Simulation of Diversified Portfolios in a Continuous Financial Market

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Platen, E. & Heath, D.: A Benchmark Approach to Quantitative Finance
Springer Finance, 700 pp., 199 illus., Hardcover, ISBN-10 3-540-26212-1 (2006).

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Benchmark Approach

Pl. & Heath (2006)

- Diversification Theorem
  well diversified portfolio approximates numeraire portfolio
- growth optimal portfolio equals numeraire portfolio, Long (1990)
- benchmark in portfolio optimization
- numeraire in derivative pricing

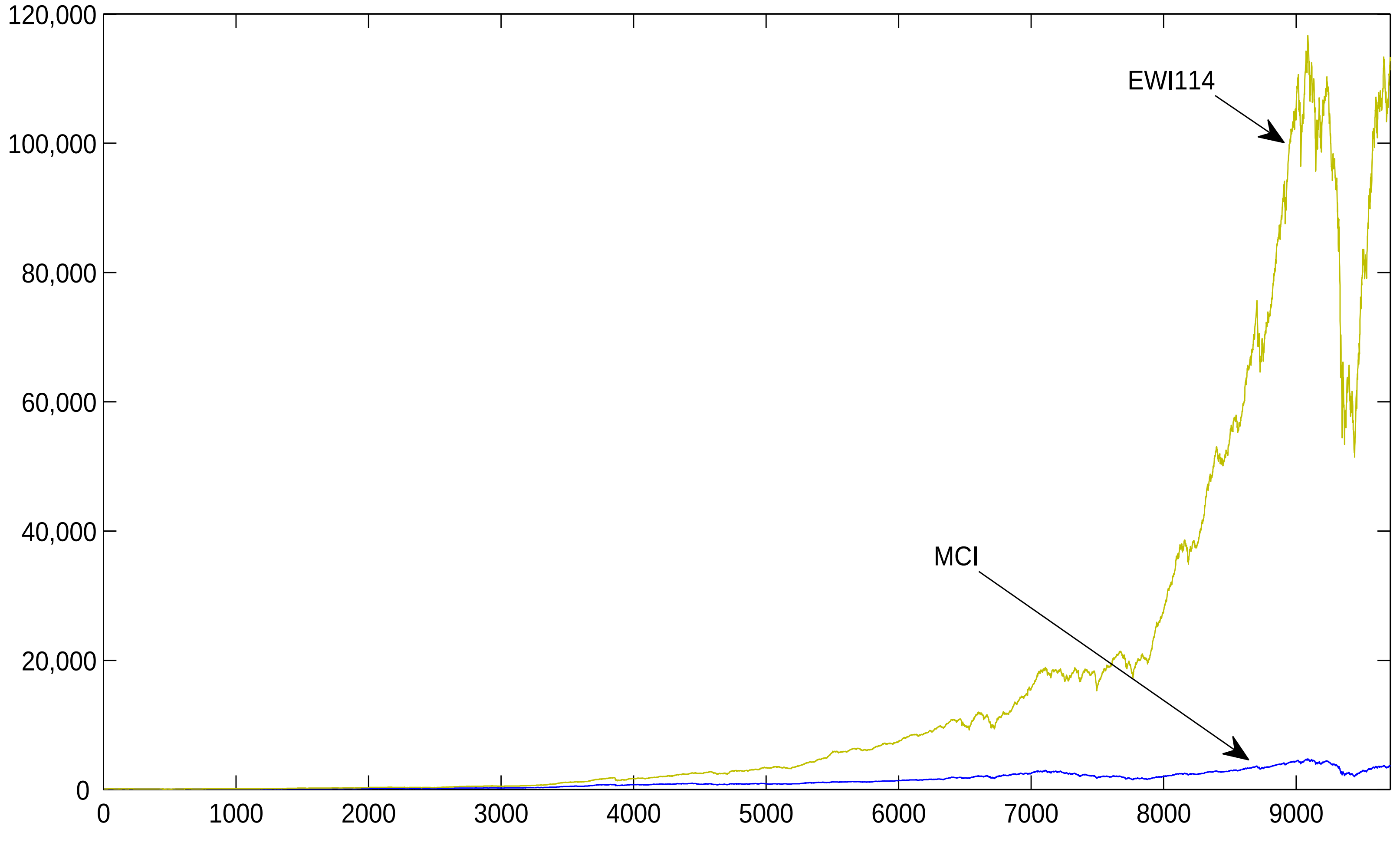

Figure 1: EWI114 and MCI.

Supermartingale Property

Assume numeraire portfolio $S^{\delta_*}_{t_n}$ as benchmark s.t.

for all nonnegative portfolios $S^{\delta}_{t_n}$

$$\frac{S^{\delta}_{t_n}}{S^{\delta_*}_{t_n}} = \hat{S}^{\delta}_{t_n} \geq E_{t_n}\left(\hat{S}^{\delta}_{t_{n+1}}\right)$$

S^{δ_*} - best performing portfolio, numeraire portfolio, growth optimal

Kelly (1956), Long (1990), Becherer (2001), Pl. (2002),

Bühlmann & Pl. (2003), Pl. & Heath (2006), Karatzas & Kardaras (2007)

- **Benchmarked numeraire portfolio**

$$\hat{S}^{\delta_*}_{t_n} = 1 \Longrightarrow$$

$$\frac{\hat{S}^{\delta_*}_{t_{n+1}} - \hat{S}^{\delta_*}_{t_n}}{\hat{S}^{\delta_*}_{t_n}} = 0 \quad \text{a.s.}$$

- approximate $\hat{S}^{\delta_*}$

  strictly positive

- sequence of strictly positive, benchmarked portfolios

$$\left(\hat{S}^{\delta_\ell}\right)_{\ell \in \{1,2,\dots\}}$$

with $\hat{S}_0^{\delta_\ell} = 1$

is a sequence of **approximate numeraire portfolios**

if for $\varepsilon > 0$

$$\lim_{\ell \to \infty} P_{t_n}\left(\frac{1}{\sqrt{t_{n+1} - t_n}}\left|\frac{\hat{S}^{\delta_\ell}_{t_{n+1}} - \hat{S}^{\delta_\ell}_{t_n}}{\hat{S}^{\delta_\ell}_{t_n}}\right| \geq \varepsilon\right) = 0$$

for all $n \in \{0, 1, \dots\}$

- financial market is **semi-regular** if for all $\varepsilon > 0$

$$\lim_{\ell \to \infty} P\left(\left| \frac{1}{\sqrt{\ell}} \sum_{j=1}^{\ell} \sigma_{t_n}^{j,k} \right| \geq \varepsilon \right) = 0$$

for all $k \in \{1, 2, \ldots\}$ and $n \in \{0, 1, \ldots\}$

Diversification Theorem:

In a semi-regular market each sequence of EWIs

is a sequence of approximate numeraire portfolios.

Pl. (2005), Pl. & Rendek (2009)

Numeraire portfolio has only non-diversifiable risk!

Equally Weighted Index

EWI

$$\pi^{j}_{\delta_{\mathrm{EWI}},t} = \frac{1}{d}$$

$j \in \{1, 2, \ldots, d\}$

Inverse Transform Method

- random variable Y - distribution function F_Y
- uniformly distributed random variable $0 < U < 1$
  F_Y distributed random variable $y(U)$

$$U = F_Y(y(U))$$

$$y(U) = F_Y^{-1}(U)$$

More generally

$$y(U) = \inf\{y : U \leq F_Y(y)\}$$

Transition Density of a Matrix SR-process

$d \times m$ matrix

$$p(s, X; t, Y) = \frac{p_\delta \left(\varphi(s), \frac{X}{s_s}; \varphi(t), \frac{Y}{s_t} \right)}{s_t}$$

- φ-time

$$\varphi(t) = \varphi(0) + \frac{\bar{b}^2}{4\bar{c}\, s_0} \left(1 - \exp\{-\bar{c}\, t\}\right)$$

$$s_t = s_0 \exp\{\bar{c}\, t\} \text{ for } t \in [0, \infty),\ s_0 > 0,\ \bar{c} < 0 \text{ and } \bar{b} \neq 0$$

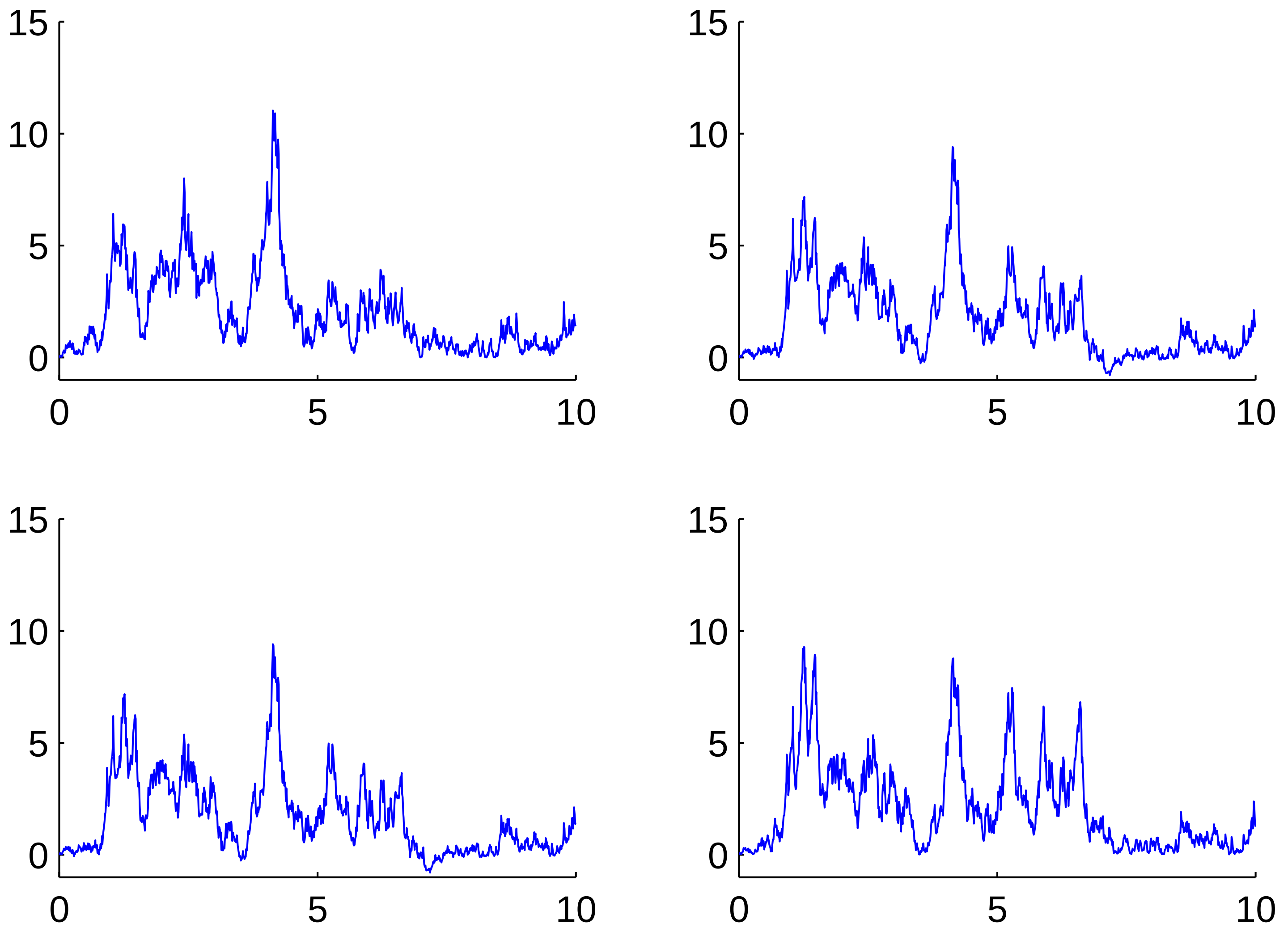

Figure 2: Matrix valued square root process

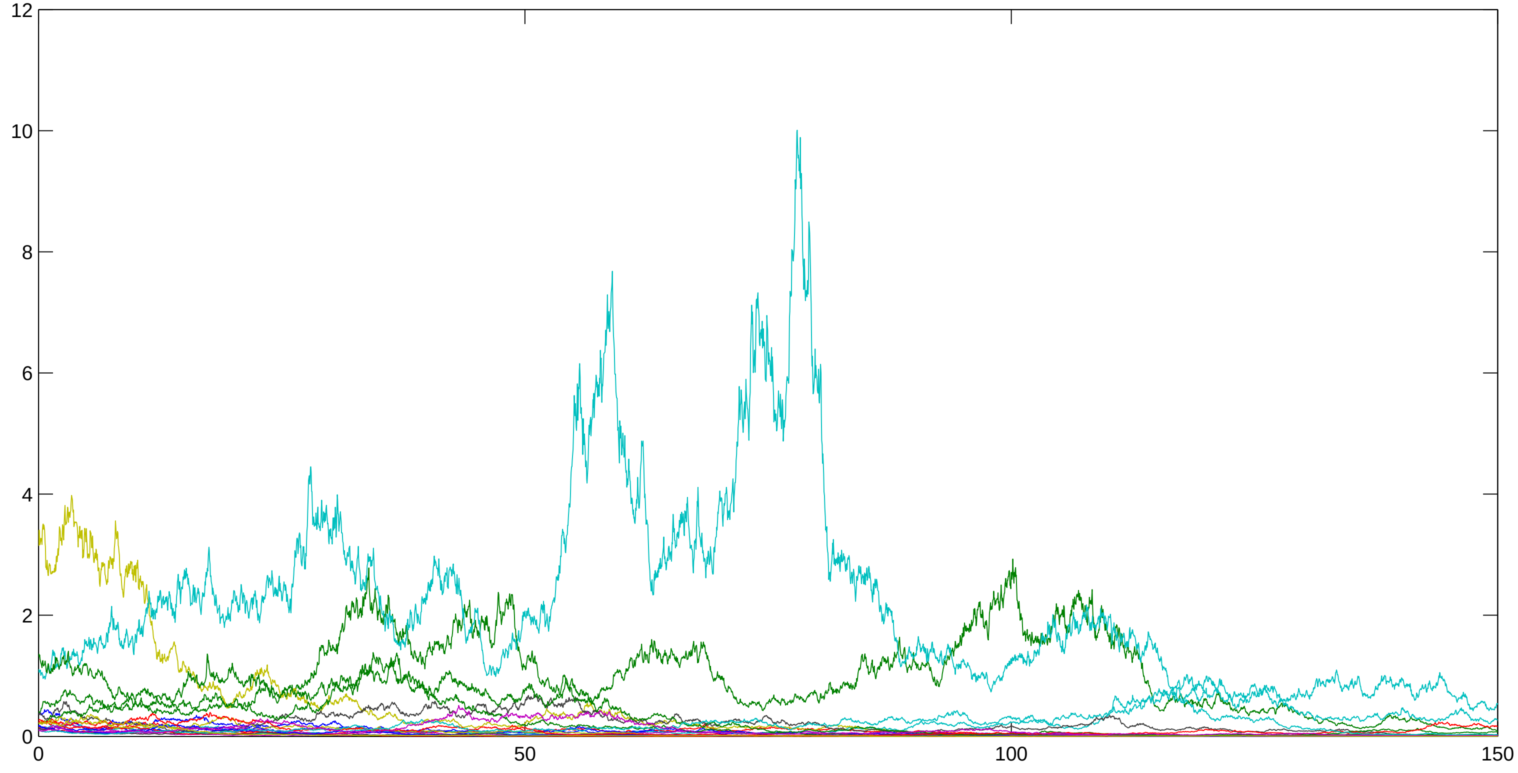

Figure 3: Simulated benchmarked primary security accounts under the Black-Scholes model

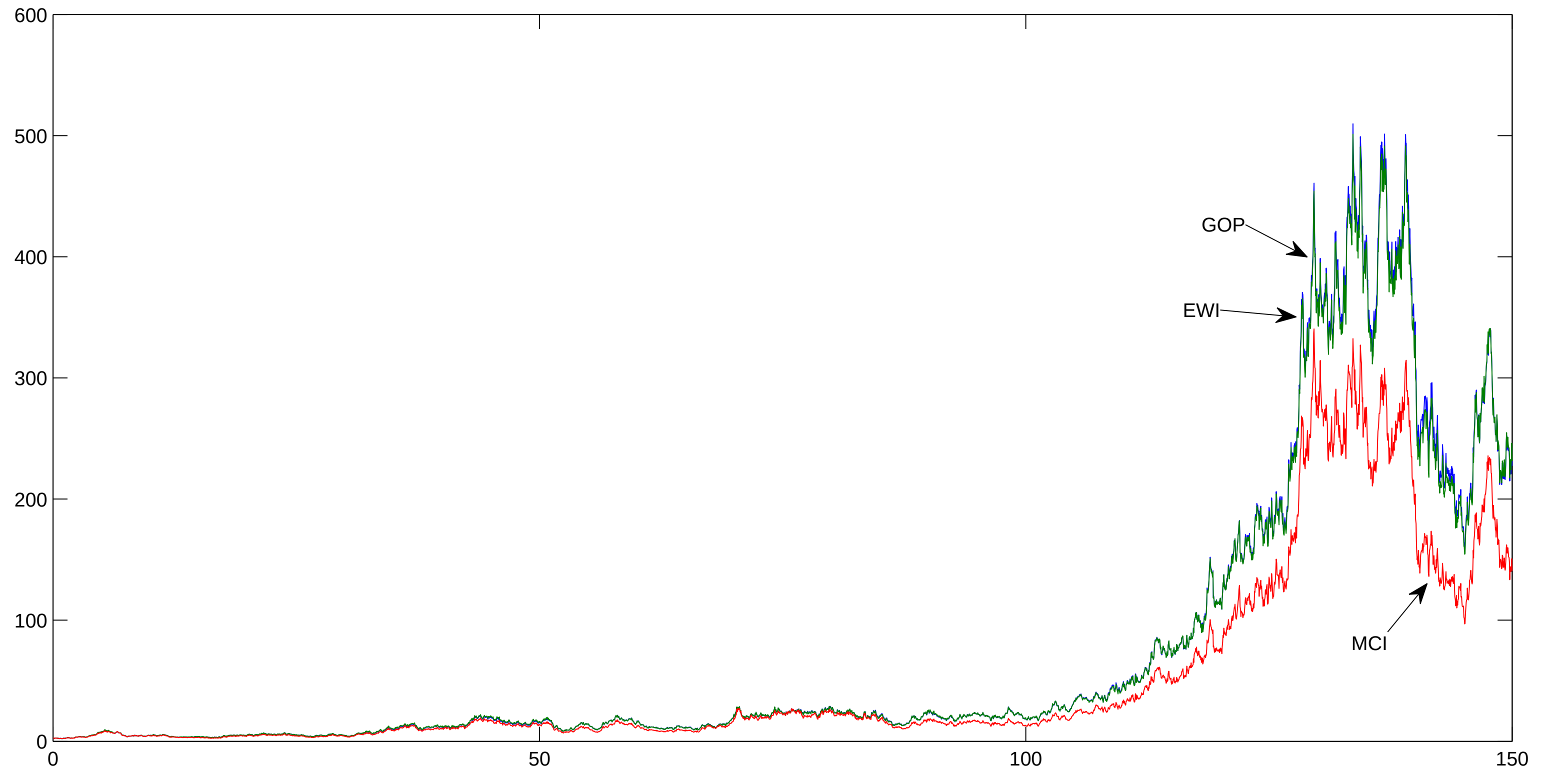

Figure 4: Simulated GOP, EWI and MCI under the Black-Scholes model

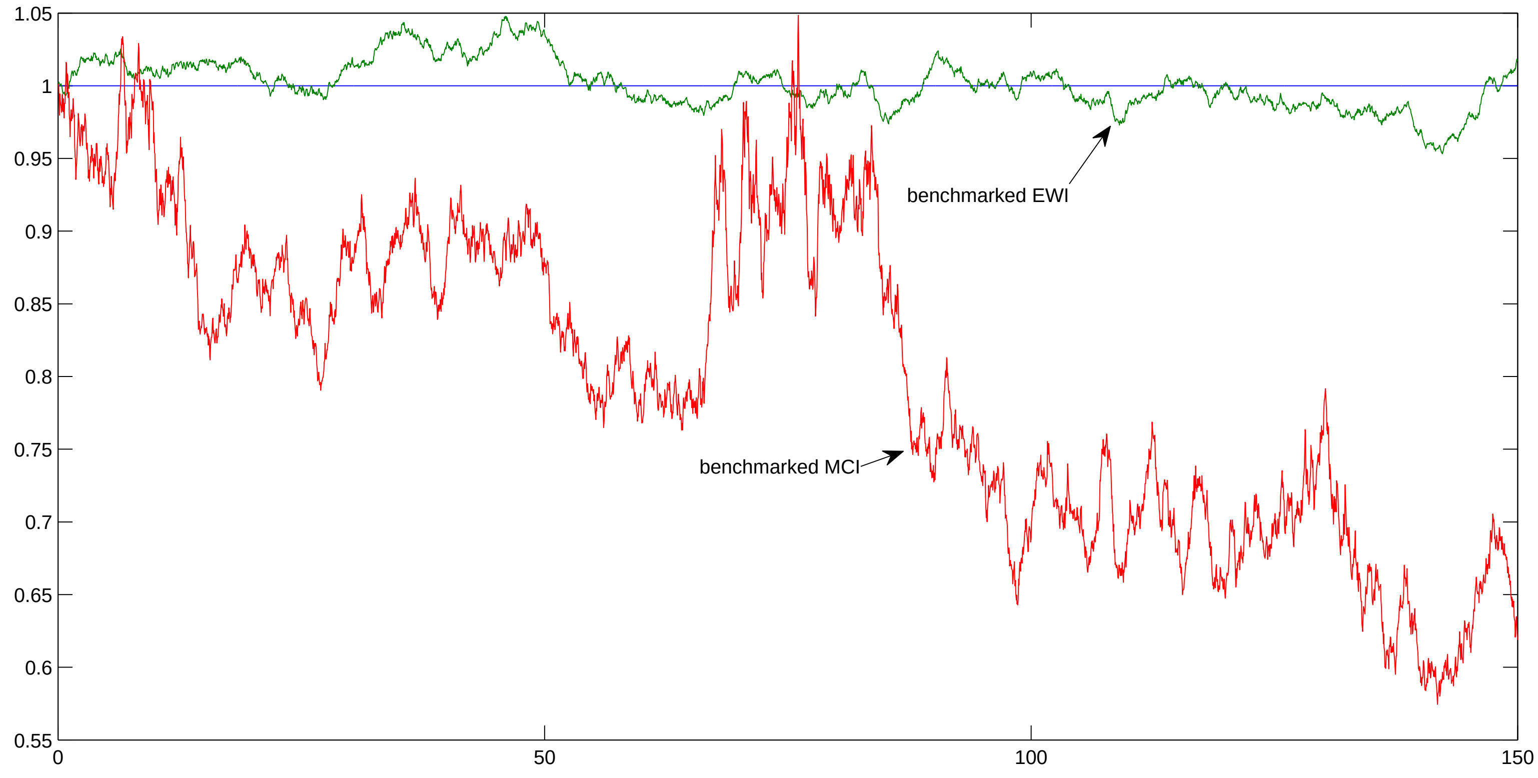

Figure 5: Simulated benchmarked GOP, EWI and MCI under the Black-Scholes model

Multi-asset Heston Model

Heston (1993)

- **matrix SDEs**

$$d\hat{\boldsymbol{S}}_t = \text{diag}\left(\sqrt{\boldsymbol{V}_t}\right) \text{diag}\left(\hat{\boldsymbol{S}}_t\right) \left(\boldsymbol{A} d\tilde{\boldsymbol{W}}_t^{\boldsymbol{1}} + \boldsymbol{B} d\tilde{\boldsymbol{W}}_t^{\boldsymbol{2}}\right)$$

$$d\boldsymbol{V}_t = (\boldsymbol{a} - \boldsymbol{E}\boldsymbol{V}_t)\, dt + \boldsymbol{F}\text{diag}\left(\sqrt{\boldsymbol{V}_t}\right) d\tilde{\boldsymbol{W}}_t^{\boldsymbol{1}}$$

$$\hat{\boldsymbol{S}}_t = (\hat{S}_t^{\boldsymbol{0}}, \hat{S}_t^{\boldsymbol{1}}, \ldots, \hat{S}_t^{\boldsymbol{d}})^\top$$

- **independent** vectors of correlated Wiener processes

$$\tilde{\boldsymbol{W}}_t^{\boldsymbol{k}} = \boldsymbol{C}^{\boldsymbol{k}} \boldsymbol{W}_t^{\boldsymbol{k}}$$

- exact simulation in Broadie & Kaya (2006) (complicated, slow)

- simplified almost exact simulation Pl. & Rendek (2010)

- **squared volatility** $V^j_{t_{i+1}}$

  sampling directly from the noncentral chi-square distribution

  exact

- **almost exact simulation**

$$X_t = \ln(\hat{S}_t)$$

- **log-asset price**

$$X^j_{t_{i+1}} = X^j_{t_i} + \frac{\varrho_j}{\gamma_j}\left(V^j_{t_{i+1}} - V^j_{t_i} - a_j\Delta\right) + \left(\frac{\varrho_j \kappa_j}{\gamma_j} - \frac{1}{2}\right)\int_{t_i}^{t_{i+1}} V^j_u\, du$$

$$+ \sqrt{1-\varrho_j^2}\int_{t_i}^{t_{i+1}} \sqrt{V^j_u} dW^{2,j}_u$$

with

$$\int_{t_i}^{t_{i+1}} \sqrt{V^j_u} dW^{2,j}_u$$

conditionally Gaussian:

mean zero, variance $\int_{t_i}^{t_{i+1}} V^j_u\, du$

- approximate

$$\int_{t_i}^{t_{i+1}} V_u^j du$$

- **trapezoidal rule**

$$\int_{t_i}^{t_{i+1}} V_u^j du \approx \frac{\Delta}{2}\left(V_{t_i}^j + V_{t_{i+1}}^j\right)$$

$\Longrightarrow$

$$\int_{t_i}^{t_{i+1}} \sqrt{V_u^j} dW_u^{2,j} \approx \mathcal{N}\left(0, \frac{\Delta}{2}\left(V_{t_i}^j + V_{t_{i+1}}^j\right)\right)$$

achieved with high accuracy

efficient almost exact simulation technique

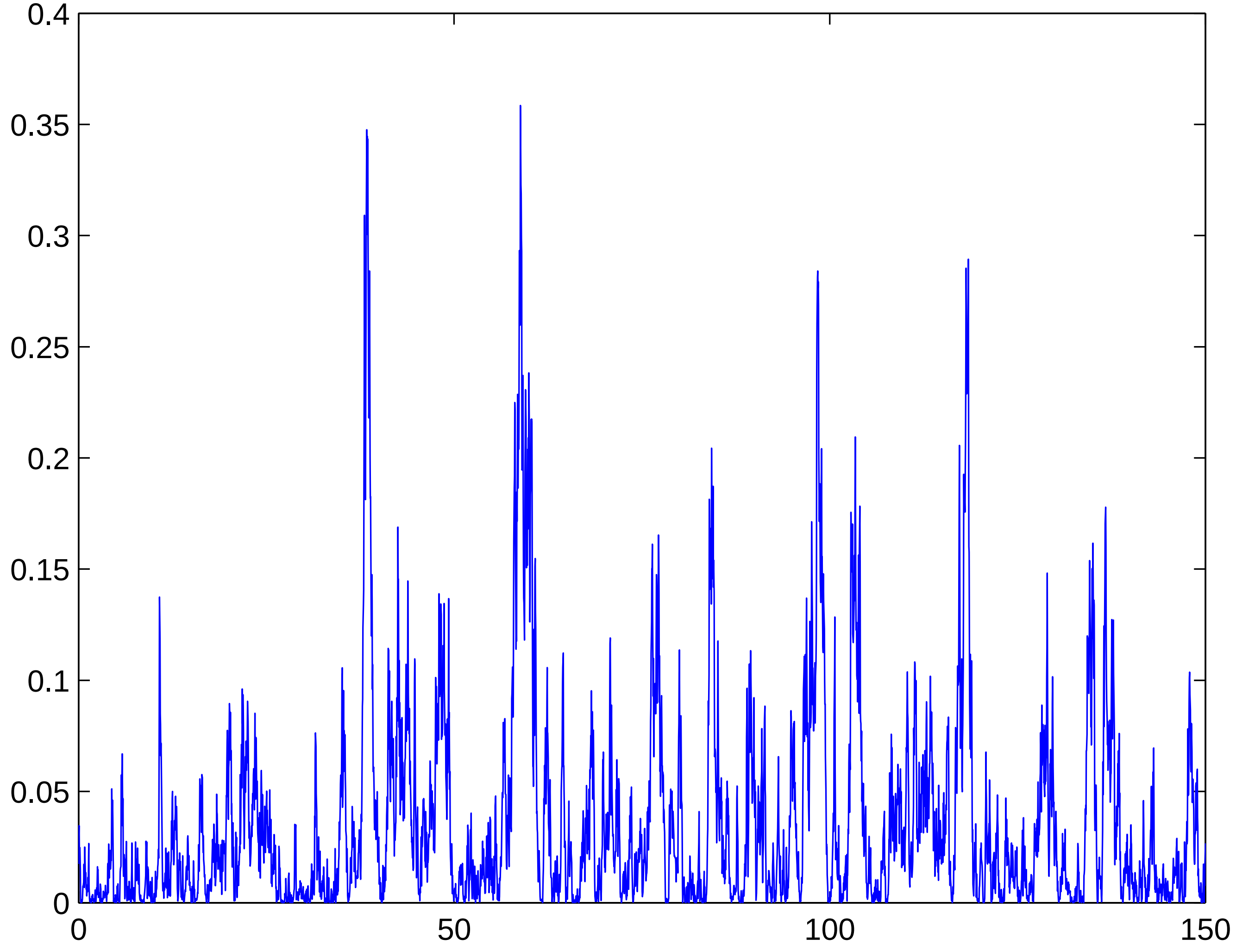

Figure 6: Simulated squared volatility under the Heston model

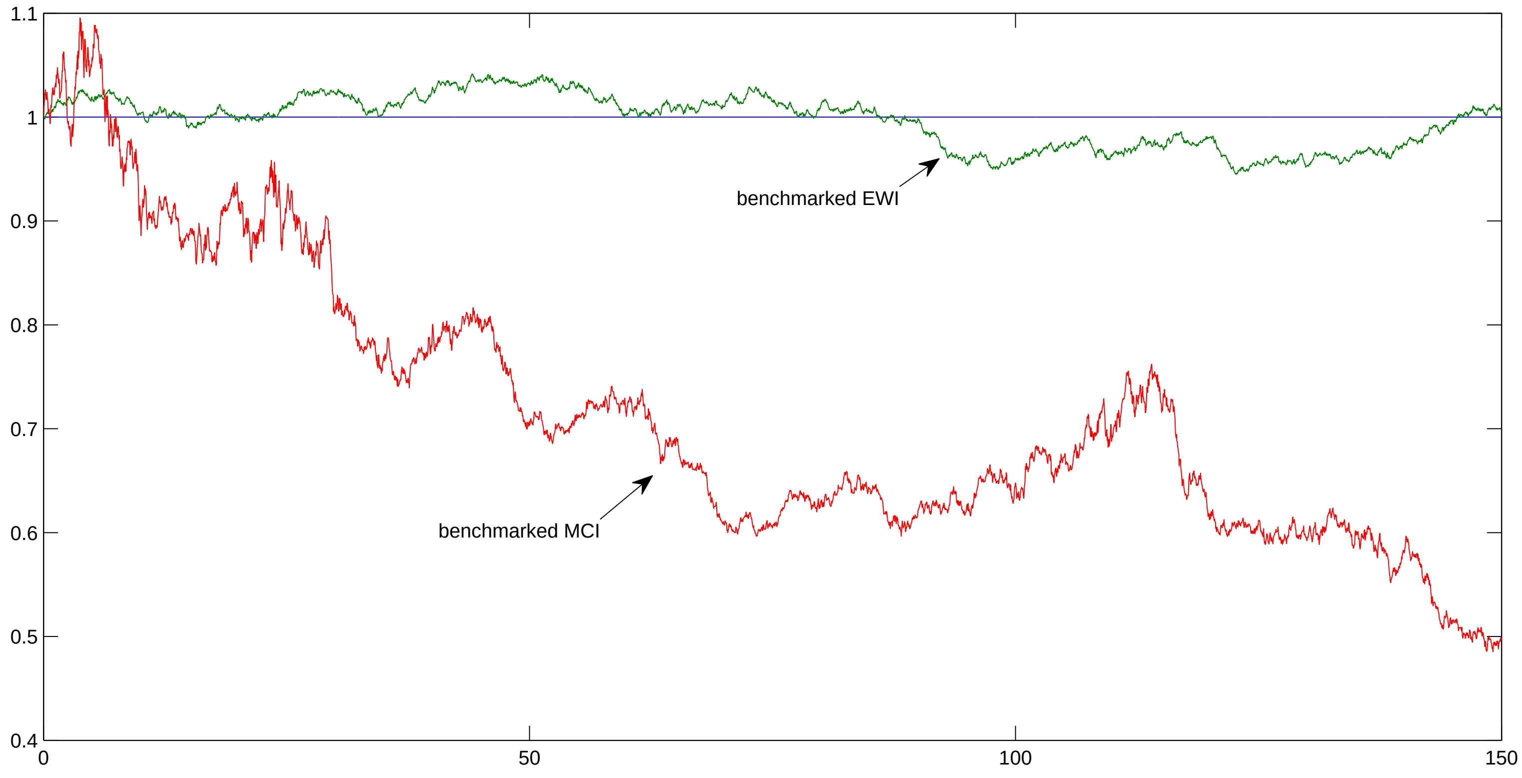

Figure 7: Simulated benchmarked GOP, EWI and MCI under the Heston model

Multi-asset ARCH-diffusion Model

Nelson (1990), Frey (1997)

$$
\begin{aligned}
d\hat{S}_t &= \operatorname{diag}\left(\sqrt{V_t}\right) \operatorname{diag}\left(\hat{S}_t\right) \left(A d\tilde{W}_t^1 + B d\tilde{W}_t^2\right) \\
dV_t &= (a - EV_t)\, dt + F \operatorname{diag}(V_t)\, d\tilde{W}_t^1
\end{aligned}
$$

- **squared volatility**

$$
\begin{aligned}
V_{t_{i+1}}^j &= \exp\left\{\left(-\kappa_j - \frac{1}{2}\gamma_j^2\right) t_{i+1} + \gamma_j W_{t_{i+1}}^{1,j}\right\} \\
&= \times\left(V_{t_0}^j + a_j \sum_{k=0}^{i} \int_{t_k}^{t_{k+1}} \exp\left\{\left(\kappa_j + \frac{1}{2}\gamma_j^2\right) s - \gamma_j W_s^{1,j}\right\} ds\right)
\end{aligned}
$$

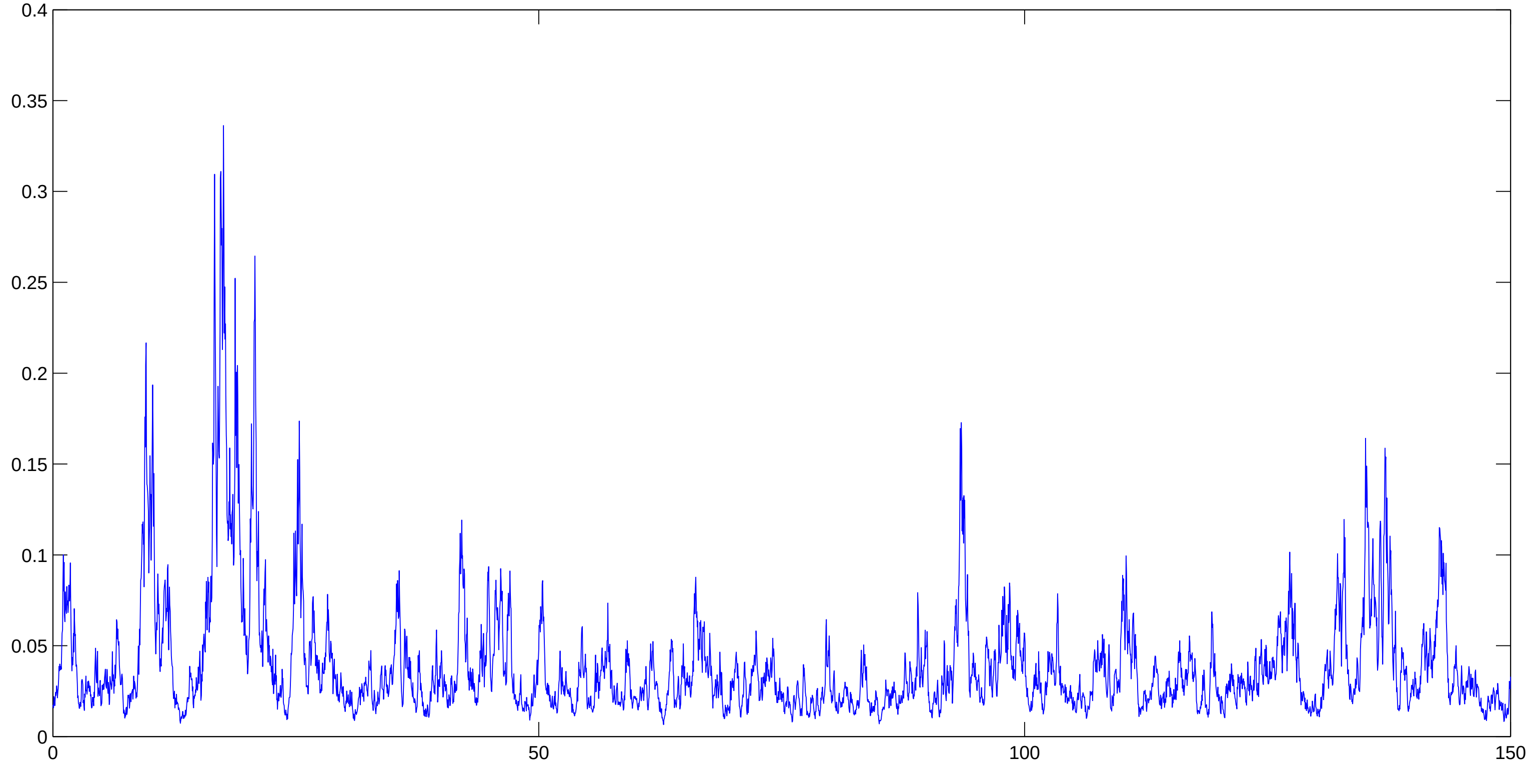

Figure 8: Simulated squared volatility under the ARCH-diffusion model

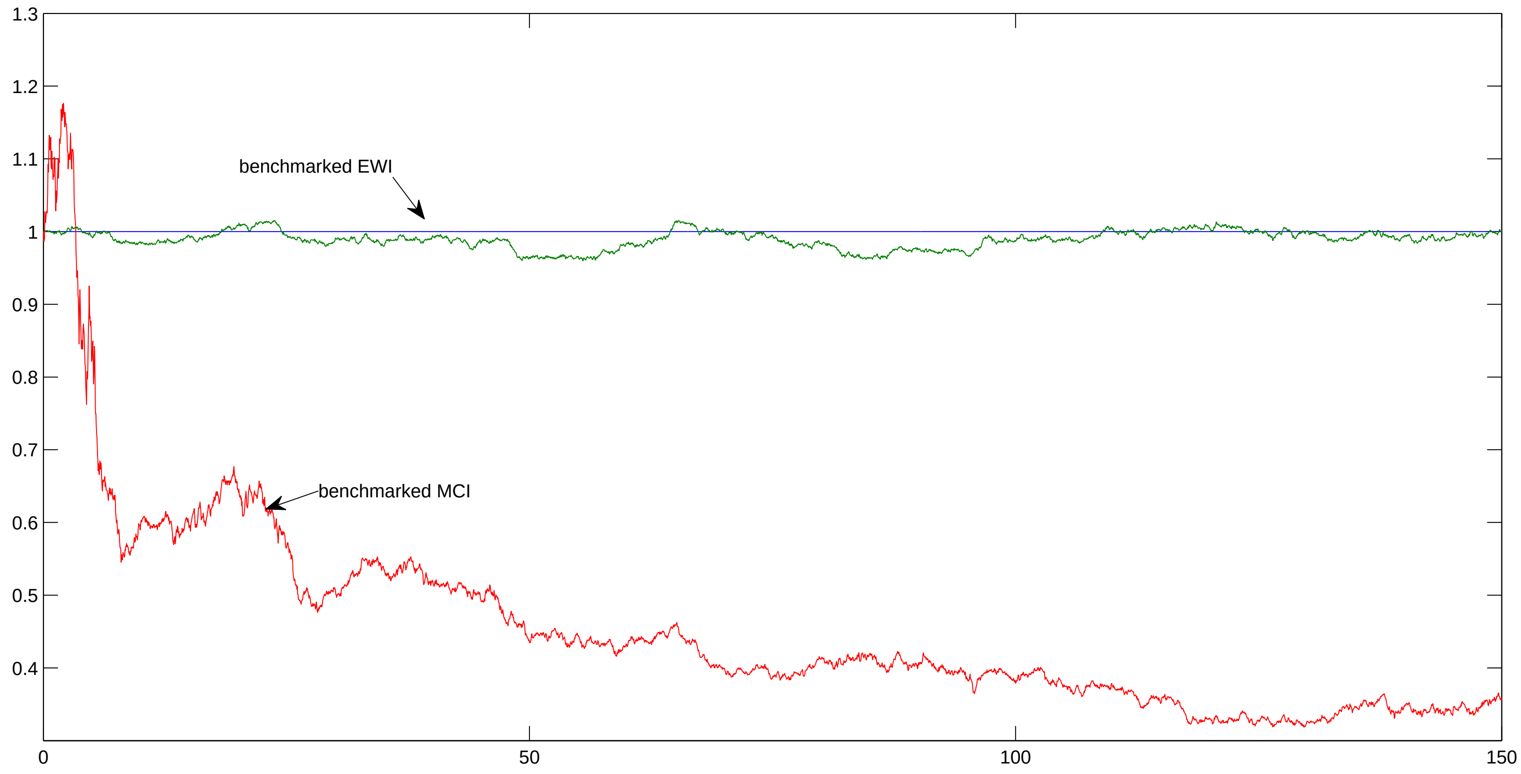

Figure 9: Simulated benchmarked GOP, EWI and MCI under the ARCH-diffusion model

Geometric Ornstein-Uhlenbeck Volatility Model

$$d\hat{\boldsymbol{S}}_t = \text{diag}\left(\exp\{\boldsymbol{V}_t\}\right) \text{diag}\left(\hat{\boldsymbol{S}}_t\right) \left(\boldsymbol{A} d\tilde{\boldsymbol{W}}_t^1 + \boldsymbol{B} d\tilde{\boldsymbol{W}}_t^2\right)$$

$$d\boldsymbol{V}_t = \left(\boldsymbol{a} - \boldsymbol{E}\boldsymbol{V}_t\right) dt + \boldsymbol{F} d\tilde{\boldsymbol{W}}_t^1$$

simulation for $\exp\{V^j\}$ exact

log-asset price as before

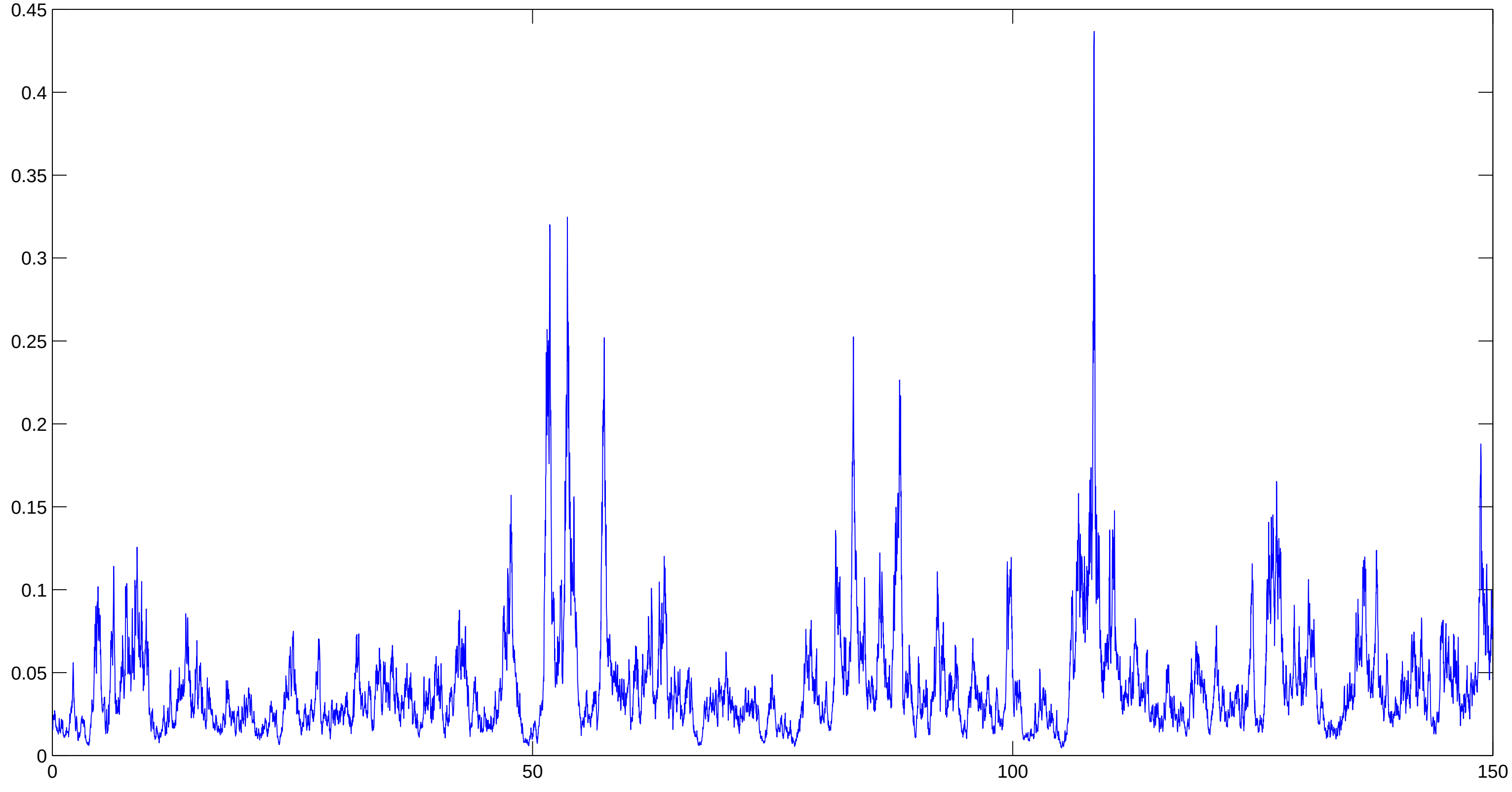

Figure 10: Simulated squared volatility under the geometric Ornstein-Uhlenbeck volatility model

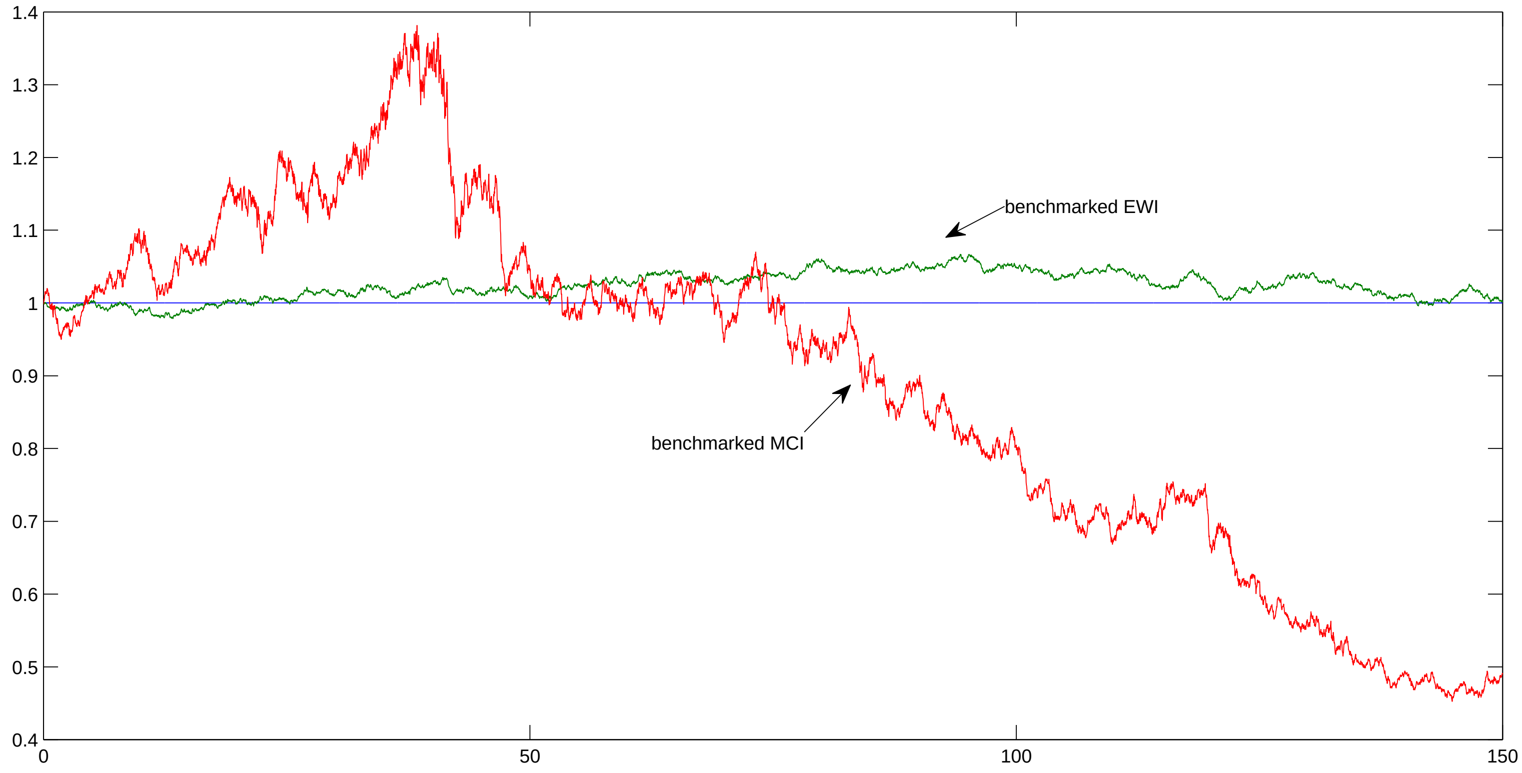

Figure 11: Simulated benchmarked GOP, EWI and MCI under the geometric Ornstein-Uhlenbeck volatility model

Minimal Market Model

Pl. (2001), Pl. & Heath (2006)

- φ-**time**

$$\varphi^j(t) = \frac{\alpha_0^j}{4\eta^j} \exp\{\eta^j t\}$$

- **benchmarked primary security account**

$$d\hat{S}^j(\varphi^j(t)) = -2\left(\hat{S}^j(\varphi^j(t))\right)^{\frac{3}{2}} d\bar{W}^j(\varphi^j(t))$$

strict supermartingale

$$\hat{S}^j\left(\varphi^j(t_{i+1})\right) = \frac{1}{\sum_{k=1}^4 \left(w^k + \bar{W}^{k,j}_{t_{i+1}}\right)^2}$$

exact simulation

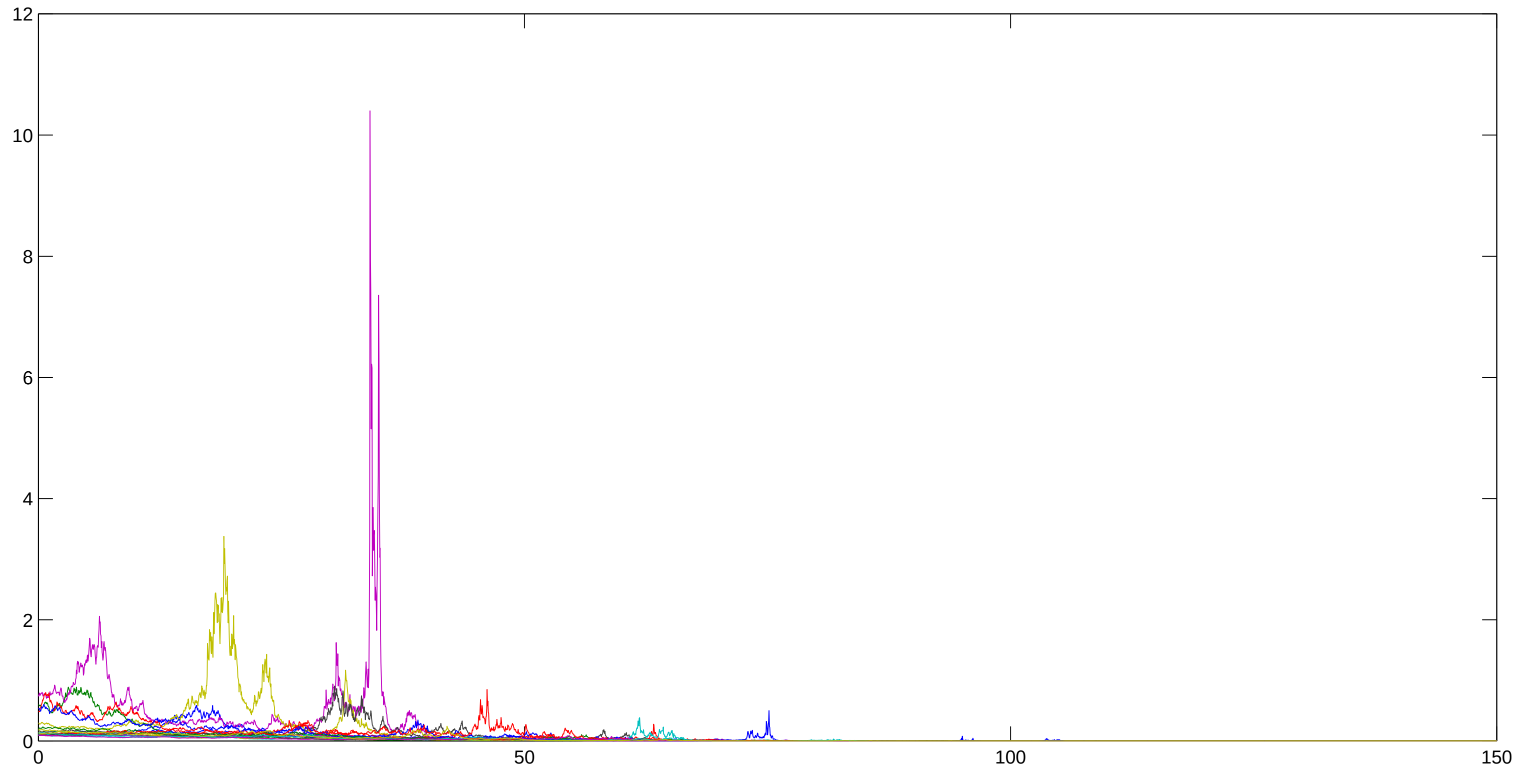

Figure 12: Simulated benchmarked primary security accounts under the MMM

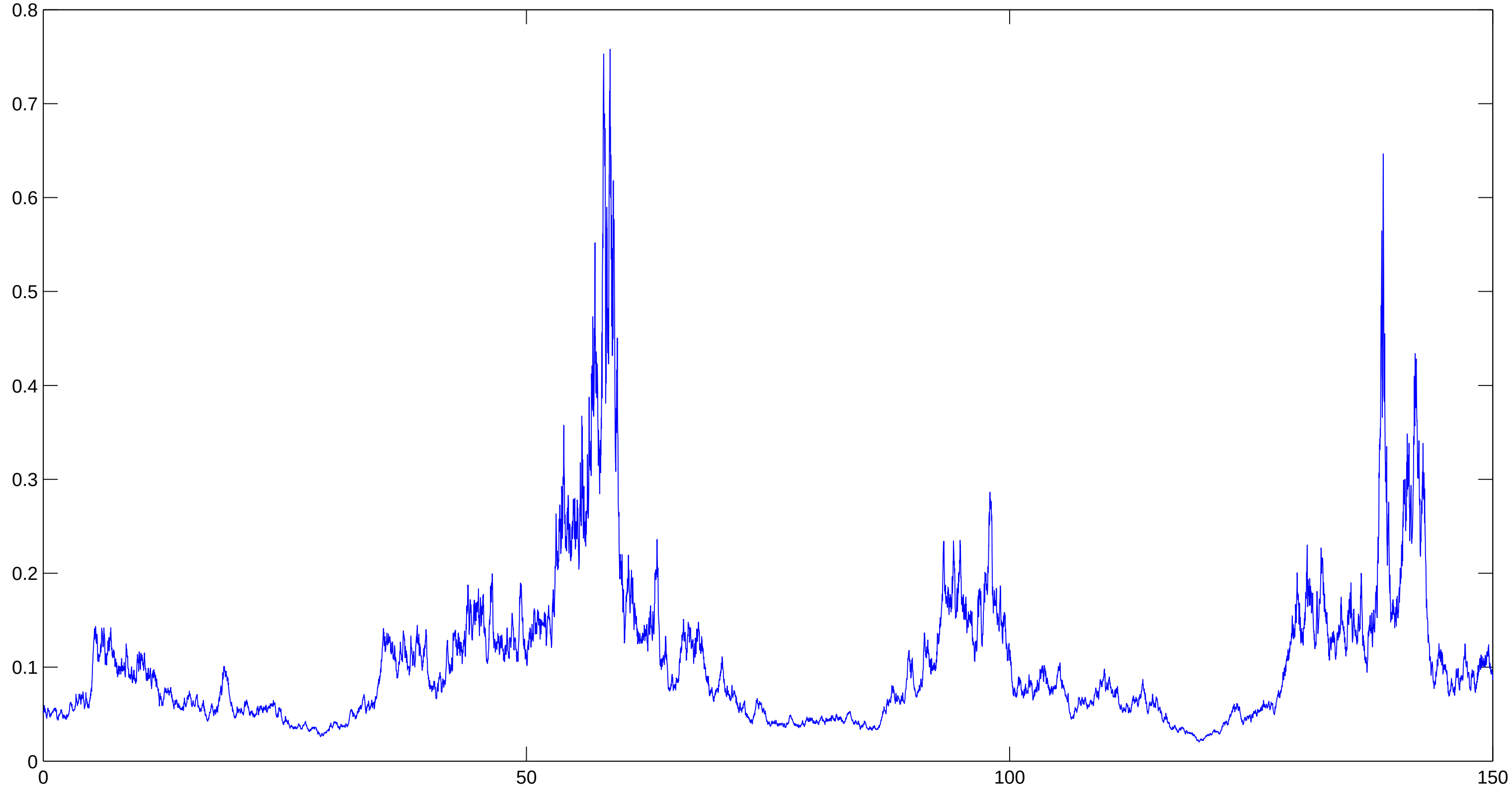

Figure 13: Simulated squared volatility under the MMM

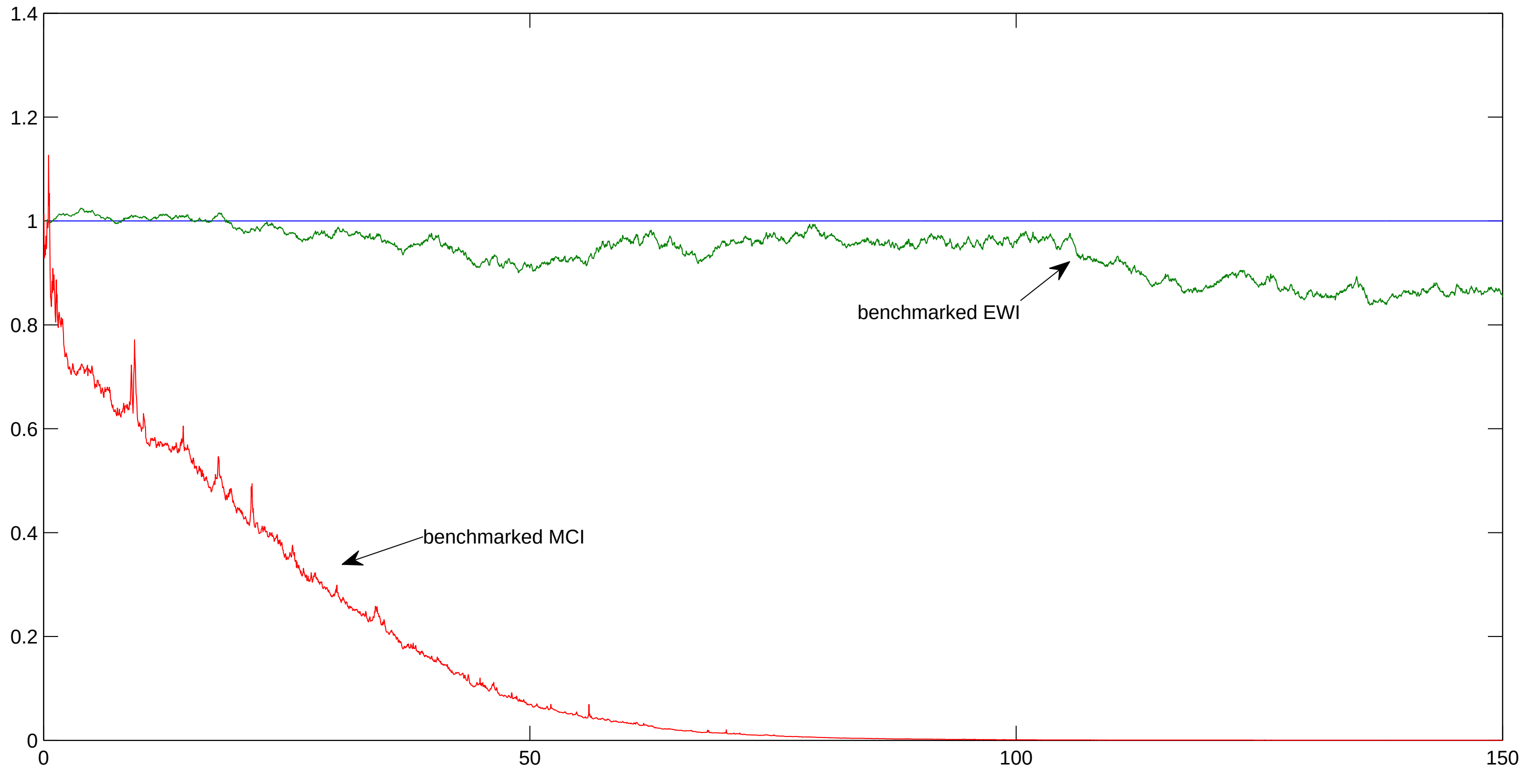

Figure 14: Simulated benchmarked GOP, EWI and MCI under the MMM model

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