# The Bellman Equation for Power Utility Maximization with Semimartingales

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# Basic Problem

- **Utility maximization:** given utility function $U(\cdot)$, consider

$$\max \quad E\left[\int_0^T U_t(c_t)\,dt + U_T(X_T(\pi,c))\right]$$

over trading and consumption strategies $(\pi, c)$.

- **Aim of our study:** describe optimal trading and consumption for the (random) power utility

$$U_t(x) := D_t \tfrac{1}{p} x^p, \quad p \in (-\infty, 0) \cup (0,1),$$

with $D > 0$ càdlàg adapted and $E[\int_0^T D_s\,ds + D_T] < \infty$.

# Outline

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# Utility Maximization Problem

- $d$ risky assets: semimartingale $R$ of stock returns, $R_0 = 0$
- spot prices $S = (\mathcal{E}(R^1), \dots, \mathcal{E}(R^d))$
- given initial capital $x_0 > 0$,

$$u(x_0) := \sup_{(\pi,c)\in\mathcal{A}} E\Big[\int_0^T \underbrace{U_t(c_t)\,dt + U_T(c_T)}_{U_t(c_t)\,\mu^\circ(dt),\text{ with } \mu^\circ := dt+\delta_{\{T\}}}\Big], \qquad U_t(x) = D_t \tfrac{1}{p} x^p$$

- assume $u(x_0) < \infty$
- $\pi \in L(R)$ trading strategy, $c \geq 0$ optional consumption
- Wealth: $X_t(\pi, c) = x_0 + \int_0^t X_{s-}(\pi, c)\pi_s\,dR_s - \int_0^t c_s\,ds$
- **Constraints:** for each $(\omega, t)$, consider a set $0 \in \mathscr{C}_t(\omega) \subseteq \mathbb{R}^d$
- **Admissibility:** $(\pi, c) \in \mathcal{A}$ if
  - $X(\pi, c) > 0$, $X_-(\pi, c) > 0$
  - $\pi_t(\omega) \in \mathscr{C}_t(\omega)$ for all $(\omega, t)$
  - $c_T = X_T(\pi, c)$.

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# Dynamic Programming

For $(\pi, c) \in \mathcal{A}$, let $\mathcal{A}(\pi, c, t) := \big\{(\tilde{\pi}, \tilde{c}) \in \mathcal{A} : (\tilde{c}, \tilde{\pi}) = (c, \pi) \text{ on } [0, t]\big\}$.

Value process:

$$J_t(\pi, c) := \operatorname*{ess\,sup}_{(\tilde{\pi},\tilde{c})\in\mathcal{A}(\pi,c,t)} E\Big[\int_0^T U_s(\tilde{c}_s)\,\mu^\circ(ds)\Big|\mathcal{F}_t\Big]$$

**Proposition (Martingale Optimality Principle)**

*Let $(\pi, c) \in \mathcal{A}$ satisfy $E[\int_0^T U_s(c_s)\,\mu^\circ(ds)] > -\infty$. Then*

- *$J(\pi, c)$ is a supermartingale*
- *$J(\pi, c)$ is a martingale if and only if $(\pi, c)$ is optimal.*

$\rightarrow$ Starting point for local description.

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## Proposition (Opportunity Process)

- *There exists a unique càdlàg process $L$ such that for any $(\pi, c) \in \mathcal{A}$,*

$$L_t \, \tfrac{1}{p}\big(X_t(\pi, c)\big)^p = \operatorname*{ess\,sup}_{(\tilde{\pi},\tilde{c})\in\mathcal{A}(\pi,c,t)} E\Big[\int_t^T U_s(\tilde{c}_s)\,\mu^\circ(ds)\Big|\mathcal{F}_t\Big].$$

- *$L$ is special: $L = L_0 + A^L + M^L$.*

**Interpretation:** $\frac{1}{p}L_t$ is the maximal amount of conditional expected utility that can be accumulated on $[t, T]$ from 1$.

**In particular:** $L_T = pU_T(1) = D_T$.

# Outline

# Local Data I

**Differential semimartingale characteristics** wrt. a fixed increasing process $A$:

- characteristics $(b^R, c^R, F^R)$ of $R$ wrt. cut-off $h(x)$.
- characteristics $(b^L, c^L, F^L)$ of $L$ wrt. identity.
- $(b^{R,L}, c^{R,L}, F^{R,L})$ joint characteristics wrt. $(h(x), x')$, $(x, x') \in \mathbb{R}^d \times \mathbb{R}$.

**Express consumption as fraction of wealth:**

- Propensity to consume $\kappa := \frac{c}{X(\pi,c)}$.
- Wealth is a stochastic exponential: $X(\pi, \kappa) = x_0 \mathcal{E}(\pi \bullet R - \kappa \bullet t)$.

# Local Data II

**Budget constraint:** Based on $\mathcal{E}(Y) \geq 0 \Leftrightarrow \Delta Y \geq -1$:

$$X(\pi,\kappa) \geq 0 \;\Leftrightarrow\; \pi \in \mathscr{C}^0 := \Big\{ y \in \mathbb{R}^d : F^R\big[x \in \mathbb{R}^d : y^\top x < -1\big] = 0 \Big\},$$

**Additional constraints:** A set-valued process $\mathscr{C}$ in $\mathbb{R}^d$, $0 \in \mathscr{C}$,

(C1) $\mathscr{C}$ is predictable,
i.e., $\{\mathscr{C} \cap F \neq \emptyset\}$ is predictable for all $F \subseteq \mathbb{R}^d$ closed.

(C2) $\mathscr{C}$ is closed and convex.

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# Bellman Equation

**Assume:** $u(x_0) < \infty$, $\exists$ optimal strategy, constraints satisfy (C1)-(C2).

## Theorem

- *Drift rate $b^L$ satisfies* $-p^{-1}b^L = \max_{k\in[0,\infty)} f(k)\,\frac{dt}{dA} + \max_{y\in\mathscr{C}\cap\mathscr{C}^0} g(y)$.
- *Optimal propensity to consume:* $\hat{\kappa} = (D/L)^{1/(1-p)}$.
- *Optimal trading strategy:* $\hat{\pi} \in \arg\max_{\mathscr{C}\cap\mathscr{C}^0} g$.

$$f(k) := U(k) - kL_-,$$

$$g(y) := L_- y^\top \Big(b^R + \frac{c^{RL}}{L_-} + \frac{(p-1)}{2} c^R y\Big) + \int_{\mathbb{R}^d\times\mathbb{R}} x' y^\top h(x)\, F^{R,L}(d(x,x'))$$
$$+ \int_{\mathbb{R}^d\times\mathbb{R}} (L_- + x')\{p^{-1}(1+y^\top x)^p - p^{-1} - y^\top h(x)\}\, F^{R,L}(d(x,x')).$$

# Bellman BSDE

Orthogonal decomposition of $M^L$ wrt. $R$:

$$L = L_0 + A^L + \varphi^L \bullet R^c + W^L * (\mu^R - \nu^R) + N^L.$$

$\varphi^L \in L^2_{loc}(R^c)$, $W^L \in G_{loc}(\mu^R)$, $N^L$ local martingale such that $\langle (N^L)^c, R^c \rangle = 0$ and $M^P_{\mu^R}(\Delta N^L | \widetilde{\mathcal{P}}) = 0$.

## Corollary

*$L$ satisfies the BSDE*

$$L = L_0 - pU^*(L_-) \bullet t - p \max_{\mathscr{C} \cap \mathscr{C}^0} g \bullet A + \varphi^L \bullet R^c + W^L * (\mu^R - \nu^R) + N^L$$

*with terminal condition $L_T = D_T$, where*

$$g(y) := L_- y^\top \Big( b^R + c^R \Big( \frac{\varphi^L}{L_-} + \frac{(p-1)}{2} y \Big) \Big) + \int_{\mathbb{R}^d} (\Delta A^L + W^L(x) - \widehat{W}^L) y^\top h(x) \, F^R(dx)$$
$$+ \int_{\mathbb{R}^d} (L_- + \Delta A^L + W^L(x) - \widehat{W}^L) \{ p^{-1}(1 + y^\top x)^p - p^{-1} - y^\top h(x) \} \, F^R(dx).$$

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# Minimality

**Theorem (Conditions of main theorem)**

*$L$ is the minimal solution of the Bellman equation.*

For any special semimartingale $\ell$, $\exists!$ orthogonal decomposition

$$\ell = \ell_0 + A^\ell + \varphi^\ell \bullet R^c + W^\ell * (\mu^R - \nu^R) + N^\ell.$$

**Definition**

*A solution of the Bellman BSDE is a càdlàg special semimartingale $\ell$,*

- $\ell, \ell_- > 0$,
- *$\exists$ $\mathscr{C} \cap \mathscr{C}^{0,*}$-valued $\check{\pi} \in L(R)$ such that $g^\ell(\check{\pi}) = \sup_{\mathscr{C} \cap \mathscr{C}^0} g^\ell < \infty$,*
- *$\ell$ (and $\varphi^\ell, W^\ell, N^\ell, \ldots$) satisfy the BSDE.*

*With $\check{\kappa} := (D/\ell)^{1/(1-p)}$, call $(\check{\pi}, \check{\kappa})$ the strategy associated with $\ell$.*

- If $R = M + \int d\langle M \rangle \lambda$ with $M$ cont. local martingale, then $\check{\pi}$ exists.

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# Verification

## Theorem

*Let $\ell$ be a solution of the Bellman equation with associated strategy $(\check{\pi}, \check{\kappa})$. Assume that $\mathscr{C}$ is convex and let*

$$\Gamma = \ell \check{X}^p + \int \check{\kappa}_s \ell_s \check{X}_s^p \, ds, \quad \check{X} := X(\check{\pi}, \check{\kappa}).$$

*Then*

- $\Gamma$ *is a local martingale,*
- $\Gamma$ *martingale* $\iff$ $u(x_0) < \infty$ *and* $(\check{\pi}, \check{\kappa})$ *is optimal and* $\ell = L$.

Contents from (N. 2009, available on ArXiv):

[a] The Opportunity Process for Optimal Consumption and Investment with Power Utility

[b] The Bellman Equation for Power Utility Maximization with Semimartingales

**Selected related literature:**

- Existence: Kramkov&Schachermayer (AAP99), Karatzas&Žitković (AoP03)
- log-utility: Goll&Kallsen (AAP03), Karatzas&Kardaras (FS07), Kardaras (MF09)
- Mean-variance: Černý&Kallsen (AoP07)
- Power utility: Mania&Tevzadze (GeorgMJ03), Hu et al. (AAP05), Muhle-Karbe (Diss09)

**Thanks for your attention!**