Linear Matrix Inequalities vs Convex Sets

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Your narrator is Bill Helton   Math Dept   UCSD

Advertisement: **Try noncommutative computation**
NCAlgebra[1]
NCSoSTools[2]

[1] deOliveira (UCSD), Helton, Stankus (CalPoly SanLObispo), Miller
[2] Igor Klep

Ingredients of Talk: LMIs and Convexity

A Linear Pencil is a matrix valued function L of the form

$$L(x) := L_0 + L_1 x_1 + \cdots + L_g x_g,$$

where $L_0, L_1, L_2, \cdots, L_g$ are symmetric matrices and $x := \{x_1, \cdots, x_g\}$ are g real parameters.

A Linear Matrix Inequality (LMI) is one of the form:

$$L(x) \succ 0 \qquad \text{means} \qquad L(x) \text{ is PosDef.}$$

Normalization: a monic LMI is one with $L_0 = I$.

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The set of solutions

$$\mathcal{G} := \{(x_1, x_2, \cdots, x_g) : L_0 + L_1 x_1 + \cdots + L_g x_g \succ 0\}$$

is a convex set. Solutions can be found numerically for problems of modest size. This is called

Semidefinite Programming SDP

Ingredients of Talk: Noncommutative polynomials

$x = (x_1, \cdots, x_g)$ algebraic noncommuting variables

Noncommutative polynomials: $p(x)$:

$$\text{Eg.} \qquad p(x) = x_1 x_2 + x_2 x_1$$

Evaluate p: on matrices $X = (X_1, \cdots X_g)$ a tuple of matrices.
Substitute a matrix for each variable $x_1 \to X_1$, $x_2 \to X_2$

$$\text{Eg.} \qquad p(X) = X_1 X_2 + X_2 X_1.$$

Noncommutative inequalities: p is positive means:

$$p(X) \text{ is PSD for all } X$$

Outline

Ingredients: Polynomials and LMIs with Matrix Unknowns

Linear Systems give NonCommutative Polynomial Inequalities

Dimension Free Convexity vs NC LMIs

Comparison to LMIs in Scalar Unknowns

Free RAG
- Convex Positivstellensatz
- Randstellensatz for Defining Polynomials

Examples of NC Polynomials

The Ricatti polynomial

$$r((a,b,c),x) = -xb^Tbx + a^Tx + xa + c$$

Here $m = (a,b,c)$ and $x = (x)$.

Evaluation of NC Polynomials

r is naturally evaluated on a $1+3=4$ tuple of matrices

$$M = (A,B,C) \in (\mathbb{R}^{n\times n})^3 \qquad X = (X) \in \mathbb{S}^{n\times n}$$

$$r((A,B,C),X) = -XB^TBX + A^TX + XA + C \in \mathbb{S}_n(\mathbb{R}).$$

Note that the form of the Riccati is independent of n.

POLYNOMIAL MATRIX INEQUALITIES

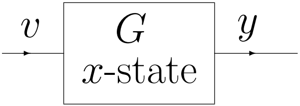
Polynomial or Rational function of matrices are PosSDef.
Example: Get Riccati expressions like

$$AX + XA^T - XBB^TX + CC^T \succ 0$$

OR Linear Matrix Inequalities (LMI) like

$$\begin{pmatrix} AX + XA^T + C^TC & XB \\ B^TX & I \end{pmatrix} \succ 0$$

which is equivalent to the Riccati inequality.



$$\frac{dx(t)}{dt} = Ax(t) + Bv(t)$$

$$y(t) = Cx(t) + Dv(t)$$

A, B, C, D are matrices

x, v, y are vectors

Asymptotically stable

$$\mathrm{Re}(\mathrm{eigvals}(A)) \prec 0 \iff$$

$$A^T\mathbf{E} + \mathbf{E}A \prec 0 \quad \mathbf{E} \succ 0$$

Energy dissipating

$$G : L^2 \to L^2$$

$$\int_0^T |v|^2 dt \geq \int_0^T |Gv|^2 dt$$

$$x(0) = 0$$

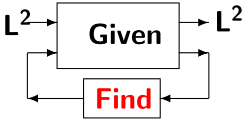
$$\exists \quad \mathbf{E} = \mathbf{E}^T \succeq 0$$

$$H := A^T\mathbf{E} + \mathbf{E}A +$$

$$+\mathbf{E}BB^T\mathbf{E} + C^TC \preceq 0$$

$\mathbf{E}$ is called a storage function

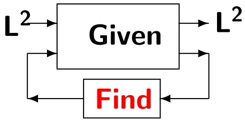
Linear Systems Problems $\rightarrow$ Matrix Inequalities



Many such problems Eg. H^{∞} control

The problem is Dimension free: since it is given only by signal flow diagrams and L^2 signals.

Linear Systems Problems $\rightarrow$ Matrix Inequalities



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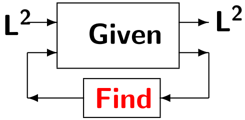
A Dim Free System Prob is Equivalent to Noncommutative Polynomial Inequalities

Linear Systems Problems → Matrix Inequalities



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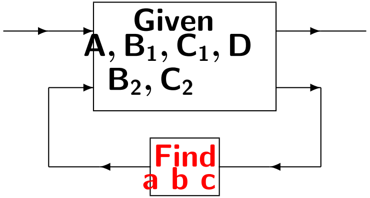
Example:

GET ALGEBRA



$$\mathbf{D} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

DYNAMICS of "closed loop" system: BLOCK matrices

$$\mathcal{A} \quad \mathcal{B} \quad \mathcal{C} \quad \mathcal{D}$$

ENERGY DISSIPATION:

$$\mathbf{H} := \mathcal{A}^T \mathbf{E} + \mathbf{E}\mathcal{A} + \mathbf{E}\mathcal{B}\mathcal{B}^T\mathbf{E} + \mathcal{C}^T\mathcal{C} \preceq 0$$

$$\mathbf{E} = \begin{pmatrix} \mathbf{E}_{11} & \mathbf{E}_{12} \\ \mathbf{E}_{21} & \mathbf{E}_{22} \end{pmatrix} \qquad \mathbf{E}_{12} = \mathbf{E}_{21}{}^T$$

$$\mathbf{H} = \begin{pmatrix} \mathbf{H}_{xx} & \mathbf{H}_{xy} \\ \mathbf{H}_{yx} & \mathbf{E}_{yy} \end{pmatrix} \qquad \mathbf{H}_{xy} = \mathbf{H}_{yx}^T$$

H^∞ Control

ALGEBRA PROBLEM:

Given the polynomials:

$$H_{xx} = E_{11} A + A^T E_{11} + C_1^T C_1 + {E_{12}}^T b C_2 + C_2^T b^T {E_{12}}^T + E_{11} B_1 b^T {E_{12}}^T + E_{11} B_1 B_1^T E_{11} + E_{12} b b^T {E_{12}}^T + E_{12} b B_1^T E_{11}$$

$$H_{xz} = E_{21} A + \frac{a^T (E_{21} + {E_{12}}^T)}{2} + c^T C_1 + E_{22} b C_2 + c^T B_2^T {E_{11}}^T + \frac{E_{21} B_1 b^T (E_{21} + {E_{12}}^T)}{2} + E_{21} B_1 B_1^T {E_{11}}^T + \frac{E_{22} b b^T (E_{21} + {E_{12}}^T)}{2} + E_{22} b B_1^T {E_{11}}^T$$

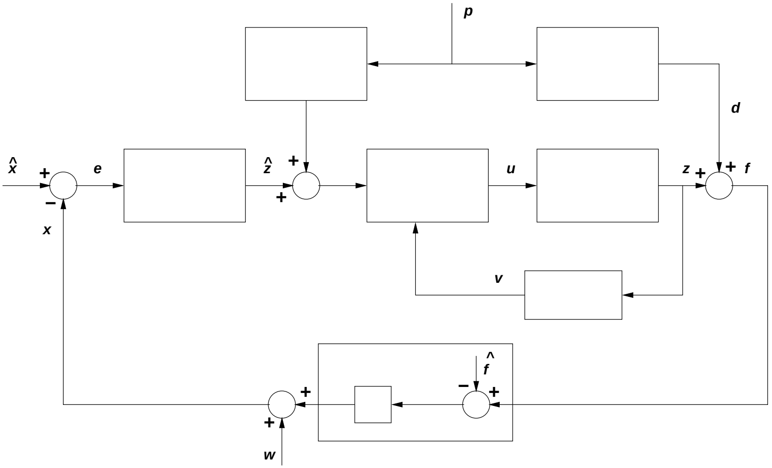
$$H_{zx} = A^T {E_{21}}^T + C_1^T c + \frac{(E_{12} + {E_{21}}^T) a}{2} + E_{11} B_2 c + C_2^T b^T {E_{22}}^T + E_{11} B_1 b^T {E_{22}}^T + E_{11} B_1 B_1^T {E_{21}}^T + \frac{(E_{12} + {E_{21}}^T) b b^T {E_{22}}^T}{2} + \frac{(E_{12} + {E_{21}}^T) b B_1^T {E_{21}}^T}{2}$$

$$H_{zz} = E_{22} a + a^T {E_{22}}^T + c^T c + E_{21} B_2 c + c^T B_2^T {E_{21}}^T + E_{21} B_1 b^T {E_{22}}^T + E_{21} B_1 B_1^T {E_{21}}^T + E_{22} b b^T {E_{22}}^T + E_{22} b B_1^T {E_{21}}^T$$

(PROB) A, B_1, B_2, C_1, C_2 are knowns.

Solve the inequality $\begin{pmatrix} H_{xx} & H_{xz} \\ H_{zx} & H_{zz} \end{pmatrix} \preceq 0$ for unknowns a, b, c and for E_{11}, E_{12}, E_{21} and E_{22}

More complicated systems give fancier nc polynomials



Engineering problems defined entirely by signal flow diagrams and L^2 performance specs are equivalent to Polynomial Matrix Inequalities

A more precise statement is on the next slide

Linear Systems and Algebra Synopsis

A Signal Flow Diagram with L^2 based performance, eg H^∞ gives precisely a nc polynomial

$$p(a, x) := \begin{pmatrix} p_{11}(a, x) & \cdots & p_{1k}(a, x) \\ \vdots & \ddots & \vdots \\ p_{k1}(a, x) & \cdots & p_{kk}(a, x) \end{pmatrix}$$

Such linear systems problems become exactly:

> **Given matrices A.**
> **Find matrices X so that $P(A, X)$ is PosSemiDef.**

BAD Typically p is a mess, until a hundred people work on it and maybe convert it to CONVEX in x Matrix Inequalities.

All known successes[3] do more: They convert to a LMI in x.

[3] about 20, plus a few thousand ad hoc compromises

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Convexity vs LMIs

QUESTIONS (Vague) :

WHICH DIM FREE PROBLEMS "ARE" LMI PROBLEMS.
Clearly, such a problem must be convex and "semialgebraic".
Which convex nc problems are NC LMIS?

WHICH PROBLEMS ARE TREATABLE WITH LMI's?
This requires some kind of change of variables theory.

The first is the main topic of this talk

A cleaner problem

Consider a cleaner problem we consider $p(x, a)$ but with no a:

$$p(x) := \begin{pmatrix} p_{11}(x) & \cdots & p_{1k}(x) \\ \vdots & \ddots & \vdots \\ p_{k1}(x) & \cdots & p_{kk}(x) \end{pmatrix}$$

WHICH DIM FREE PROBLEMS "ARE" LMI PROBLEMS?

Linear Pencil

RECALL

- For symmetric matrices $L_0, L_1, \ldots, L_g \in \mathbb{S}^{s \times s}$ and $x = (x_1, \ldots, x_g)$, the expression

$$L(x) = L_0 + L_1 x_1 + \cdots + L_s x_g$$

is called a $s \times s$ **linear pencil**.

If $L_0 = I$, we say that $L(x)$ is **monic**.

Linear Pencil, Linear Matrix Inequality (LMI)

RECALL

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  is called a $s \times s$ **linear pencil**.

  If $L_0 = I$, we say that $L(x)$ is **monic**.

- A **linear matrix inequality (LMI)** is of the form $L(x) \succeq 0$.

  Its **solution set**

$$\begin{aligned} \mathcal{D}_L(1) &= \{x \in \mathbb{R}^g \mid L(x) \succeq 0\} \\ &= \{x \in \mathbb{R}^g \mid L_0 + L_1 x_1 + \cdots + L_g x_g \succeq 0\} \end{aligned}$$

  is called a **spectrahedron** or also an **LMI domain**.

Pencils in Matrix Variables

Given a $s \times s$ linear pencil

$$L(x) = L_0 + L_1 x_1 + \cdots + L_g x_g,$$

it is natural to substitute symmetric matrices X_j for the variables x_j.

Pencils in Matrix Variables

Given a $s \times s$ linear pencil

$$L(x) = L_0 + L_1 x_1 + \cdots + L_g x_g,$$

it is natural to substitute symmetric matrices X_j for the variables x_j:

- For $X = (X_1, \ldots, X_g) \in (\mathbb{S}^{n \times n})^g$, the evaluation $L(X)$ is

$$L(X) := L_0 \otimes I_n + L_1 \otimes X_1 + \cdots + L_g \otimes X_g \in \mathbb{S}^{sn \times sn}.$$

  The tensor product in this expression is the standard (Kronecker) tensor product of matrices.

- The positivity set of L is

$$\mathcal{D}_L(n) := \{X \in (\mathbb{S}^{n \times n})^g : \ L(X) \succ 0\} \qquad \mathcal{D}_L := \cup_n \mathcal{D}_L(n)$$

Convex Matrix Inequalities vs Linear Matrix Inequalities

Let p be a symmetric nc polynomial denote the principal component of the positivity domain

$$\mathcal{D}_p(n) := \{X \in (\mathbb{S}^{n\times n})^g : \ p(X) \succ 0\}.$$

by $\mathcal{D}_p^\circ(n)$.

Theorem H-McCullough (Annals 2012)

SUPPOSE p is a nc symmetric noncommutative polynomial with $p(0) = 1$ and $\mathcal{D}_p^0$ bounded.
THEN

$\mathcal{D}_p^0$ is a convex set for each n
if and only if
there is a monic linear pencil L such that $\mathcal{D}_p^\circ = \mathcal{D}_L$.

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This is also true if p is a symmetric matrix of nc polynomials.

The (SAD) MORAL OF THE STORY

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Advice to engineers from this (and other theorems).
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Advice to engineers from this (and other theorems).
It looks like:

A CONVEX problem specified entirely by a signal flow diagram and L^2 performance of signals is equivalent to some LMI.

Looking for LMIs is what they already do. SAD there is no other way to get convexity.

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Comparison to the case when all unknowns are scalars.

Which Sets in $\mathbb{R}^g$ have LMI REPRESENTATIONS?

QUESTION (Vague):
ARE CONVEX PROBLEMS ALL TREATABLE WITH LMI's?

DEFINITION: A set $\mathcal{C} \subset R^g$

has an **Linear Matrix Inequality (LMI) Representation** provided that there are sym matrices $L_1, L_2, \cdots, L_g$ for which the **monic Linear Pencil**, $L(x) := I + L_1x_1 + \cdots + L_gx_g$, has positivity set,

$$\mathcal{D}_L := \{x : \ L_0 + L_1x_1 + \cdots + L_gx_g \quad \text{is PosSD}\} \quad \subset \mathbb{R}^g$$

equals the set $\mathcal{C}$; that is,

$$\mathcal{C} = \mathcal{D}_L.$$

EXAMPLE

$$\mathcal{C} := \{(x_1, x_2) : 1 + 2x_1 + 3x_2 - (3x_1 + 5x_2)(3x_1 + 2x_2) > 0\}$$

has the LMI Rep

$$\mathcal{C} = \{x : L(x) \succ 0\} \qquad \text{here } x := (x_1, x_2)$$

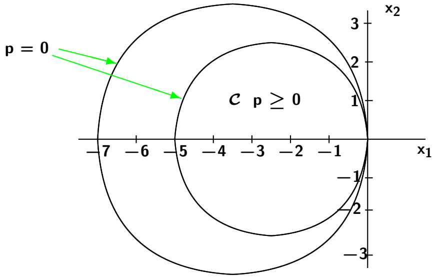
with

$$L(x) = \begin{pmatrix} 1 + 2x_1 + 3x_2 & 3x_1 + 5x_2 \\ 3x_1 + 2x_2 & 1 \end{pmatrix}$$

Pf: The determinant of $L(x)$ is pos iff $L(x)$ is PosDef.

QUESTION 1

Does this set $\mathcal{C}$ which is the inner component of



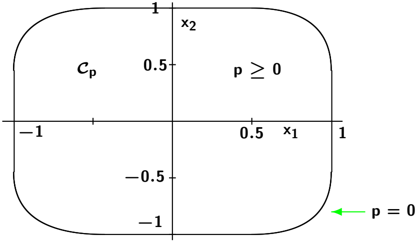
have an LMI representation?

$$p(x_1, x_2) = (x_1{}^2 + x_2{}^2)(x_1{}^2 + x_2{}^2 + 12x_1 - 1) + 36x_1{}^2 > 0$$

$$\mathcal{C} := \text{inner component of}\{x \in R^2 : p(x) > 0\}$$

QUESTION 2

Does this set have an LMI representation?



$$p(x_1, x_2) = 1 - {x_1}^4 - {x_2}^4 > 0$$

$\mathcal{C}_p := \{x \in \mathrm{R}^2 : p(x) > 0\}$ **has degree 4.**

Rigid Convexity-Line Test

DEFINITION: A convex set $\mathcal{C}$ in $\mathbf{R}^{\mathbf{g}}$ with minimal degree defining polynomial p passes the the "line test" means:

For every point $\mathbf{x}^{\mathbf{0}}$ in $\mathcal{C}$ and almost every line ℓ through $\mathbf{x}^{\mathbf{0}}$ the line ℓ intersects the the zero set

$$\{\mathbf{x} \in \mathbf{R}^{\mathbf{g}} : \mathbf{p}(\mathbf{x}) = 0\} \quad \text{of} \quad \mathbf{p}$$

in exactly d points [4] **where d = degree of p.**

[4] In this counting one ignores lines which go thru $\mathbf{x}^{\mathbf{0}}$ and hit the boundary of $\mathcal{C}$ at ∞.

IN $\mathbf{R}^2$ THE LINE TEST RULES

THM [Vinnikov + H, CPAM 2007].
IF $\mathcal{C}$ is a bounded open convex set in R^g with an LMI representation, THEN $\mathcal{C}$ must pass the line test.

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When $g = 2$, the converse is true, namely, a convex set which passes the line test has a LMI representation with symmetric matrices $L_j \in \mathbf{R}^{d \times d}$ and $L_0 = I$.

Lewis-Parrilo-Ramana showed our determinantal representation solves a conjecture (1958) by Peter Lax about constant coefficient linear hyperbolic PDE, one time and 2 space dim.

Snippets of Free RAG.

Convex (perfect) Positivstellensatz

Suppose:

- **$L(x)$ is a monic linear pencil;**
- **$q(x)$ is a noncommutative polynomial.**

Is $q(X)$ PosSemiDef if $L(X)$ is PosSemiDef?

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THEOREM (H-Klep- McCullough; Advances 2012)
$q \succeq 0$ where $L \succeq 0$ if and only if

$$q(x) = s(x)^* s(x) + \sum_j v_j(x)^* L(x) v_j(x),$$

where s, v_j are vectors of polynomials each of degree $\leq \left\lfloor \frac{\deg(q)}{2} \right\rfloor$.

If $\mathcal{D}_L$ is bounded, then we may take $s = 0$.

Arveson -Stinespring vs PosSS

Convex PosSS: Suppose L monic linear pencil and $\mathcal{D}_L$ is bounded.
$q \succeq 0$ on $\mathcal{D}_L$ **iff**

$$q(x) = \sum_j f_j(x)^* L(x) f_j(x),$$

where f_j are vectors of polynomials each of degree $\leq \left\lfloor \frac{\deg(q)}{2} \right\rfloor$.

Take $q(x) = \tilde{L}(x)$ an affine linear nc function, $q(0) = I$.
Then $\left\lfloor \frac{\deg(\tilde{L})}{2} \right\rfloor = 0$, so f_j are constants. The PosSS becomes:

Free LMI domination Theorem:
$\tilde{L} \succeq 0$ on $\mathcal{D}_L$ if and only if $\tilde{L}(x) = \sum_j^{\mu} V_j^* L(x) V_j$

if and only if $\tilde{L}(x) = V^* \big(I_\mu \otimes L(x)\big) V$ V isometery

(**EQUIVALENT** to finite dim Arveson Extension plus Steinspring.)

Recall Real Nullstellensatz addresses

$$\mathbf{q(X)v = 0 \text{ if } p(X)v = 0}$$

Now there is good theory of it: Cimpric, McCullough , Nelson -H (Proc London Math Soc – to appear)

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Now there is good theory of it: Cimpric, McCullough , Nelson -H (Proc London Math Soc – to appear)

For classical polynomials on $\mathbb{R}^g$ there is an algebraic certificate equivalent to any list of polynomial inequalities-equalities.
For NC polynomials open.

Currently doing mixtures of PosSS and NullSS.

Randstellensatz for Defining Polynomials: Zariski Nice

Given p a d $\times$ d matrix of nc polynomials defining a domain by

$$\mathcal{D}_{\mathbf{p}}^{\circ} := \text{principal component of } \{\mathbf{X} : \mathbf{p}(\mathbf{X}) \succ \mathbf{0}\}$$

with (detailed) boundary

$$\widehat{\partial}\mathcal{D}_{\mathbf{p}}^{\circ} := \{(\mathbf{X}, \mathbf{v}) : \quad \mathbf{X} \in \text{closure } \mathcal{D}_{\mathbf{p}}^{\circ}, \ \mathbf{p}(\mathbf{X})\mathbf{v} = \mathbf{0}\}$$

Theorem in preparation
SUPPOSE:
p(x) is a d $\times$ d symmetric nc polynomial, and
L(x) is a d $\times$ d monic linear pencil for which
"the free Zariski closure" of $\widehat{\partial}\mathcal{D}_{\mathbf{L}}^{\circ}$ equals "the Zero Set of L",
THEN

Randstellensatz for Defining Polynomials: Zariski Nice ... continued

Theorem (continued)

$$\mathcal{D}_L \subseteq \mathcal{D}_p \qquad \text{and} \qquad \widehat{\partial}\mathcal{D}_L^\circ \subseteq \widehat{\partial}\mathcal{D}_p^\circ$$

if only if

$$p = L\left(\sum_i q_i^* q_i\right)L + \sum_j (r_j L + C_j)^* L (r_j L + C_j),$$

where q_i, r_j are matrices of polynomials, and C_j are real matrices satisfying $C_j L = L C_j$.

Current ventures

Free convex hulls of free semialgebraic sets

Free change of variables to achieve free convexity. – Obsession
(Motivates recent PosSS work)

MANY THANKS

from

Igor Klep	**Math Dept**	**NewZealand**
Scott McCullough	**Math Dept**	**University of Florida**
Chris Nelson	**Math Dept**	**UCSD $\rightarrow$ NSA**
Victor Vinnikov	**Ben Gurion U of the Negev**	

Bill

Polys in a and x

Partial Convexity of NC Polynomials

The polynomial $p(a, x)$ is convex in x for all A if for each X, Y and $0 \le \alpha \le 1$,

$$p(A, \alpha X + (1-\alpha)Y) \preceq \alpha p(A, X) + (1-\alpha)p(A, Y).$$

The Riccati $r(a, x) = c + a^T x + x a - x b^T b x$ is concave, meaning $-r$ is convex in x (everywhere).

Can localize A to an nc semialgebraic set.

Structure of Partially Convex Polys

THM (Hay-Helton-Lim- McCullough)

SUPPOSE $p \in \mathbb{R}\langle a, x\rangle$ is convex in x THEN

$$p(a,x) = L(a,x) + \tilde{L}(a,x)^T Z(a) \tilde{L}(a,x),$$

where,

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where,
- $L(a, x)$ has degree at most one in x;
- $Z(a)$ is a symmetric matrix-valued NC polynomial;
- $Z(A) \succeq 0$ for all A;
- $\tilde{L}(a, x)$ is linear in x. $\tilde{L}(a, x)$ is a (column) vector of

. NC polynomials of the form $x_j m(a)$.

Structure of Partially Convex Polys

THM (Hay-Helton-Lim- McCullough)

SUPPOSE $p \in \mathbb{R}\langle a, x\rangle$ is convex in x THEN

$$p(a,x) = L(a,x) + \tilde{L}(a,x)^T Z(a) \tilde{L}(a,x),$$

where,

- $L(a,x)$ has degree at most one in x;
- $Z(a)$ is a symmetric matrix-valued NC polynomial;
- $Z(A) \succeq 0$ for all A;
- $\tilde{L}(a,x)$ is linear in x. $\tilde{L}(a,x)$ is a (column) vector of
. NC polynomials of the form $x_j m(a)$.

This also works fine if p is a matrix of nc polynomials.

This also works fine if A only belongs to an open nc semi-algebraic set (will not be defined here) .

Structure of Partially Convex Polys

COR SUPPOSE $p \in \mathbb{R}\langle a, x\rangle$ is convex in x

THEN there is a linear pencil $\Lambda(a, x)$ such that the set of all solutions to $\{X : p(A, X) \succeq 0\}$ equals $\{X : \Lambda(A, X) \succeq 0\}$.

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Proof: p is a Schur Complement of some Λ by the previous theorem.

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The (SAD) MORAL OF THE STORY

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The (SAD) MORAL OF THE STORY

A CONVEX problem specified entirely by a signal flow diagram and L^2 performance of signals is equivalent to some LMI.

Context: Related Areas

Convex Algebraic Geometry (mostly commutative)
NSF FRG: Helton -Nie- Parrilo- Strumfels- Thomas

One aspect: Convexity vs LMIs.

Now there is a roadmap with some theorems and conjectures.

Three branches:

1. Which convex semialgebraic sets in $\mathbb{R}^g$ have an LMI rep? (Line test) Is it necessary and sufficient? Ans: Yes if $g \leq 2$.

2.Which convex semialgebraic sets in $\mathbb{R}^g$ lift to a set with an LMI representation? Ans: Most do.

3. Which noncommutative semialgebraic convex sets have an LMI rep? Ans: All do. (like what you have seen.), see Helton-McCullough Annals of Mathematics Sept 2012.

NC Real Algebraic Geometry (since 2000)

We have a good body of results in these areas.
Eg. Positivestellensatz – Saw in Tuesday Morning Tutorial session.