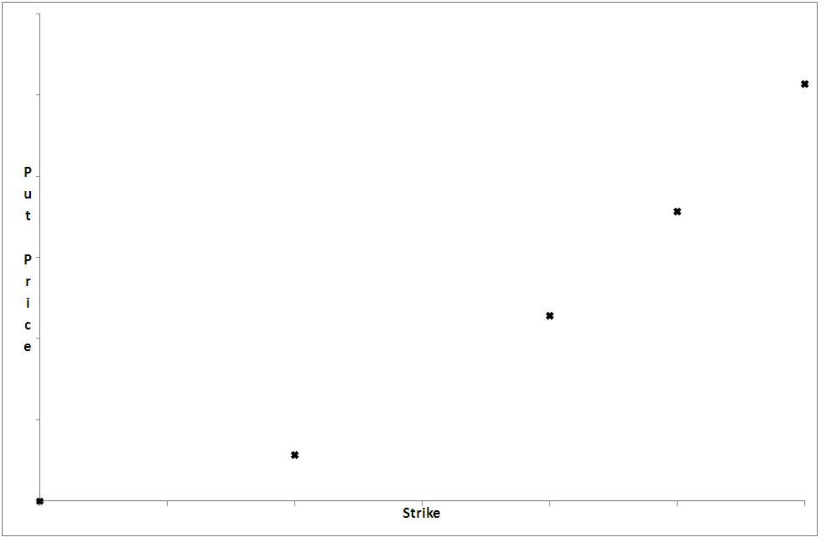
$$\bar{n} := \inf\{i \geq 0 : \; r_i = k_i - 1\} \quad \& \quad \underline{\mathrm{n}} = \max\{i \geq 0 : \; r_i = 0\}. \qquad (4)$$

Theorem (Davis and Hobson (2005))

Under A1 the following statements are equivalent:

1. The market prices $(\mathcal{P}, \mathbb{X}_E)$ do not admit a weak-arbitrage.
2. The option prices satisfy: $r_0 = 0$, $r_i \geq (k_i - 1)^+ \; \forall i$, and the piecewise linear interpolation over $[k_{\underline{\mathrm{n}}}, k_{\bar{n}\wedge n}]$ of the points $(k_{\underline{\mathrm{n}}}, r_{\underline{\mathrm{n}}}), \ldots, (k_{\bar{n}}, r_{\bar{n}})$ is increasing, convex and with slope strictly bounded by +1.
3. There exists a $(\mathcal{P}, \mathbb{X}_E)$ – market model.

Price Data and Market Making



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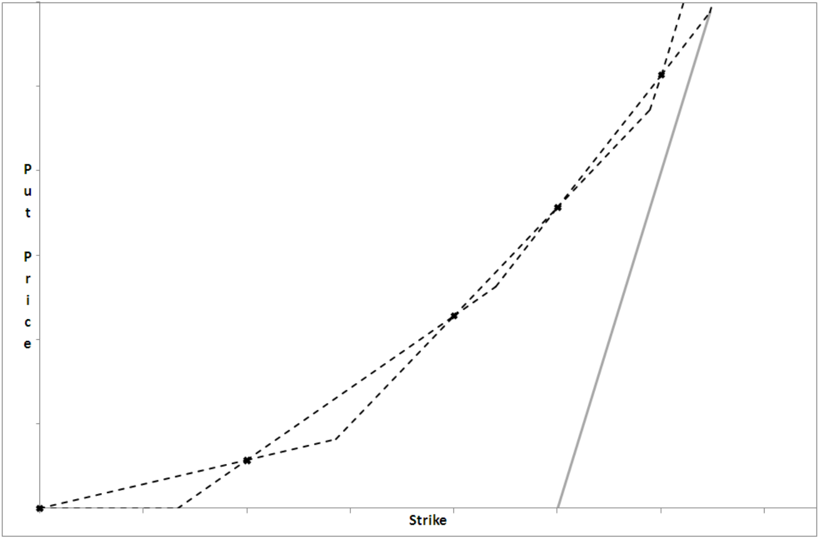

Points to Consider

1. Should the range change if a *variance swap* is traded? ....yes!
   - From Breeden & Litzenberger note that if $r_2 = \frac{r_1}{k_1} k_2$, then any model matching these prices places atomic mass on the event $\{S_T = 0\}$, however $-\ln S_T$ diverges here.
2. If so, how does it change?

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Arbitrage Bounds for the Price of a Variance Swap

A Weak Arbitrage Restriction

For this section, assume stock price trajectories $t \to S_t$ are continuous. Suppose in addition to the n European options, a *variance swap* with maturity T is also traded with swap-rate k_{vs}, $0 < k_{vs} < \infty$.

To illustrate a weak-arbitrage, assume the put prices satisfy $r_2 = \frac{r_1}{k_1} k_2 > 0$.

Recall in any model in $\mathbb{M}$, the log-contract is synthesized with price $\mathcal{P}(-\ln S_T) = \frac{k_{vs}}{2}$. We work with this portfolio.

For models in which $\mathbb{Q}[S_T \in [0, k_2)] = 0$ an arbitrage is realised by selling the put with strike k_1, to realise a profit of $p_1 > 0$.

This leaves models in which $\mathbb{Q}[S_T \in [0, k_2)] > 0$ and we partition these according to whether $\{S_T \in (0, k_2)\}$ is a null-set or not.

In a model for which $\mathbb{Q}[S_T \in (0, k_2)] > 0$, an arbitrage is realised (at zero cost) by buying the put with strike k_2 and selling $\frac{k_2}{k_1}$ units of the put with strike k_1.

On the other hand if $\mathbb{Q}[S_T \in (0, k_2)] = 0$, then selling

$$\frac{1}{r_1}\left[\mathcal{P}(-\ln(S_T/k_2)) + \frac{1}{k_2}(1 - k_2)\right] > 0$$

units of the put with strike k_1 and constructing (at no additional cost) the portfolio with positive payoff

$$-\ln(S_T/k_2) + \frac{1}{k_2}(S_T - k_2),$$

yields an arbitrage. Though this portfolio may not dominate the put payoff $(k_1 - S_T)^+$ for $S_T \in (0, k_1)$, this does not matter for this class of models since the assumption was $\mathbb{Q}[S_T \in (0, k_2)] = 0$ and so $\mathbb{Q}[S_T = 0] > 0$. Hence these market prices admit weak-arbitrage.

Special Case of Davis, Obłój & R

Define

$$\begin{aligned} LB &= \sup_{(\pi,\phi,\psi)\in\mathbb{R}^{n+2}} \mathcal{P}X_T^{(\pi,\phi,\psi)} \\ \text{s.t.} &\quad X_T^{(\pi,\phi,\psi)} \leq -\ln S_T, \end{aligned}$$

and

$$\begin{aligned} UB &= \inf_{(\pi,\phi,\psi)\in\mathbb{R}^{n+2}} \mathcal{P}X_T^{(\pi,\phi,\psi)} \\ \text{s.t.} &\quad X_T^{(\pi,\phi,\psi)} \geq -\ln S_T, \end{aligned}$$

...clearly $UB = +\infty$, whereas LB in general is non-trivial to compute.

Finally denote the enlarged set of traded options is

$$\mathbb{X} = \mathbb{X}_V \cup \{\langle \ln S\rangle_T - k_{vs}\}.$$

Theorem

Let $(\mathcal{P}, \mathbb{X})$ be the market input. Suppose $(\mathcal{P}, \mathbb{X}_V)$ do not admit a weak arbitrage. The following are equivalent:

1. There exists a $(\mathcal{P}, \mathbb{X})$–market model.
2. The market prices $(\mathcal{P}, \mathbb{X})$ do not admit a weak-arbitrage.
3. The market prices $(\mathcal{P}, \mathbb{X}_V \cup \{-\ln S_T\})$, with $\mathcal{P}(-\ln S_T) = \frac{k_{vs}}{2}$, do not admit a weak-arbitrage.

In particular if $k_{vs} \in [LB, \infty)$ then there exists a $(\mathcal{P}, \mathbb{X})$–market model and if $k_{vs} \notin [LB\infty]$ then market prices admit weak arbitrage.

Key result for the proof is a Theorem by Karlin & Isii, that establishes there exists a solution that attains LB, i.e. a portfolio $(\pi^\dagger, \phi^\dagger, \psi^\dagger)$ such that $LB = \mathcal{P}X_T^{(\pi^\dagger, \phi^\dagger, \psi^\dagger)}$. Moreover

$$LB = \inf_{\mathcal{M} \in \mathbb{M}^V} \mathbb{E}_{\mathcal{M}}[-\ln S_T],$$

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Market Making in a Variance Swap Market

Suppose now we write a put option with strike $\hat{k} \notin \{k_1, \ldots, k_n\} \cup \{0\}$ and price $\hat{r}$. The requirement is that this price should not create an arbitrage opportunity. Define $\hat{k}_1, \ldots, \hat{k}_{n+1}$ to be the augmented strikes in increasing order, so that if $\hat{k} \in (k_{i-1}, k_i)$, then $\hat{k}_i = \hat{k}$.

Lemma

Assume $(\mathcal{P}, \mathbb{X})$ does not admit a weak-arbitrage. Then, the market with the additional put is consistent with absence of weak-arbitrage if and only if:

1. the linear interpolation of

$$\{(\hat{k}_1, \hat{r}_1), \ldots, (\hat{k}_{n+1}, \hat{r}_{n+1})\}$$

   is increasing, convex, has slope bounded by $+1$ and $\hat{r}_2 > \frac{\hat{k}_2}{\hat{k}_1}\hat{r}_1$. In addition to this,

2. $\hat{LB} \leq k_{vs}$, where $\hat{LB}$ is the new lower bound with the inclusion of the additional put option.

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2. $L\hat{B} \leq k_{vs}$, where $L\hat{B}$ is the new lower bound with the inclusion of the additional put option.

Implied Volatility and Asymptotics

Let $\mathcal{M}$ be a market model, and define

$$r(k) = \mathbb{E}(k - S_T)^+, \; k \geq 0.$$

Define the *implied variance* for log-strike $x = \ln(k)$, $I^2(x)$, as the unique root to

$$r(e^x) = p_B(x, I(x)),$$

where

$$p_B(x, \sigma) = e^x \Phi[-d(x)] - \Phi[-d(x) - \sigma] \tag{5}$$

and

$$d = -\frac{x}{\sigma\sqrt{T}} - \frac{\sigma\sqrt{T}}{2}. \tag{6}$$

Lee (2004): let $\tilde{q} = \sup\{q : \mathbb{E}S_T^{-q} < \infty\}$ and $\beta_L := \limsup_{x \to -\infty} \frac{I^2(x)}{|x|/T}$.
Then $\beta_L \in [0, 2]$ and $\beta_L = 2 - 4\left(\sqrt{\tilde{q}^2 + \tilde{q}} - \tilde{q}\right)$

Thus implied-variance is *at most* linear in log-strike. However there is no market information about how many inverse moments a stock-price admits.

Trading a (continuous or discrete) variance swap entails $\mathbb{Q}[S_T = 0] = 0$. In this case Lee's work tells us there exists $x^* < 0$ such that for all $x < x^*$,

$$I(x) < \sqrt{2|x|/T}.$$

What is the bound when S_T admits finite log-moments? (which does not imply any finite inverse moments)

The Log-Moment Formula

Theorem

Let

$$\tilde{q} = \sup\{q : \mathbb{E}|\ln S_T|^q < \infty\},$$

then

$$\liminf_{x \to -\infty} \frac{d(x)}{\sqrt{2 \ln |x|}} = \sqrt{\tilde{q}},$$

where recall $d(x) = -\frac{x}{I(x)\sqrt{T}} - \frac{I(x)\sqrt{T}}{2}$.

Consequently, there exists $x_{\tilde{q}} < 0$ such that $x \leq x_{\tilde{q}}$ implies

$$I(x) \leq \sqrt{2/T}\left(\sqrt{-x + \tilde{q}\ln(-x)} - \sqrt{\tilde{q}\ln(-x)}\right).$$

Proof: Via some analysis show:

Lemma

Let $q \geq 0$ and assume $\mathbb{E}|\ln S_T|^q < \infty$. Then for $x < (q-1)\mathbf{1}_{q<1}$,

$$p(x, I(x)) \leq e^x |x|^{-q} \mathbb{E}|\ln S_T|^q.$$

Then establish a bound for implied volatility by proving

$$\lim_{x \to -\infty} \frac{e^x |x|^{-q}}{p\left(x, \sqrt{2/T}\left(\sqrt{-x + Q\ln(-x)} - \sqrt{Q\ln(-x)}\right)\right)} = \begin{cases} 0 & \text{if } Q < q \\ \infty & \text{if } Q \geq q \end{cases} . \tag{7}$$

Contradiction arguments of the kind used by Lee yields the result.

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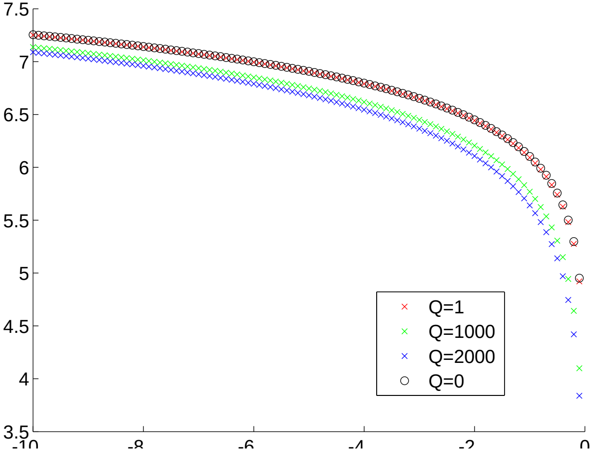
Illustration of

$$\ln\left(\sqrt{2/T}\left(\sqrt{-x+Q\ln(-x)}-\sqrt{Q\ln(-x)}\right)\right)$$

with $T = 1$.



Conclusions

- Under the umbrella of continuous sample paths of the underlying, a finite number of European option prices imply model-free arbitrage bounds for the price of a variance swap. (In fact this is true for *weighted variance swaps*, e.g. corridor and gamma swaps.)
- In a market with no variance swap, put prices lie within $r(k) \in [(k-1)^+, k]$ and implied volatility satisfies $I(x) = O(\sqrt{2|x|/T})$ as $x \to -\infty$.
- The price of a variance swap gives information about the risk-neutral distribution of the underlying asset. In particular mass near the origin is 'restricted.'
- The work on arbitrage bounds yields the restrictions imposed on the prices of vanilla put options.
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Thank You!