A BSDE approach to Curve Following in Limit Order Markets

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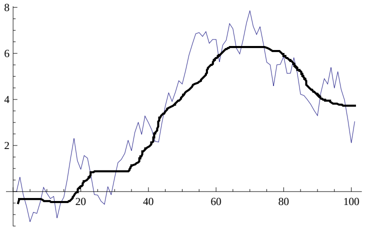
Outline

Motivation

- We are given a target function and want to minimise the deviation of stock holdings to this function.

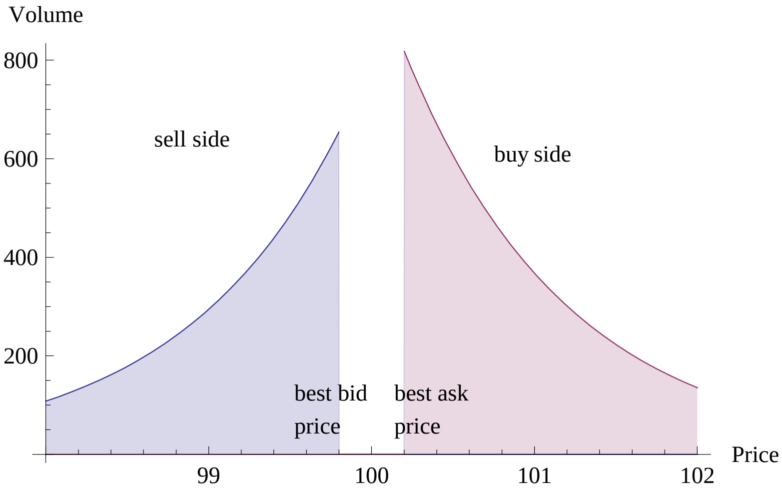


- This is a classical problem in stochastic control and related to
  - Tracking Brownian motion, e.g. Beneš, Shepp, and Witsenhausen (1980).
  - Finite fuel problems, e.g. Karatzas (1985).

Link to Finance

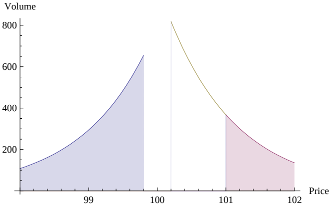
- Applications in finance:
  - Index tracking,
  - Portfolio liquidation,
  - Delta hedging,
  - Trading at volume weighted average prices (VWAP).
- There is a tradeoff between accuracy and cost.
- We trade in a limit order market.

Diagram of a Limit Order Market

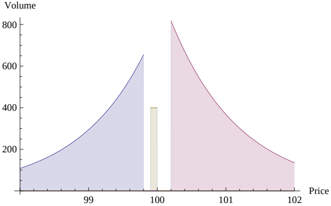


Two Types of Orders

- The investor may submit a market order and consume volume in the book...



- ... or he may place a limit order and wait for execution.



The Minimisation Problem

- Given a control u, we assume that the stock holdings satisfy

$$dX^u(t) = u_1(t)N(dt) + u_2(t)dt, \quad X^u(0) = x.$$

- The investor wants to minimise the performance functional

$$J(t,x,z,u) \triangleq \mathbb{E}\Big[\int_t^T g\left(u_2(s), Z(s)\right) + h\big(X^u(s) - \alpha(s, Z(s))\big)ds \\ + f\big(X^u(T) - \alpha(T, Z(T))\big)\Big]$$

with cost function g, penalty functions h and f, target function α and a vector of stochastic signals Z, e.g. spread or index.

The Minimisation Problem ctd

- We assume the following dynamics for the process Z:

$$
\begin{aligned}
dZ(t) =& \mu(t, Z(t))dt + \sigma(t, Z(t))dW(t) \\
&+ \int \gamma(t, Z(t), \theta)\tilde{M}(d\theta, dt), \quad Z(0) = z.
\end{aligned}
$$

- The value function is defined as

$$
v(t, x, z) \triangleq \inf_{u \in \mathcal{U}} J(t, x, z, u).
$$

- Assumptions:
  - Limit orders only on reference price, only full execution.
  - f, g and h strictly convex, nonnegative, smooth and of quadratic growth
  - α of polynomial growth, μ, σ and γ Lipschitz.

Existence and Uniqueness

Theorem (N. and Westray (2010) Theorem 3.1)

There is a unique optimal control $\hat{u}$.

- The proof combines the following a priori estimate with a Komlos argument.

Lemma

1. There are constants $K_1 \in \mathbb{R}, K_2 > 0$ such that

$$J(t, x, z, u) \geq K_1 + K_2 \left\| u_2 \right\|_{L^2}.$$

2. There is a constant $K_3 > 0$ such that if $\left\| u \right\|_{L^2} \geq K_3$ then u cannot be optimal.

Characterisation via BSDE

Lemma (Cadenillas (2002) Lemma 4.1)

The functional J is Gâteaux differentiable.

- It is known that $\hat{u}$ is optimal iff $\langle J'(\hat{u}), u - \hat{u} \rangle \geq 0$ for all $u \in \mathcal{U}$.
- This yields the following characterisation in terms of the adjoint equation (P, Q, R) (see next slide).

Theorem

A control $\hat{u}$ is optimal if and only if

1. $\hat{u}_2$ maximises $u_2 \mapsto g(u_2, z) - P(t)u_2$
2. $P(t-) + R_1(t) = 0$.

The Coupled Forward-Backward System

- The adjoint equation is the following backward SDE

$$
\begin{aligned}
dP(t) =& h'\left(X^{\hat{u}}(t) - \alpha(t, Z(t))\right) dt + Q(t)dW(t) + R_1(t)\tilde{N}(dt) \\
&+ \int_{\mathbb{R}^k} R_2(t,\theta)\tilde{M}(dt, d\theta), \\
P(T) =& -f'\left(X^{\hat{u}}(T) - \alpha(T, Z(T))\right).
\end{aligned}
$$

- It is coupled with the forward SDE

$$
\begin{aligned}
dX^{\hat{u}}(t) =& \hat{u}_1(t)N(dt) + \hat{u}_2(t)dt, \\
dZ(t) =& \mu(t, Z(t))dt + \sigma(t, Z(t))dW(t) + \int_{\mathbb{R}^k} \gamma\left(t, Z(t-), \theta\right) \tilde{M}(dt, d\theta), \\
X^{\hat{u}}(0) =& x, Z(0) = z,
\end{aligned}
$$

- via the optimality conditions

$$
\hat{u}_2(t, Z(t)) = \arg\max_{u_2} \left\{ g(u_2, Z(t)) - P(t)u_2 \right\} \text{ and } P(t-) + R_1(t) = 0.
$$

The Cost Adjusted Target Function

- We define the cost-adjusted target function as

$$\tilde{\alpha}(t,z) = \arg\min_{x\in\mathbb{R}} v(t,x,z)$$

- Analysing the FBSDE, we show that trading is directed towards $\tilde{\alpha}$.

Theorem

1. The optimal limit order is $u_1 = \tilde{\alpha}(t,z) - x$.
2. In the *buy region* $\{x < \tilde{\alpha}\}$ we have $u_1, u_2 > 0$.
   In the *sell region* $\{x > \tilde{\alpha}\}$ we have $u_1, u_2 < 0$.
   In the *no trade region* $\{x = \tilde{\alpha}\}$ we have $u_1, u_2 = 0$.

Properties of the Cost Adjusted Target Function

- Further analysis of the FBSDE yields that the map $\alpha \mapsto \tilde{\alpha}$ is monotone, translation invariant and bounded.

Proposition

- If $\alpha \geq \beta$ then $\tilde{\alpha} \geq \tilde{\beta}$.
- If $\beta = \alpha + K$ then $\tilde{\beta} = \tilde{\alpha} + K$ for any constant K.
- $\inf \alpha \leq \tilde{\alpha} \leq \sup \alpha$.

Example: Curve Following with Signal

For simple dynamics, we have closed form solutions.

Proposition

Let $g(u_2, z) = \kappa u_2^2$ and $f(y) = h(y) = y^2$ and

$$dZ = \mu(t)dt + \sigma(t)dW(t).$$

Then

$$\tilde{\alpha}(t,z) = -\frac{b}{a}, \quad \hat{u}_1 = -\frac{b}{a} - x \text{ and } \hat{u}_2 = -\frac{a}{2\kappa}\left(-\frac{b}{a} - x\right),$$

where a and b solve some linear PDEs involving α and are known explicitly.

Conclusion

- We proved a version of the SMP and applied it to the problem of curve following in illiquid markets, allowing for limit and market orders.
- We analysed the corresponding adjoint equation and derived the existence of buy and sell regions.
- Explicit solution in special cases.

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