Stochastic Calculus of Heston's Stochastic–Volatility Model

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Overview

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1. Introduction and Heston Model for Stochastic Volatility.

1.1 Heston Model for Stochastic Volatility:

The stochastic–variance is modeled with the often used Heston (1993) mean–reverting stochastic–variance $V(t)$ and square–root diffusion from $\sqrt{V(t)}$, with a triplet of parameters $\{\kappa_v(t), \theta(t), \sigma_v(t)\}$:

$$dV(t) = \kappa_v(t)\,(\theta_v(t) - V(t))\,dt + \sigma_v(t)\sqrt{V(t)}dW_v(t), \qquad (1)$$

with $V(0) = V_0 > 0$, where

- Log–rate: $\kappa_v(t) > 0$;
- Reversion–level: $\theta_v(t) > 0$;
- Volatility of variance (*volatility of volatility*): $\sigma_v(t) > 0$;
- Standard Brownian motion for $V(t)$: $W_v(t)$.

Equation (1) comprises the underlying stochastic–volatility (SV) model, which will be called the Heston model here, but often the term Heston model applies to the system of underlying and its stochastic–volatility.

1.2 Heston Model Background:

The mean–reverting, square–root–diffusion, stochastic–volatility model of Heston (1993) is frequently used. Heston's model derives from the *CIR* model of Cox, Ingersoll and Ross (1985b) for interest rates. The CIR paper also cites earlier and seminal work of Feller (1951), including proper (Feller) boundary conditions, process nonnegativity and the distribution for the general square-root diffusions.

Andersen, Benzoni and Lund (2002), as well as others, have statistically confirmed the importance of both stochastic–volatility and jumps in equity returns. Bates (1996) used stochastic-volatility, jump-diffusion models for exchange rates.

1.3 Simulations for Heston and Related Models:

Higham and Mao (2005) have established strong convergence and other results for the Euler–Maruyama discretization of several versions of the mean–reverting, square–root model.

Also, Lord, Koekkoek and Dijk (2007) carry out extensive comparisons of a number of Euler discretization models of the more general CEV (constant elasticity of volatility) models to force nonnegativity.

See these papers for other sources.

1.4 Singular Diffusion Complications:

The Heston model (1993) of stochastic–volatility is a square–root diffusion model for the stochastic–variance. According to Feller(1951) the model is a singular diffusion for the distribution. Unlike a regular diffusion, there is a constrained relationship between the limit that the variance goes to zero and the limit that time–step goes to zero, so that any nontrivial transformation of the Heston model leads to a transformed diffusion in the Itô stochastic calculus.

Due to the square–root term, the singular nature of the diffusion is intrinsic. Geometric Brownian motion is nominally a singular diffusion, since the diffusion vanishes when the diffusion coefficient vanishes. However, by the well–known logarithmic transformation the singular nature of geometric Brownian motion in the state is removable, resulting in additive Brownian motion, but singular in volatility. In general, singular diffusions can be sensitive to slight changes in the model, e.g., Hanson and Wazwaz (1988).

1.5 Hints at Stochastic Calculus Problems:

However, here we are interested in the properties of the Heston model alone, in simple methods of revealing its non-negativity and the consistency of the Itô diffusion approximation under transformation of the stochastic variance when the stochastic–variance can be small. As Jäckel (2005) states regarding the Heston variance process model:

> ***In an infinitesimal neighbourhood of zero, Itô's lemma cannot be applied to the variance process. The transformation of the variance process to a volatility formulation results in a structurally different process!***

Similarly, Lord et al. (2008) briefly follows up on Jäckel's warning.

Their comments suggest that a more thorough investigation of the problem is merited.

2. Verification of Nonnegativity of Stochastic Variance by Transformation to Perfect-square Form:

- Feller (1951) showed long ago that the continuos-time square-root diffusion had a nonnegative solution, i.e., $V(t)$ using very elaborate Laplace transform techniques on the classical Kolmogorov forward equation for the distribution. Also, in the time-independent form notation here, positivity and uniqueness of the distribution is assured if $\kappa_v\theta_v/\sigma_v^2 > 1/2$ with zero boundary conditions in value and flux at $v = 0$, while if $0 < \kappa_v\theta_v/\sigma_v^2 < 1/2$ then only positivity can be assured for the distribution is the flux vanishes at $v = V(t) = 0$.
- However, various simulations using the Euler-Maruyama discretization of the Heston stochastic-volatility model can produce negative values of some $V(t_i)$.
- One objective here is to confirm nonnegativity by straight-forward stochastic calculus methods.

2.1. Perfect-Square Solution by Stochastic Calculus Transformation Techniques:

Using the general transformation techniques in Hanson (2007), let $Y(t) = F(V(t), t)$. Using Itô's lemma for truncation to a diffusion approximation, the following transformed SDE is obtained,

$$\begin{aligned} dY(t) &= F_t(V(t), t)dt + F_v(V(t), t)dV(t) \\ &+ \tfrac{1}{2} F_{vv}(V(t), t)\sigma_v^2(t)V(t)dt, \end{aligned} \quad (2)$$

to dt–precision. Then a simpler form is sought with volatility-independent noise term, i.e.,

$$\begin{aligned} dY(t) &= \Big(\mu_y^{(0)}(t) + \mu_y^{(1)}(t)\sqrt{V(t)} \\ &+ \mu_y^{(2)}(t)\Big/\sqrt{V(t)}\Big)\, dt + \sigma_y(t)dW_v(t), \end{aligned} \quad (3)$$

with $Y(0) = F(V_0, 0)$, where $\mu_y^{(0)}(t)$, $\mu_y^{(1)}(t)$, $\mu_y^{(2)}(t)$ and $\sigma_y(t)$ are time-dependent coefficients to be determined.

Equating the coefficients of $dW_v(t)$ terms between (2) and (3), given $V(t)=v\geq 0$, leads to

$$F_v(v,t)=\left(\frac{\sigma_y}{\sigma_v}\right)(t)\frac{1}{\sqrt{v}}, \tag{4}$$

and then partially integrating (4) yields

$$F(v,t)=2\left(\frac{\sigma_y}{\sigma_v}\right)(t)\sqrt{v}+c_1(t), \tag{5}$$

which is the desired transformation with a function of integration $c_1(t)$. Additional differentiations of (4) produce

$$F_t(v,t)=2\left(\frac{\sigma_y}{\sigma_v}\right)'(t)\sqrt{v}+c_1'(t)$$

and

$$F_{vv}(v,t)=-\frac{1}{2}\left(\frac{\sigma_y}{\sigma_v}\right)(t)v^{-3/2}.$$

Terms of order $v^0dt=dt$ imply that $c_1'(t)=\mu_y^{(0)}(t)$, but this equates two unknown coefficients, so we set $\mu_y^{(0)}(t)=0$ and $c_1(t)=0$ for convenience.

Equating terms of order $\sqrt{v}dt$,

$$\mu^{(1)}(t) = \left(2\left(\frac{\sigma_y}{\sigma_v}\right)' - \kappa_v \left(\frac{\sigma_y}{\sigma_v}\right)\right)(t) \tag{6}$$

and for order $dt/\sqrt{v}$,

$$\mu^{(2)}(t) = \left(\kappa_v \theta_v - \sigma_v^2/4\right)(t)\left(\frac{\sigma_y}{\sigma_v}\right)(t). \tag{7}$$

However, there are more unknown functions than equations, so $\mu^{(1)}(t) = 0$ is set in (6) since that leads to an exact differential for σ_y/σ_v with solution

$$\left(\frac{\sigma_y}{\sigma_v}\right)(t) = \left(\frac{\sigma_y}{\sigma_v}\right)(0)e^{\overline{\kappa}_v(t)/2},$$

where

$$\overline{\kappa}_v(t) \equiv \int_0^t \kappa_v(s)ds.$$

For convenience, we set $\sigma_y(0) = \sigma_v(0)$. Thus (6) becomes

$$\mu_y^{(2)}(t) = e^{\overline{\kappa}_v(t)/2} \left(\kappa_v\theta_v - \sigma_v^2/4\right)(t),$$

completing the coefficient determination.

Assembling these results we form the solution as follows,

$$Y(t) = 2e^{\overline{\kappa}_v(t)/2}\sqrt{V(t)}$$

from (5), and

$$dY(t) = e^{\overline{\kappa}_v(t)/2}\left(\left(\frac{\kappa_v\theta_v - \sigma_v^2/4}{\sqrt{V}}\right)(t)dt + (\sigma_v dW_v)(t)\right)$$

from (3) and inverting for $V(t)$ yields the *transparent nonnegativity result*:

Theorem 1A. Nonnegativity of Variance:

*Let $V(t)$ be the solution to the Heston model (1), subject to conditions on the diffusion approximation truncation (2) to be determined (**Theorem 1B**), then*

$$V(t) = e^{-\overline{\kappa}_v(t)} \left(\frac{Y(t)}{2}\right)^2 \geq 0, \tag{8}$$

due to the perfect square form, where

$$Y(t) = 2\sqrt{V_0} + 2I_g(t) \tag{9}$$

and

$$I_g(t) = 0.5 \int_0^t e^{\overline{\kappa}_v(s)/2} \left(\left(\frac{\kappa_v \theta_v - \sigma_v^2/4}{\sqrt{V}} \right)(s) ds + (\sigma_v dW_v)(s) \right). \tag{10}$$

This is an implicit form that is singular unless the solution $V(t)$ is bounded away from zero, $V(t) > 0$. More generally it is desired that the solution is such that $1/\sqrt{V(t)}$ is integrable in t as $V(t) \to 0^+$, so the singularity will be ignorable in theory.

3. Model Consistency for Itô Lemma Diffusion Approximation Truncation Under Transformation and Limit of Vanishing Variance:

In general, we will assume v is both positive and bounded, i.e., $0 < \varepsilon_v \leq v \leq B_v$, where B_v is a realistic rather than theoretical upper bound. It is necessary to check the consistency of the Itô lemma diffusion approximation truncation specified in (2) because of the competing time-variance limits. As the time-increment $\Delta t \to 0^+$ in the mean square limit for the Itô approximation and as the variance singularity is approached, $V(t) \to 0^+$, i.e., $\varepsilon_v \to 0^+$, difficulties arise. Hence, it no longer make sense to assume that the state variable $V(t)$ is fixed if a uniform approximation in Δt and $V(t)$ is needed for model consistency and robustness.

3.1. Partial Derivative Power Rules:

The $F(v,t)$ given in (5) has the form $F(v,t)=\beta_0(t)\sqrt{v}+c_1(t)$, and the partial derivatives satisfy the power law

$$\frac{\partial^k F}{\partial v^k}(v,t)=\beta_k(t)v^{-(2k-1)/2},$$

where the coefficient $\beta_k(t)$ satisfies the recursion $\beta_{k+1}(t)=-(k-0.5)\beta_k(t)$ when $k\geq 0$ with $\beta_1(t)=(\sigma_y/\sigma_v)(t)$. Hence, the partial derivatives will be bounded as long as v is positive.

3.2. Taylor Series Expansion:

For $\Delta t \ll 1$, the corresponding increment in F will be expandable as a Taylor series depending on the relative sizes of Δt and $v = V(t)$, as

$$\begin{aligned}\Delta F(V(t),t) = &F(V(t)+\Delta V(t), t+\Delta t) - F(V(t),t)\\ &= \frac{\partial F}{\partial v}\Delta V(t) + \frac{\partial F}{\partial t}\Delta t + \frac{1}{2!}\frac{\partial^2 F}{\partial v^2}(\Delta V)^2(t)\\ &+ \frac{\partial^2 F}{\partial v \partial t}\Delta V(t)\Delta t + \frac{1}{2!}\frac{\partial^2 F}{\partial t^2}(\Delta t)^2\\ &+ \frac{1}{3!}\frac{\partial^3 F}{\partial v^3}(\Delta V)^3(t) + \frac{1}{2!}\frac{\partial^3 F}{\partial v^2 \partial t}(\Delta V)^2(t)\Delta t\\ &+ \frac{1}{2!}\frac{\partial^3 F}{\partial v \partial t^2}\Delta V(t)(\Delta t)^2 + \frac{1}{3!}\frac{\partial^3 F}{\partial t^3}(\Delta t)^3\\ &+ \cdots .\end{aligned}$$

3.3. Estimation of Dominant Terms:

If $\Delta t \ll 1$ with conditioning the current variance on v, letting $\mu_v(t) = \kappa_v(t)(\theta_v(t) - v)$ and $\Delta W_v(t) = \sqrt{\Delta t} Z_v(t)$ with standard normal $Z_v(t) \overset{\text{dist}}{=} \mathcal{N}(0, 1)$, then

$$[\Delta V(t)|V(t) = v] \simeq \sigma_v(t)\sqrt{v\Delta t} Z_v(t) + \mu_v(t)\Delta t.$$

In terms of small Δt when $k > 1$, the pure variance derivatives, i.e., those having only v-derivatives, will dominate the cross variance-time derivatives, the mixed v and t derivatives, as well as the pure time derivatives, since for $\Delta t \ll 1$ then $\Delta t \ll \sqrt{\Delta t} \ll 1$, while considering v fixed. Thus for $k > 1$, only the powers of diffusion part of $[\Delta V(t)|V(t) = v]$ need be considered. The mean estimate of the absolute value of dominant diffusion power is

$$\mathrm{E}\left[\left|\sigma_v(t)\sqrt{v}\Delta W_v\right|^k \middle| V(t) = v \right] = \alpha_k(t)(v\Delta t)^{k/2},$$

where $\alpha_k(t) = \sigma_v^k(t)\mathrm{E}[Z_v^k(t)]$.

Estimates of the products of these terms produce are,

$$\left|\frac{\partial^k F}{\partial v^k}(v,t)\right| \mathrm{E}[(\sigma_v(t)\sqrt{v}\Delta W_v)^k] = \gamma_k(t)\frac{\Delta t}{\sqrt{v}}\left(\frac{\Delta t}{v}\right)^{(k-2)/2}$$
$$\leq \gamma_k(t)\frac{\Delta t}{\sqrt{\varepsilon_v}}\left(\frac{\Delta t}{\varepsilon_v}\right)^{(k-2)/2},$$

for $\gamma_k(t) = \alpha_k(t)|\beta_k(t)|$, separated into the order $\Delta t/\sqrt{v}$ of the Itô diffusion approximation ($k = 2$) term and the factor relative to it. Hence, to eliminate all terms of higher order than $k = 2$, we need $\Delta t/v \ll 1$, i.e., $\Delta t \ll \varepsilon_v \ll 1$ to obtain a proper Itô diffusion approximation (2) for the transformation $Y(t) = F(V(t), t)$ in (5). Summarizing:

Theorem 1B. Conditions for a Consistent Itô Lemma Diffusion Approximation Truncation for Transforming the Heston Model (1) to one of Additive Noise: *Let* $V(t)$ *be positive and finite such that* $0 < \varepsilon_v \leq V(t) \leq B_v$, *then the additive noise model (2) is a consistent Itô diffusion approximation to the Heston model (1) uniform for* $\Delta t \ll 1$ *and* $\varepsilon_v \ll 1$, *provided* $\Delta t \ll \varepsilon_v \ll 1$.

4. Solution Consistency for Singular Limit Formulation Suitable for Theory and Computation:

However, as $V(t) \to 0^+$, it is necessary to **verify that the solution (11) satisfies the Heston model (1) in the limit**, due to the questions involving the validity of the Itô lemma and the singular integral $I_g(t)$ in (10).[a]

First recall that from (8)–(10)

$$V(t) = e^{-\overline{\kappa}_v(t)} \left(\sqrt{V_0} + I_g(t)\right)^2. \qquad (11)$$

Modifying the ***method of ignoring the singularity*** (Davis and Rabinowitz, 1965) to this implicit singular formulation, let

$$V^{(\varepsilon_v)}(t) \equiv \max(V(t), \varepsilon_v) \qquad (12)$$

where $\varepsilon_v > 0$ such that $\Delta t \ll \varepsilon_v \ll 1$. This ensures that the time–step goes to zero faster than the cutoff singular denominator.

[a]Recall that Zabusky and Kruskal (1965) showed that the well-known discretization of the Fermi-Pasta-Ulam problem numerically solved the Korteweg-deVries problem instead.

Next (11)–(10) is reformulated as a recursion for the next time increment Δt, i.e.,

$$V^{(\varepsilon_v)}(t+\Delta t) = \max\Big(e^{-\Delta\overline{\kappa}_v(t)}\Big(\sqrt{V^{(\varepsilon_v)}(t)} + e^{-\overline{\kappa}_v(t)/2}\Delta I_g^{(\varepsilon_v)}(t)\Big)^2, \varepsilon_v\Big), \tag{13}$$

where

$$\Delta\overline{\kappa}_v(t) \equiv \int_t^{t+\Delta t} \kappa_v(s)ds \to \kappa_v(t)\Delta t \ \text{ as } \ \Delta t \to 0^+.$$

Similarly, a scaled increment of an integral is defined by

$$e^{-\overline{\kappa}_v(t)/2}\Delta I_g^{(\varepsilon_v)}(t) \equiv 0.5\int_t^{t+\Delta t} e^{(\overline{\kappa}_v(s)-\overline{\kappa}_v(t))/2} \left(\left(\frac{\kappa_v\theta_v - \sigma_v^2/4}{\sqrt{V^{(\varepsilon_v)}}}\right)(s)ds + (\sigma_v dW_v)(s)\right)$$
$$\to 0.5\left(\left(\frac{\kappa_v\theta_v - \sigma_v^2/4}{\sqrt{V}}\right)(t)\Delta t + (\sigma_v\Delta W_v)(t)\right),$$

such that $\Delta t/\varepsilon_v \to 0^+$ as $\Delta t \to 0^+$ & $\varepsilon_v \to 0^+$.

An Itô–Taylor expansion to precision dt or small Δt confirms that (13)–(14) yields the Heston (1993) model, **verifying solution consistency**.

Thus, the square in (13) formally justifies the nonnegativity of the variance and the volatility of the Heston (1993) model, for a **proper computational nonnegativity–preserving procedure**.

5. *Nonsingular, Explicit, Exact Solution:*

In any event, the **singular term in (11)–(10) vanishes in the special parameter case**, such that

$$\kappa_v(t)\theta_v(t) = \sigma_v^2(t)/4, \quad \forall\, t. \tag{14}$$

Hence, we obtain a nonnegative, nonsingular exact solution

$$V(t) = e^{-\overline{\kappa}_v(t)}\left(\sqrt{V_0} + 0.5\int_0^t e^{\overline{\kappa}_v(s)/2}(\sigma_v dW_v)(s)\right)^2, \tag{15}$$

with the recursive numerical form corresponding to the ε_v-truncated forms (13)–(14),

$$\begin{aligned} V^{(\varepsilon_v)}(t+\Delta t) = \max\Bigg(& e^{-\Delta\overline{\kappa}_v(t)}\Bigg(\sqrt{V^{(\varepsilon_v)}(t)} \\ &+ \frac{1}{2}\int_t^{t+\Delta t} e^{(\overline{\kappa}_v(s) - \overline{\kappa}_v(t))/2} \\ &\cdot (\sigma_v dW_v)(s)\Bigg)^2, \varepsilon_v\Bigg), \end{aligned} \tag{16}$$

which is **useful for testing simulation algorithms**.

6. *Alternate Solution Relative to Deterministic Solution Using Integrating Factor Transformation:*

Similarly, the chain rule for the **linear integrating factor form**

$$X(t) = \exp(\overline{\kappa}_v(t))V(t) \tag{17}$$

for the general stochastic–volatility (1) leads to a somewhat simpler integrated form,

$$V(t) = V^{(\text{det})}(t) + e^{-\overline{\kappa}_v(t)} \int_0^t e^{\overline{\kappa}_v(s)} \left(\sigma_v \sqrt{V} dW_v\right)(s), \tag{18}$$

suppressing the maximum with respect to zero, where

$$V^{(\text{det})}(t) = e^{-\overline{\kappa}_v(t)} \left(V_0 + \int_0^t \theta_v(s) d\left(e^{\overline{\kappa}_v(s)}\right)\right) \tag{19}$$

is the **deterministic part of $V(t)$**.

So the deterministic part is easily separated out from the square-root dependence and replaces the mean-reverting drift term. The $V^{(\mathrm{det})}(t)$ will be positive for positive parameters. For simulation purposes, the incremental recursion is useful,

$$\begin{aligned} V^{(\mathrm{det})}(t+\Delta t) = e^{-\Delta\overline{\kappa}_v(t)} \Big(& V^{(\mathrm{det})}(t) \\ & + e^{-\overline{\kappa}_v(t)} \int_t^{t+\Delta t} \Big(\theta_v d\Big(e^{\overline{\kappa}_v}\Big)\Big)(s) \Big), \end{aligned} \tag{20}$$

yielding

$$\begin{aligned} V(t+\Delta t) = e^{-\Delta\overline{\kappa}_v(t)} \Big(& V(t) + e^{-\overline{\kappa}_v(t)} \int_t^{t+\Delta t} e^{\overline{\kappa}_v(s)} \\ & \cdot \Big(\kappa_v\theta_v + \sigma_v\sqrt{V}dW_v\Big)(s)\Big). \end{aligned} \tag{21}$$

Note that with **constant coefficients**, $\theta_v(t)=\theta_0$, $\kappa_v(t)=\kappa_0$ and $\sigma_v(t)=\sigma_0$, then (19–21) become

$$V^{(\mathrm{det})}(t)=V_0e^{-\kappa_0 t}+\theta_0\left(1-e^{-\kappa_0 t}\right), \tag{22}$$

$$V^{(\mathrm{det})}(t+\Delta t)=\theta_0+e^{-\kappa_0\Delta t}\left(V^{(\mathrm{det})}(t)-\theta_0\right), \tag{23}$$

and

$$V(t+\Delta t)=\theta_0+e^{-\kappa_0\Delta t}\left(V(t)-\theta_0+\sigma_0\sqrt{V(t)}\Delta W(t)\right). \tag{24}$$

However, as Lord et al. (2008) point out, a sufficiently accurate simulation scheme and a **large number of simulation nodes** are required so that the right-hand side of (1) generates nonnegative values. Otherwise, $V(t) = \max(V(t), \varepsilon_v)$ can be used.

7. Selected Numerical Simulations:

In **Figure 1**, the simulations for the Euler approximation to Heston's stochastic–variance equation (1), truncating any negative values to zero, compared to the perfect square solution simulations in (13) using $\varepsilon_v = 0$. The parameter values used are $\{\kappa_v = 2.00$, $\theta_v = 0.01$, $\sigma_v = 0.25\}$, or the exact parameter ratio $\kappa_v \theta_v / \sigma_v^2 = 0.25$.

The number of negative values K_{neq} before truncation are plotted in **Figure 2** against the Heston model parameter ratio $\kappa_v \theta_v / \sigma_v^2$.

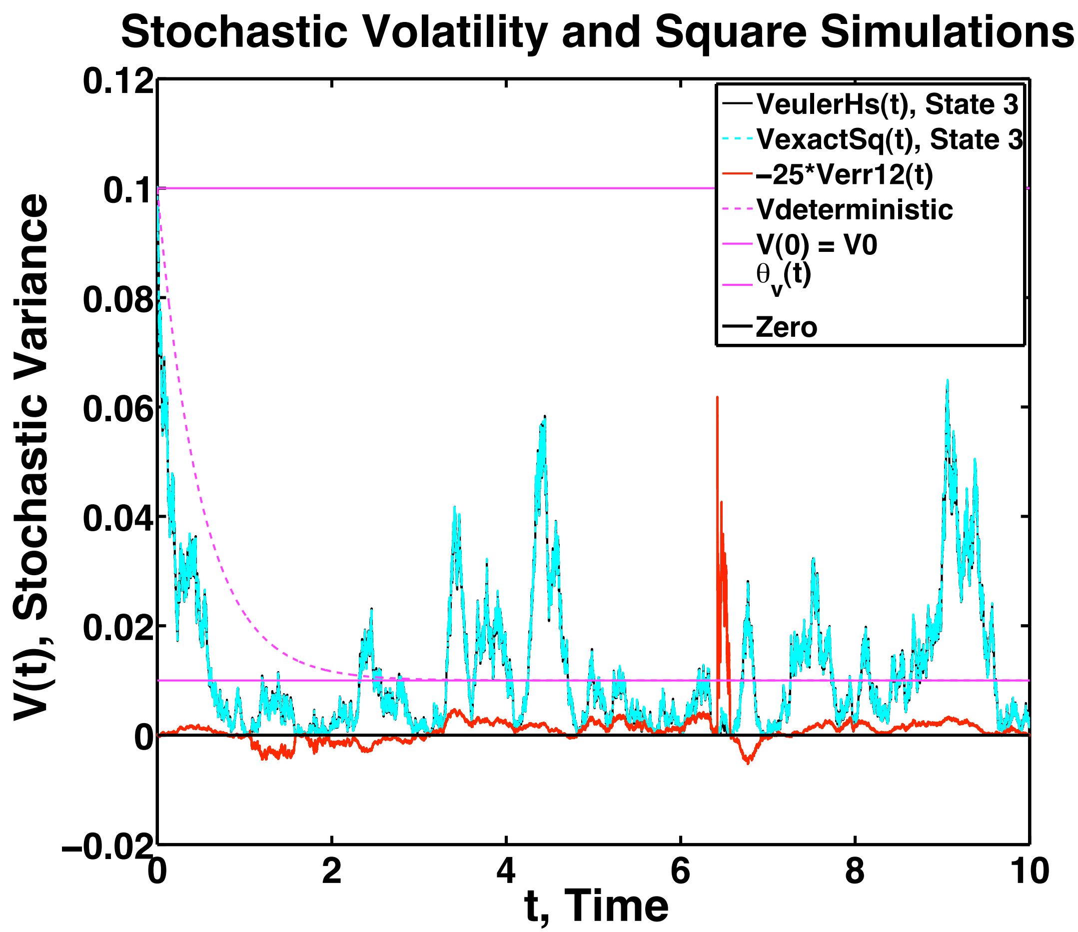

Figure 1: Comparison of Heston-Euler simulation of **VeulerHs** (1) with $\boldsymbol{\varepsilon_v = 0}$ and the perfect square solution **VexactSq** (13), barely distinguishable with the maximum error **2.46e-3** of **Verr12(t)**, ***magnified 25 times***. Also shown are the deterministic solution (20). The Heston model parameter ratio is $\boldsymbol{\kappa_v \theta_v / \sigma_v^2 = 0.25}$, $\boldsymbol{N = 10^6}$ sample points and $\boldsymbol{K_{neq} = 76}$ prior to truncation.

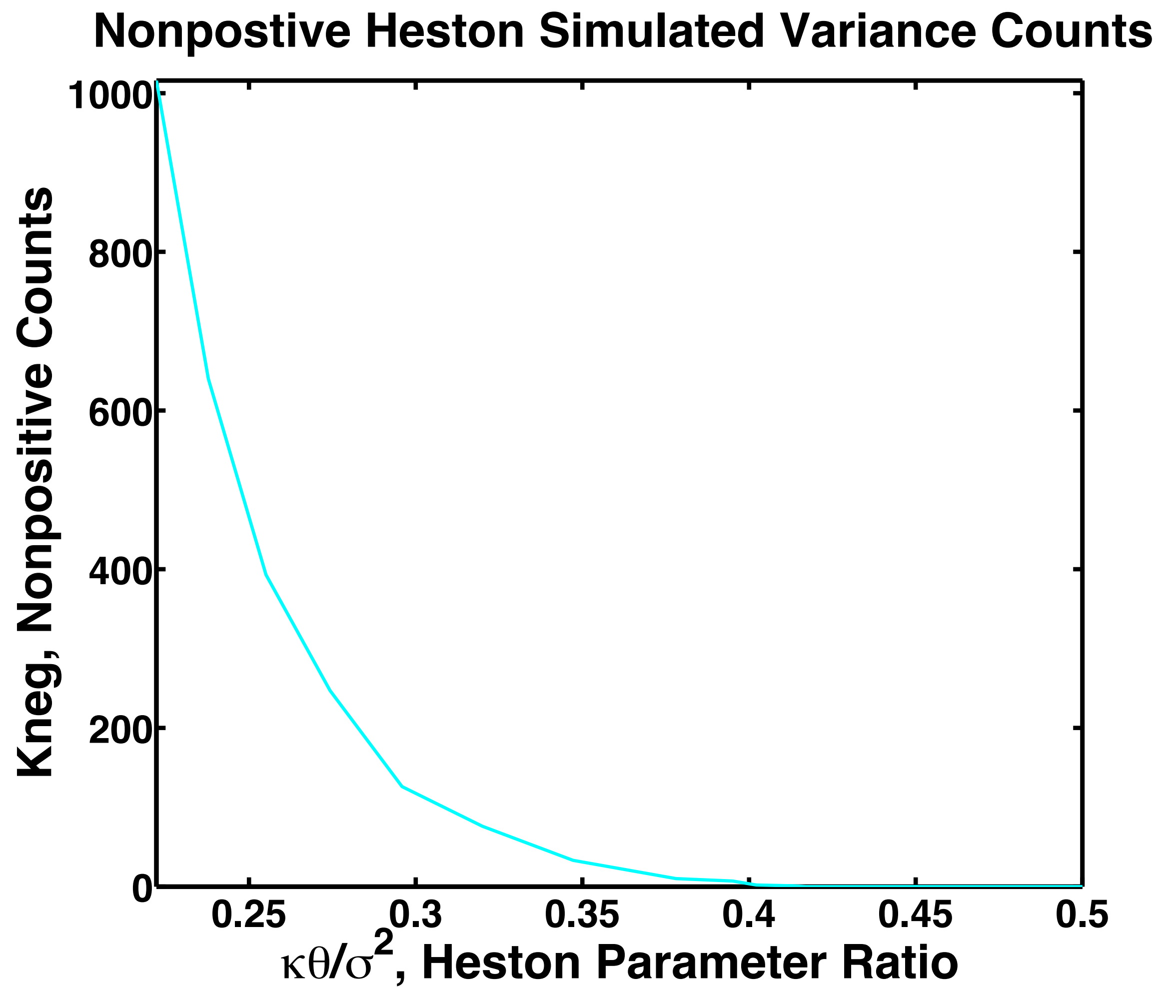

Figure 2: Nonpositive variance counts for the Heston Euler and Alternate solution simulations, counted prior to truncation to zero. The coordinate axis is the Heston model parameter ratio $\kappa_v\theta_v/\sigma_v^2$, where $\kappa_v = 2.$ and $\theta_v = 0.01$, while $\sigma_v \in [0.2, 0.3]$.

8. Conclusions:

- Derived a transformation of the Heston stochastic volatility $V(t)$ square-root noise model to a **additive noise model**, leading to a **perfect-square, nonnegative model** for $V(t)$;
- Derived the **Itô lemma consistency** conditions in the time-step Δt and lower bound $\varepsilon_v = \min[V(t)]$ to uniformly ensure truncation to a diffusion for the transformed model;
- Confirmed that the transformed, truncated model formal solution **reduces back to the Heston model** in the joint small Δt and ε_v limit.
- Specified the special case of Heston parmeters with an **explicit, nonsingular, exact solution**.
- Produced another transformed solution that **separates out the deterministic solution**.
- Showed **numerical simulations** compared and the dependence of the spurious nonpositive counts on the Heston parameter ratio.