Normal forms and desingularization

1. Analytic differential equations in the complex domain

For an open domain $U \subseteq \mathbb{C}^n$ we denote by $\mathcal{O}(U)$ the complex linear space of functions holomorphic in U (see Appendix). The space of vector-valued holomorphic functions is denoted by

$$\mathcal{O}^m(U) = \underbrace{\mathcal{O}(U) \times \cdots \times \mathcal{O}(U)}_{m \text{ times}} = \mathcal{O}(U) \otimes_{\mathbb{C}} \mathbb{C}^m.$$

1A. Differential equations, solutions, initial value problems. Let $U \subseteq \mathbb{C} \times \mathbb{C}^n$ be an open domain and $F = (F_1, \dots, F_n) \colon U \to \mathbb{C}^n$ a holomorphic vector function. An *analytic ordinary differential equation* defined by F on U is the vector equation (or the *system* of n scalar equations)

$$\frac{dx}{dt} = F(t,x), \qquad (t,x) \in U \subseteq \mathbb{C} \times \mathbb{C}^n, \quad F \in \mathcal{O}^n(U). \tag{1.1}$$

The *solution* of this equation is a parameterized holomorphic curve, the holomorphic map $\varphi = (\varphi_1, \dots, \varphi_n) \colon V \to \mathbb{C}^n$, defined in an open subset $V \subseteq \mathbb{C}$, whose graph $\{(t, \varphi(t)) \colon t \in V\}$ belongs to U and whose complex "velocity vector" $\frac{d\varphi}{dt} = \left(\frac{d\varphi_1}{dt}, \dots, \frac{d\varphi_n}{dt}\right) \in \mathbb{C}^n$ at each point t coincides with the vector $F(t, \varphi(t)) \in \mathbb{C}^n$.

The graph of φ in U is called the *integral curve*. From the real point of view it is a 2-dimensional smooth surface in $\mathbb{R}^{2n+2}$. Note that from the beginning we consider only holomorphic solutions which may be, however, defined on domains of different size.

The equation is *autonomous*, if F is independent of t. In this case the image $\varphi(V) \subseteq \mathbb{C}^n$ is called the *phase curve*. Any differential equation (1.1) can be "made" autonomous by adding a fictitious variable $z \in \mathbb{C}$ governed by the equation $\frac{dz}{dt} = 1$.

If $(t_0, x_0) = (t_0, x_{0,1}, \dots, x_{0,n}) \in U$ is a specified point, then the *initial value problem*, sometimes also called the *Cauchy problem*, is to find an integral curve of the differential equation (1.1) passing through the point (t_0, x_0), i.e., a solution satisfying the condition

$$\varphi\colon V \to \mathbb{C}^n, \qquad \varphi(t_0) = x_0 \in \mathbb{C}^n. \tag{1.2}$$

In what follows we will often denote by a dot the derivative with respect to the complex variable t, $\dot{x}(t) = \frac{dx}{dt}(t)$.

The first fundamental result is the local existence and uniqueness theorem.

Theorem 1.1. *For any holomorphic differential equation* (1.1) *and every point* $(t_0, x_0) \in U$ *there exists a sufficiently small polydisk* $D_\varepsilon = \{|t - t_0| < \varepsilon,\ |x_j - x_{0,j}| < \varepsilon,\ j = 1, \dots, n\} \subseteq U$*, such that the solution of the initial value problem* (1.2) *exists and is unique in this polydisk.*

This solution depends holomorphically on the initial value $x_0 \in \mathbb{C}^n$ *and on any additional parameters, provided that the vector function* F *depends holomorphically on these parameters.*

From the real point of view, Theorem 1.1 asserts existence of $2n$ functions of *two* independent real variables whose graph is a *surface* in $\mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$, with the tangent plane spanned by two real vectors $\operatorname{Re} F, \operatorname{Im} F$. To derive this theorem from the standard results on existence, uniqueness and differentiability of solutions of *smooth* ordinary differential equations in the *real domain*, one should use rather deep results on *integrability of distributions*; see Remark 2.10 below. Rather unexpectedly, the direct proof is *simpler* than in the real case in the part concerning dependence on initial conditions. This proof is given in the next subsection. The main idea of this proof, as well as many other proofs below, is the *contracting map principle.*

1B. Contracting map principle. Consider the linear space $\mathcal{A}(D_\rho)$ of functions holomorphic in the polydisk D_ρ and continuous on its closure,

$$\mathcal{A}(D_\rho) = \{f\colon D_\rho \to \mathbb{C} \text{ holomorphic in } D_\rho \text{ and continuous on } \overline{D_\rho}\}. \tag{1.3}$$

This space is naturally equipped with the supremum-norm,

$$\|f\|_\rho = \max_{z \in D_\rho} |f(z)|, \qquad z = (z_1, \dots, z_n) \in \mathbb{C}^n, \tag{1.4}$$

and thus naturally a subspace of the *complete normed* (i.e., Banach) space $C(\overline{D_\rho})$ of continuous complex-valued functions. Though holomorphic functions may have very complicated boundary behavior and thus $\mathcal{A}(U) \subsetneq \mathcal{O}(U)$, they are continuous and therefore for any smaller domain U' relatively compact in U (i.e., when $\overline{U'} \Subset U$), there is an obvious inclusion $\mathcal{A}(U') \supset \mathcal{O}(U)$.

Theorem 1.2. *The space $\mathcal{A}(D_\rho)$ and its vector counterparts $\mathcal{A}^m(D_\rho) = \bigoplus_{m \text{ times}} \mathcal{A}(D_\rho)$ are complete (Banach) spaces.*

Proof. Any fundamental sequence in $\mathcal{A}(D_\rho)$ is by definition fundamental in the Banach space $C(\overline{D_\rho})$ and has a uniform limit in the latter space. By the Weierstrass compactness principle [**Sha92**], this limit is again holomorphic in D_ρ, i.e., belongs to $\mathcal{A}(D_\rho)$. □

A map F of a metric space $\mathcal{M}$ into itself is called *contracting*, if for some positive real number $\lambda < 1$ and all $u, v \in \mathcal{M}$ the inequality $\operatorname{dist}(F(u), F(v)) \leqslant \lambda \operatorname{dist}(u, v)$ holds. A point $w \in \mathcal{M}$ is *fixed* (by F), if $F(w) = w$.

Theorem 1.3 (Contracting map principle). *Any contracting map $F \colon M \to M$ of a complete metric space M has a unique fixed point in $\mathcal{M}$.*

This fixed point is the limit of any sequence of iterations $u_{k+1} = F(u_k)$, $k = 0, 1, 2, \ldots$ beginning with an arbitrary initial point $u_0 \in M$.

Proof. For any initial point $u_0 \in M$, the sequence u_k, $k = 1, 2, \ldots$ is fundamental, since $\operatorname{dist}(u_k, u_{k+1}) \leqslant \lambda^k \operatorname{dist}(u_0, u_1)$ and by the triangle inequality $\operatorname{dist}(u_k, u_l) \leqslant \operatorname{dist}(u_0, u_1)\lambda^k/(1-\lambda)$ for any $k < l$. By completeness assumption, the sequence u_k converges to a limit $w \in M$. Since F is continuous, passing to the limit in the identity $u_{k+1} = F(u_k)$ yields $w = F(w)$. If w_1, w_2 are two fixed points, then $\operatorname{dist}(w_1, w_2) \leqslant \lambda \operatorname{dist}(F(w_1), F(w_2)) = \lambda \operatorname{dist}(w_1, w_2)$ which is possible only if $\operatorname{dist}(w_1, w_2) = 0$, i.e., when $w_1 = w_2$. □

1C. Picard operators and their contractivity. The exposition below is based on [**Arn78**, §31] with minor modifications.

Consider the equation (1.1) defined in a domain U. Denote by $D_\varepsilon = \{|z - x_0| < \varepsilon,\ |t - t_0| < \varepsilon\} \subset \mathbb{C}^{n+1}$ a polydisk centered at the point $(t_0, x_0) \in U$ and small enough to belong to U.

Definition 1.4. The *Picard operator* $\mathbf{P}$ associated with the differential equation (1.1) and the initial value $(t_0, z_0) \in U$, is the operator $f \mapsto \mathbf{P}f$ defined by the integral formula

$$(\mathbf{P}f)(s, z) = z + \int_{t_0}^{s} F(t, f(t, z))\, dt \tag{1.5}$$

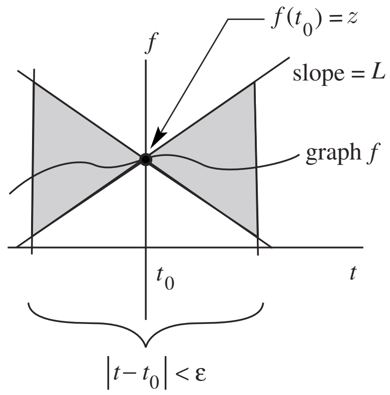

Figure I.1. Domain of definition of Picard iterations (in the intersection with the hyperplane $z = \text{const}$)

for all vector functions $f(t, z)$ the expression in the right hand side makes sense.

We will now construct a complete metric space invariant by $\mathbf{P}$, on which this operator is contracting. Denote by L_0 and L_1 the bounds for the magnitude of F and its Lipschitz constant in U: for any $(t, x), (t, x') \in U$,

$$|F(t, z)| \leqslant L_0, \qquad |F(t, z) - F(t, z')| \leqslant L_1 \, |z - z'|. \tag{1.6}$$

Denote by $\mathcal{M}$ the subspace of the space $\mathcal{A}^n(D_\varepsilon)$ which consists of the functions satisfying the additional inequality

$$|f(t, z) - z| \leqslant L_0 \, |t - t_0|. \tag{1.7}$$

This space is complete in the metric induced by the norm $\| \cdot \|_\varepsilon$ inherited from the ambient space $\mathcal{A}^n(D_\varepsilon)$ (Exercise 1.3).

Lemma 1.5. *If the polydisk D_ε is sufficiently small, the Picard operator $\mathbf{P}$ given by the integral* (1.5)*, is well defined and contracting on $\mathcal{M}$.*

More precisely, for sufficiently small ε its contraction factor λ does not exceed εL_1, where L_1 is the Lipschitz constant for F in U.

Proof. Explicit majorizing of the integral shows that

$$|\mathbf{P}f(s, z) - z| \leqslant L_0 \int_{t_0}^{s} |dt| \leqslant L_0 \, |s - t_0| \leqslant L_0 \varepsilon,$$

so if ε is chosen sufficiently small, the operator $\mathbf{P}$ is well defined on $\mathcal{M}$ and maps this space into itself. For any two vector functions f, f' defined on such a small polydisk D_ε, we have by virtue of the same estimate

$$\|\mathbf{P}f - \mathbf{P}f'\| = \sup_{|s - t_0| < \varepsilon} \int_{t_0}^{s} L_1 \, |f(t, z) - f'(t, z)| \, |dt| \leqslant \varepsilon L_1 \, \|f - f'\|.$$

If $\varepsilon L_1 < 1$, the operator $\mathbf{P}$ is contracting. □

Proof of Theorem 1.1. Assume ε is so small that the $\varepsilon L_1 < 1$ so that by Lemma 1.5, the Picard operator $\mathbf{P}$ is contracting. By Theorem 1.2 the fixed point of this operator (which exists by Theorem 1.3 and Lemma 1.5) is a *holomorphic* vector function $f\colon D_\varepsilon \to \mathbb{C}^n$ that satisfies the integral equation

$$f(s,z) = z + \int_{t_0}^{s} F(t, f(t,z))\,dt, \qquad |s - t_0| < \varepsilon, \quad |z - x_0| < \varepsilon. \tag{1.8}$$

For each fixed z, the function $\varphi_z(t) = f(t,z)$ clearly satisfies both the initial condition (1.2) with $x_0 = z$ and the differential equation (1.1). By construction, it depends holomorphically on the initial condition z.

To prove holomorphic dependence on additional parameters, one can treat them as fictitious dependent variables. Assume that the vector function $F = F(t,x,y)$ depends holomorphically on additional parameters $y \in \mathbb{C}^m$, and consider the initial value problem (recall that the dot means the derivative $\frac{d}{dt}$)

$$\begin{cases} \dot{x} = F(t,x,y), & x(t_0) = x_0, \\ \dot{y} = 0, & y(t_0) = y_0. \end{cases} \tag{1.9}$$

The solution of this initial value problem is a function $f(t,x,y,x_0,y_0)$ holomorphically depending on all variables. □

Remark 1.6. For a differential equation with the right hand side $F(t,x)$ the *shifted solution* $x'(t) = x(t-y)$, $y \in \mathbb{C}^1$, satisfies the shifted equation $\dot{x}' = F(t-y, x')$ which analytically depends on the parameter y. By Theorem 1.1, this shows that solutions of the initial value problem depend holomorphically also on the t-component of the initial point $(t_0, x_0) \in U$.

1D. Principal example: exponential formula for linear systems. The proof of the existence theorem is *constructive*: the solution of a differential equation is obtained as the uniform limit of its *Picard approximations*, iterations of the Picard operator.

In the simplest case of a differential equation with *constant* (i.e., independent of t,x,y) right hand side $F = \text{const} \in \mathbb{C}^n$ the Picard approximations stabilize immediately: if $f_0(t,v) = v$, then $f_1(t,v) = f_2(t,v) = \cdots = v + (t - t_0)F$.

A *linear system with constant coefficients* is the system of equations

$$\dot{x} = Ax, \qquad x \in \mathbb{C}^n, \quad A \in \operatorname{Mat}(n, \mathbb{C}) \tag{1.10}$$

where $A = \|a_{ij}\|$ is a constant (independent of t and x) $(n \times n)$-matrix with complex entries. Reasoning by induction, one can see that the Picard approximations for the solution of (1.10) which start with the constant initial term $f_0(t,v)=v$, have the form

$$f_k(t,v)=\Big(E+tA+\tfrac{t^2}{2!}A^2+\cdots+\tfrac{t^k}{k!}A^k\Big)\,v. \tag{1.11}$$

Indeed,

$$\begin{aligned}\mathbf{P}f_k(t,v)&=v+\int_0^t A\cdot\big(E+sA+\cdots+\tfrac{s^k}{k!}A^k\big)v\,ds\\&=Ev+\big(tA+\cdots+\tfrac{t^{k+1}}{(k+1)!}A^{k+1}\big)\,v=f_{k+1}(t,v).\end{aligned}$$

These formulas motivate the following fundamental object.

Definition 1.7 (matrix exponential)**.** For an arbitrary constant matrix $A\in\operatorname{Mat}(n,\mathbb{C})$ its *exponential* $\exp A$ is the sum of the infinite (matrix) series

$$\exp A=E+A+\frac{1}{2!}A^2+\cdots+\frac{1}{k!}A^k+\cdots. \tag{1.12}$$

Since $|A^k|\leqslant|A|^k$ and since the factorial series $\sum_{k\geqslant0}r^k/k!$ converges absolutely for all values $r\in\mathbb{R}$, the matrix series (1.12) converges absolutely on the complex linear space $\operatorname{Mat}(n,\mathbb{C})\cong\mathbb{C}^{n^2}$ for any finite n.

Note that for any two *commuting* matrices A,B their exponents satisfy the *group identity*

$$\exp(A+B)=\exp A\cdot\exp B=\exp B\cdot\exp A. \tag{1.13}$$

This can be proved by substituting A,B instead of two scalars a,b into the formal identity obtained by expansion of the law $e^ae^b=e^{a+b}$.

The explicit formula (1.11) for Picard approximations for the linear system (1.10) immediately proves the following theorem.

Theorem 1.8. *The solution of the linear system $\dot x=Ax$, $A\in\operatorname{Mat}(n,\mathbb{C})$, with the initial value $x(0)=v$ is given by the matrix exponential,*

$$x(t)=(\exp tA)\,v,\qquad t\in\mathbb{C},\quad v\in\mathbb{C}^n.\quad\square \tag{1.14}$$

Remark 1.9. Computation of the matrix exponential can be reduced to computation of a matrix polynomial of degree $\leqslant n-1$ and exponentials of eigenvalues of A. Indeed, assume that A has a Jordan normal form $A=\Lambda+N$, where $\Lambda=\operatorname{diag}\{\lambda_1,\dots,\lambda_n\}$ is the diagonal part and N the upper-triangular (nilpotent) part *commuting* with Λ. Then $\exp\Lambda$ is a diagonal matrix with the exponentials of the eigenvalues of Λ on the diagonal, $N^n=0$

by nilpotency, and therefore

$$\begin{aligned}\exp[t(\Lambda + N)] &= \exp t\Lambda \cdot \exp tN \\ &= \begin{pmatrix} \exp t\lambda_1 & & \\ & \ddots & \\ & & \exp t\lambda_n \end{pmatrix} \cdot \left(E + tN + \frac{t^2}{2!}N^2 + \cdots + \frac{t^{n-1}}{(n-1)!}N^{n-1} \right).\end{aligned} \tag{1.15}$$

This provides a practical way of solving linear systems with constant coefficients: components of any solution in any basis are linear combinations of *quasipolynomials* $t^k \exp t\lambda_j$, $0 \leqslant k \leqslant n-1$ with complex coefficients.

Remark 1.10 (Liouville–Ostrogradskii formula)**.** By direct inspection of the formula (1.15) we conclude that

$$\forall A \in \operatorname{Mat}(n, \mathbb{C}) \qquad \det \exp A = \exp \operatorname{tr} A. \tag{1.16}$$

Indeed, $\det \exp A = \det \exp \Lambda \cdot \det \exp N = \prod_{i=1}^n \exp \lambda_i \cdot 1 = \exp \operatorname{tr} \Lambda = \exp \operatorname{tr} A$, since the matrix polynomial $\exp N$ is upper triangular with units on the diagonal.

1E. Flow box theorem. Let $f(t, x_0)$ be the holomorphic vector function solving the initial value problem (1.2) for the differential equation (1.1).

Definition 1.11. The *flow map* for a differential equation (1.1) is the vector function of $n+2$ complex variables (t_0, t_1, v) defined when $(t_0, x) \in U$ and $|t_0 - t_1|$ is sufficiently small, by the formula

$$(t_0, t_1, v) \mapsto \Phi_{t_0}^{t_1}(v) = f(t_1, v), \tag{1.17}$$

where $f(t, v)$ is the fixed point of the Picard operator $\mathbf{P}$ as in (1.8) associated with the initial point t_0.

In other words, $\Phi_{t_0}^{t_1}(v)$ is the value $\varphi(t)$ which takes the solution of the initial value problem with the initial condition $\varphi(t_0) = v$, at the point t_1 sufficiently close to t_0.

Example 1.12. For a linear system (1.10) with constant coefficients, the flow map is linear:

$$\Phi_{t_0}^{t_1}(v) = [\exp(t_1 - t_0)A]\, v.$$

This map is defined for *all* values of t_0, t_1, v.

By Theorem 1.1, Φ is a holomorphic map. Since the solution of the initial value problem is unique, it obviously must satisfy the functional equation

$$\Phi_{t_1}^{t_2}(\Phi_{t_0}^{t_1}(x)) = \Phi_{t_0}^{t_2}(x) \tag{1.18}$$

for all t_1, t_2 sufficiently close to t_0 and all x sufficiently close to x_0. Since for any x the vector function $t \mapsto \varphi_x(t) = \Phi_{t_0}^t(x)$ is a solution of (1.1), we have

$$\frac{\partial}{\partial t}\bigg|_{t=t_0,\, x=x_0} \Phi_{t_0}^t(x) = -\frac{\partial}{\partial t_0}\bigg|_{t=t_0,\, x=x_0} \Phi_{t_0}^t(x) = F(t_0, x_0).$$

From the integral equation (1.8) it follows that

$$\Phi_{t_0}^t(x_0) = x_0 + (t - t_0)F(t_0, x_0) + o(|t - t_0|), \tag{1.19}$$

and therefore the Jacobian matrix of Φ with respect to the x-variable is

$$\left(\frac{\partial \Phi_{t_0}^t(x)}{\partial x}\right)_{t=t_0,\, x=x_0} = E. \tag{1.20}$$

Differential equations can be transformed to each other by various transformations. The most important is the (bi)holomorphic equivalence, or holomorphic conjugacy.

Definition 1.13. Two differential equations, (1.1) and another such equation

$$\dot{x}' = F'(t', x'), \qquad (t', x') \in U', \tag{1.21}$$

are *conjugated* by the biholomorphism $H\colon U \to U'$ (the *conjugacy*), if H sends any integral trajectory of (1.1) into an integral trajectory of (1.21).

Two systems are *holomorphically equivalent* in their respective domains, if there exists a biholomorphic conjugacy between them.

Clearly, biholomorphically conjugate systems are indistinguishable in everything that concerns properties invariant by biholomorphisms. Finding a simple system biholomorphically equivalent to a given one, is therefore of paramount importance.

Theorem 1.14 (Flow box theorem)**.** *Any holomorphic differential equation* (1.1) *in a sufficiently small neighborhood of any point is biholomorphically conjugated by a suitable biholomorphic conjugacy* $H\colon (t, x) \mapsto (t, h(t, x))$ *preserving the independent variable* t*, to the trivial equation*

$$\dot{x}' = 0. \tag{1.22}$$

Proof of the theorem. Consider the map $H'\colon \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ which is defined near the point (t_0, x_0) using the flow map (1.17) for the equation (1.1),

$$H'\colon (t, x') \mapsto (t, \Phi_{t_0}^t(x')), \qquad (t, x') \in (\mathbb{C}^{n+1}, (t_0, x_0)).$$

By construction, it takes the lines $x' = \text{const}$ parallel to the t-axis, into integral trajectories of the equation (1.1), while preserving the value of t.

The Jacobian matrix $\partial H'(t, x')/\partial(t, x')$ of the map H' at the point (t_0, x_0) has by (1.20) the form $\left(\begin{smallmatrix} 1 & \\ * & E \end{smallmatrix}\right)$ and is therefore invertible.

Thus H' restricted on a sufficiently small neighborhood of the point (t_0, x_0), is a biholomorphic conjugacy between the trivial system (1.21), whose solutions are exactly the lines $x' = \text{const}$, and the given system (1.1). The inverse map also preserves t and conjugates (1.1) with (1.21). □

1F. Vector fields and their equivalence. The above constructions after small modification become more transparent in the *autonomous* case, when the vector function $x \mapsto F(x)$ which is now independent of t, can be considered as a *holomorphic vector field* on its domain $U \subseteq \mathbb{C}^n$. The space of vector fields holomorphic in a domain $U \subseteq \mathbb{C}^n$ will be denoted by $\mathcal{D}(U)$, while the notation $\mathcal{D}(\mathbb{C}^n, x_0)$ is reserved for the space of germs of holomorphic vector fields at a specific point $x_0 \in \mathbb{C}^n$, usually the origin, $x_0 = 0$.

In the autonomous case, translation of the independent variable preserves solutions of the equation

$$\dot{x} = F(x), \qquad F \colon U \to \mathbb{C}^n, \tag{1.23}$$

so the flow map $\Phi_{t_0}^{t_1}$ actually depends only on the difference $t = t_1 - t_0$ and hence will be denoted simply by $\Phi^t(\cdot) = \Phi_0^t(\cdot)$. The functional identity (1.18) takes the form

$$\Phi^t(\Phi^s(x)) = \Phi^{t+s}(x), \qquad t, s \in (\mathbb{C}, 0),\ x \in (\mathbb{C}^n, x_0), \tag{1.24}$$

which means that the maps $\{\Phi^t\}$ form a one-parametric *pseudogroup* of biholomorphisms. ("Pseudo" means that the composition in (1.24) is not always defined. The pseudogroup is a true group, $\Phi^t \circ \Phi^s = \Phi^{t+s}$, if the maps Φ^t are globally defined for all $t \in \mathbb{C}$. For more details on pseudogroups see §6**D**).

For autonomous equations it is natural to consider biholomorphisms that are *time-independent*.

Definition 1.15. Two holomorphic vector fields, $F \in \mathcal{D}(U)$ and $F' \in \mathcal{D}(U')$ defined in two domains $U, U' \subseteq \mathbb{C}^n$, are *biholomorphically equivalent* if there exists a biholomorphic map $H \colon U \to U'$ conjugating their respective flows,

$$H \circ \Phi^t = \Phi'^t \circ H \tag{1.25}$$

whenever both sides are defined. The biholomorphism H is said to be a *conjugacy* between F and F'.

A conjugacy H maps *phase* curves of the first field into phase curves of the second field; in a similar way the suspension

$$\mathrm{id} \times H \colon (\mathbb{C}, 0) \times U \to (\mathbb{C}, 0) \times U', \qquad (t, x) \mapsto (t, H(x)),$$

maps *integral* curves of the two fields into each other. Differentiating the identity (1.25) in t at $t = 0$, we conclude that the differential $dH(x)$ of a holomorphic conjugacy sends the vector $v = F(x)$ of the first field, attached to a point $x \in U$, to the vector $v' = F'(x')$ of the second field *at the appropriate point* $x' = H(x)$. In the coordinates this property takes the form of the identity

$$H_*(x) \cdot F(x) = F'(H(x)), \qquad H_*(x) = \left(\tfrac{\partial H}{\partial x}\right) = \left(\tfrac{\partial h_i}{\partial x_j}\right), \tag{1.26}$$

in which the Jacobian matrix $H_*(x) = \left(\frac{\partial H}{\partial x}\right)$ is involved. The formula (1.26) is sometimes used as the alternative *definition* of the holomorphic equivalence. The third (algebraic, in some sense most natural) way to introduce this equivalence is explained in the next section.

1G. Vector fields as derivations. It is sometimes convenient to define vector fields in a way independent of the coordinates. Each vector field $F = (F_1, \dots, F_n)$ in a domain $U \subset \mathbb{C}^n$ defines a *derivation* $\mathbf{F} \in \operatorname{Der} \mathcal{O}(U)$ of the $\mathbb{C}$-algebra $\mathcal{O}(U)$ of functions holomorphic in U, by the formula

$$\mathbf{F} f(x) = \sum_{j=1}^{n} F_j(x) \, \frac{\partial f}{\partial x_j}. \tag{1.27}$$

We often identify the holomorphic vector field F with the components F_i with the corresponding differential operator $\mathbf{F} = \sum F_j \frac{\partial}{\partial x_j}$.

Derivations can be defined in purely algebraic terms as $\mathbb{C}$-linear maps of the algebra $\mathcal{O}(U)$ satisfying the Leibniz identity,

$$\mathbf{F}(fg) = f(\mathbf{F}g) + (\mathbf{F}f)g.$$

Indeed, any $\mathbb{C}$-linear operator with this property in any coordinate system $(x_1, \dots, x_n)$ defines n functions $F_j = \mathbf{F}x_j$ and (obviously) sends all constants to zero. Any analytic function f can be written $f(x) = f(a) + \sum_1^n h_j(x)\,(x_j - a_j)$ with $h_j(a) = \frac{\partial f}{\partial x_j}(a)$. Applying the Leibniz rule, we conclude that $(\mathbf{F}f)(a) = \sum_j F_j h_j(a) + 0 \cdot \mathbf{F}h_j = \sum_j F_j \frac{\partial f}{\partial x_j}(a)$, as claimed.

A similar algebraic description can be given for holomorphic maps. With any holomorphic map $H \colon U \to U'$ between two domains $U, U' \subseteq \mathbb{C}^n$ one can associate the *pullback operator* $\mathbf{H} \colon \mathcal{O}(U') \to \mathcal{O}(U)$, acting on $f' \in \mathcal{O}(U')$ by composition, $(\mathbf{H}f')(x) = f'(H(x))$. This operator is a *homomorphism* of commutative $\mathbb{C}$-algebras, a $\mathbb{C}$-linear map respecting multiplication (i.e., $\mathbf{H}(f'g') = \mathbf{H}f' \cdot \mathbf{H}g'$ for any $f', g' \in \mathcal{O}(U')$). Conversely, any continuous homomorphism $\mathbf{H}$ between the two algebras is induced by a holomorphic map $H = (h_1, \dots, h_n)$ with $h_i = \mathbf{H}x_i$, where $x_i \in \mathcal{O}(U')$ are the coordinate functions (restricted on U'). The map H is a biholomorphism if and only if $\mathbf{H}$ is an isomorphism of $\mathbb{C}$-algebras.

In this language the action of biholomorphisms on vector fields can be described as a simple *conjugacy of operators*: two derivations $\mathbf{F}$ and $\mathbf{F}'$ of the algebras $\mathcal{O}(U)$ and $\mathcal{O}(U')$ respectively, are said to be conjugated by the biholomorphism $H\colon U \to U'$, if

$$\mathbf{F} \circ \mathbf{H} = \mathbf{H} \circ \mathbf{F}' \tag{1.28}$$

as two $\mathbb{C}$-linear operators from $\mathcal{O}(U')$ to $\mathcal{O}(U)$.

Another advantage of this invariant description is the possibility of defining the *commutator* of two vector fields naturally, as the commutator of the respective differential operators. One can immediately verify that $[\mathbf{F}, \mathbf{F}'] = \mathbf{F}\mathbf{F}' - \mathbf{F}'\mathbf{F}$ satisfies the Leibnitz identity as soon as $\mathbf{F}, \mathbf{F}'$ do, and hence corresponds to a vector field. In coordinates the commutator takes the form

$$[F, F'] = \left(\frac{\partial F'}{\partial x}\right) F - \left(\frac{\partial F}{\partial x}\right) F'. \tag{1.29}$$

Example 1.16. For any two $\mathbf{F} = Ax$, $\mathbf{F}' = A'x$ *linear* vector fields, their commutator $[\mathbf{F}, \mathbf{F}']$ is again a linear vector field with the linearization matrix $A'A - AA'$. It coincides (modulo the sign) with the usual matrix commutator $[A, A']$.

1H. Rectification of vector fields. A straightforward counterpart of the Flow box Theorem 1.14 for holomorphic vector fields holds only if the field is nonvanishing.

Definition 1.17. A point x is a *singular point* (*singularity*) of a holomorphic vector field F, if $F(x_0) = 0$. Otherwise the point is *nonsingular.*

Theorem 1.18 (Rectification theorem)**.** *A holomorphic vector field F is holomorphically equivalent to the constant vector field $F'(x') = (1, 0, \dots, 0)$ in a sufficiently small neighborhood of any nonsingular point.*

Proof. The flow Φ' of the constant vector field F' can be immediately computed: $(\Phi')^t(x') = x' + t \cdot (1, 0, \dots, 0)$. Consider any affine hyperplane $\Pi \subset U$ passing through x_0 and transversal to $F(x_0)$ and the hyperplane $\Pi' = \{x'_1 = 0\}$. Let $t = x'_1 \colon \mathbb{C}^n \to \mathbb{C}$ be the function equal to the first coordinate in $\mathbb{C}^n$, so that $(\Phi')^{-t}(x') \in \Pi'$. Let $h' \colon \Pi' \to \Pi$ be *any* biholomorphism (e.g., linear invertible map). Then the map

$$H' = \Phi^t \circ h \circ (\Phi')^{-t}, \qquad t = t(x'),$$

is a holomorphic map that sends any (parameterized) trajectory of F', passing through a point $x' \in \Pi'$, to the parameterized trajectory of F passing through $x = h(x')$. Being composition of holomorphic maps, H' is also holomorphic, and coincides with h' when restricted on Π'. It remains to notice that the differential $dH'(x_0)$ maps the vector $(1, 0, \dots, 0)$ transversal to Π', to the vector $F(x_0)$ transversal to Π. This observation proves that H' is invertible in some sufficiently small neighborhood U of x_0, and the inverse map H conjugates F in U with F' in $H(U)$. $\square$

1I. One-parametric groups of holomorphisms. The Rectification theorem from §1 can be formulated in the language of germs as follows: *Two germs of holomorphic vector fields at nonsingular points are always holomorphically equivalent to each other.* In particular, any germ of a holomorphic vector field at a nonsingular point is holomorphically equivalent to the germ of a nonzero constant vector field.

Because of this "triviality" of local description of nonsingular vector fields, we will mostly be interested in germs of vector fields at the *singular points*. The first result is existence of germs of the flow maps Φ^t at the singular point, for all values of $t \in \mathbb{C}$.

Denote by $\operatorname{Diff}(\mathbb{C}^n, 0)$ the group of germs of holomorphic self-maps $H\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ equipped with the operation of composition (which is always defined).

Proposition 1.19. *If $F \in \mathcal{D}(\mathbb{C}^n, 0)$ is the germ of a holomorphic vector field which is singular (i.e., $F(0) = 0$), then the germs of the flow maps $\Phi^t(\cdot)$ are defined for all $t \in \mathbb{C}$ and form a one-parametric subgroup of the group $\operatorname{Diff}(\mathbb{C}^n, 0)$ of germs of biholomorphic self-maps: $\Phi^t \circ \Phi^s = \Phi^{t+s}$ for any $t, s \in \mathbb{C}$.*

Proof. The existence of the flow maps Φ^t for all sufficiently small $t \in (\mathbb{C}, 0)$, the possibility of their composition, and validity of the group identity for such small t all follow from Theorem 1.1 and the fact that $\Phi^t(x_0) = x_0$.

For an arbitrary large value of $t \in \mathbb{C}$ we may define Φ^t as the composition of germs of the flow maps Φ^{t_i}, $i = 1, \dots, N$, taken in any order, where the complex numbers t_i are sufficiently small to satisfy conditions of Theorem 1.1 but added together give t. From the local group identity it follows that the definition does not depend on the particular choice of t_i and preserves the group property. $\square$

Remark 1.20. Every germ of a self-map $H \in \operatorname{Diff}(\mathbb{C}^n, 0)$ uniquely defines an *auto*morphism $\mathbf{H} \in \operatorname{Aut} \mathcal{O}(\mathbb{C}^n, 0)$ of the commutative algebra of holomorphic germs acting by substitution, $\mathbf{H}f = f \circ H$.

Proposition 1.19 translates into the algebraic language as follows: for any derivation $\mathbf{F} \in \operatorname{Der} \mathcal{O}(\mathbb{C}^n, 0)$ of the algebra of holomorphic germs there exist a one-parametric subgroup $\{\mathbf{H}^t \colon t \in \mathbb{C}\} \subset \operatorname{Aut} \mathcal{O}(\mathbb{C}^n, 0)$ of automorphisms of this algebra, such that $\frac{d}{dt}\big|_{t=0} \mathbf{H}^t = \mathbf{F}$.

For the reasons to be explained below in §3**C**, the subgroup of automorphisms $\mathbf{H}^t$ is often referred to as the *exponent*, $\mathbf{H}^t = \exp(t\mathbf{F})$, of the derivation **F**. Respectively, the flow (germs of self-maps) will be sometimes denoted by the exponent, $\Phi^t = \exp(tF)$, of the corresponding vector field F.

Exercises and Problems for §1.

Exercise 1.1. Let $a \in U$ be a nonsingular point of a holomorphic vector field $F \in \mathcal{D}(U)$. A trajectory of the vector field is the projection of the graph of the solution into the domain of the field along the time axis.

Prove that the trajectory passing through a is either the line $x = a$, or can be represented as the graph of a function $y = \varphi_a(x)$ having an *algebraic* ramification point of some finite order ν. Express ν in terms of orders of the components of the field F at a.

Exercise 1.2. Let $P\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^{n-1}, 0)$ be a holomorphic epimorphism (i.e., map of rank $n - 1$) constant along trajectories of an analytic vector field $F \in \mathcal{D}(\mathbb{C}^n, 0)$. Construct explicitly the rectifying chart for F.

Exercise 1.3. Prove that the space $\mathcal{M}$ of functions satisfying the inequality (1.7), is indeed complete.

Exercise 1.4. Two linear vector fields in $\mathbb{C}^n$ are holomorphically equivalent in some domains containing the origin. Prove that these fields are *linear* equivalent, i.e., that there exists a linear map $H \in \mathrm{GL}(n, \mathbb{C})$ conjugating them.

Exercise 1.5. Prove that if two germs of vector fields at a singular point are analytically equivalent, then the eigenvalues of these fields at the singular point coincide.

Exercise 1.6. Prove that the vector field $F(z) = z^2 \frac{\partial}{\partial z}$ is holomorphic on the Riemann sphere $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. Compute the flow of this field.

Problem 1.7. Give a complete analytic classification of the holomorphic flows on the Riemann sphere $\mathbb{P}^1$ (i.e., construct a list, finite or infinite, of flows such that every holomorphic flow in analytically equivalent to one of the flows from the list, while any two different flows in the list are *not* holomorphically equivalent.

Exercise 1.8. Prove that the constant holomorphic vector fields $\frac{\partial}{\partial z}$ on the two tori $\mathbb{T}_1 = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ and $\mathbb{T}_2 = \mathbb{C}/(\mathbb{Z} + 2i\mathbb{Z})$, are not holomorphically equivalent.

2. Holomorphic foliations and their singularities

By the Existence/Uniqueness Theorem 1.1, any open connected domain $U \subseteq \mathbb{C}^n$ with a holomorphic vector field F defined on it, can be represented as the disjoint union of connected phase curves passing through all points of U. The Rectification Theorem 1.18 provides a local model for the geometric object called *foliated space* of simply *foliation*. A systematic treatment of foliations can be found, for instance, in [**Tam92, CC03**].

2A. Principal definitions. Speaking informally, a foliation is a partition of the phase space into a continuum of connected sets called *leaves*, which locally look as the family of parallel affine subspaces.

Definition 2.1. The *standard holomorphic foliation* of dimension n (respectively, of codimension m) of a polydisk $B = \{(x, y) \in \mathbb{C}^n \times \mathbb{C}^m : |x| < 1, |y| < 1\}$ is the representation of B as the disjoint union of n-disks, called (standard) *plaques*,

$$B = \bigsqcup_{|y|<1} L_y, \qquad L_y = \{|x| < 1\} \times \{y\} \subseteq B. \tag{2.1}$$

Definition 2.2. A holomorphic foliation $\mathcal{F}$ of a domain $U \subset \mathbb{C}^{n+m}$ (or, more generally, a complex analytic manifold U of dimension $n + m$) is the partition $U = \bigsqcup_\alpha L_\alpha$ of the latter into the disjoint union *connected* subsets L_α, called *leaves*, which locally is biholomorphically equivalent to the standard holomorphic foliation by plaques.

The latter phrase means that each point $a \in U$ admits an open neighborhood $B' \ni a$ and a biholomorphism $H : B' \to B$ of B' onto the standard polydisk B, which sends the *local leaves*, the connected components of the intersections $L_\alpha \cap B'$, to the plaques of the standard foliation,

$$\forall\alpha\ \exists Y = Y(\alpha) : \qquad H(L_\alpha \cap B') = \bigsqcup_{y \in Y(\alpha)} L_y. \tag{2.2}$$

Sometimes the local leaves will also be referred to as the *plaques* of the foliation near a point a: the plaques constitute biholomorphic images of n-disks, parameterized by a small m-disk. Note that different plaques may belong to the same leaf of the global foliation.

Remark 2.3. The definition of foliation admits several flavors. In the weakest settings the standard foliations are families of parallel balls slicing the real cylinder in $\mathbb{R}^{n+m}$ (the formulas remain the same as in (2.1)), while the local equivalencies H are simply homeomorphisms or smooth maps of low or high differentiability (up to C^∞ or even real analytic). In particular, we will call the *topological foliation* a partition of the space U into disjoint subsets L_α which is locally *homeomorphic* to the standard foliation (in the sense (2.2) with H being a homeomorphism).

Moreover, one can require *different* regularity of H along the leaves and in the transversal direction. We will not deal with such exotic cases until §28.

Remark 2.4 (important)**.** The space of plaques of a foliation is naturally parameterized by points of a polydisk. Yet the index set $Y(\alpha)$ in (2.2) can be rather complicated (e.g., dense), since the global behavior of leaves outside the ball B' can be rather complicated. Yet in all of our applications all sets $Y(\alpha)$ will be at most countable.

The global space of leaves may have a very complicated structure even topologically (non-Hausdorff), therefore for indexing the leaves we use "abstract" sets without any additional structure.

Definition 2.5. Two holomorphic foliations $\mathcal{F}$ and $\mathcal{F}'$ defined on the respective holomorphic manifolds U, U', are called *holomorphically equivalent* or *topologically equivalent*, if there exists a biholomorphism $H\colon U \to U'$ (respectively, a homeomorphism) which maps (necessarily biholomorphically or homeomorphically, depending on the context) the leaves of $\mathcal{F}$ to those of $\mathcal{F}'$: $H(L_\alpha) = L'_{\alpha'}$ for some indices α, α'.

Note that this definition is *global*.

Everywhere below U stands for a holomorphic manifold or an open domain in $\mathbb{C}^n$. The following result is an obvious reformulation of the Rectification theorem in the language of foliations.

Proposition 2.6. *For any holomorphic vector field $F \in \mathcal{D}(U)$ without singularities in U, the partition of U into maximal integral curves of F forms a holomorphic foliation $\mathcal{F}_F$ of (complex) dimension* 1 *and codimension $n-1$.* □

We say that the foliation $\mathcal{F}_F$ is generated by the vector field F. Speaking about foliations rather than about vector fields means that the parametrization of solutions by the (complex) time is to be ignored.

Proposition 2.7. *Two holomorphically equivalent vector fields $F \in \mathcal{D}(U)$ and $F' \in \mathcal{D}(U')$ generate two holomorphically equivalent one-dimensional foliations.*

Conversely, if the foliations $\mathcal{F}, \mathcal{F}'$ generated by two nonsingular vector fields, are holomorphically equivalent by a biholomorphism $H\colon U \to U'$, then there exists a nonvanishing holomorphic function $\rho \in \mathcal{O}(U)$ such that

$$\rho(x) \cdot H_*(x) \cdot F(x) = F'(H(x)), \qquad \rho(x) \neq 0 \quad \text{in } U; \tag{2.3}$$

cf. with (1.26) *and Definition 1.15.*

Proof. The first assertion is obvious immediately. To prove the second, it is sufficient to show that two vector fields generating *the same* holomorphic one-dimensional foliation, differ by a nonvanishing holomorphic scalar factor ρ. This is obvious for the standard foliation: the first component must be nonzero while all other components are identically zero. □

2B. Foliations and integrable distributions. For a given holomorphic foliation $\mathcal{F}$ of dimension n and codimension m, the tangent spaces to leaves at different points are n-dimensional complex spaces in an obvious sense analytically depending on the point.

Such a geometric object is called *distribution*. To define formally subspaces analytically depending on parameters, one can choose between the language of holomorphic vector fields and that of holomorphic differential forms.

Definition 2.8. A (holomorphic nonsingular) n-dimensional distribution in a domain $U \subseteq \mathbb{C}^{n+m}$ is either

- a tuple of n holomorphic vector fields $F_1, \dots, F_n \in \mathcal{D}(U)$, linearly independent at every point of U, or
- tuple of m holomorphic 1-forms $\omega_1, \dots, \omega_m \in \Lambda^1(U)$, linearly independent at every point of U so that $\omega_1 \wedge \cdots \wedge \omega_m \in \Lambda^k(U)$ is nonvanishing.

Two tuples of the same type $\{F_j\}$ and $F'_j\}$ (resp., $\{\omega_i\}$ and $\{\omega'_i\}$ define the same distribution, if $F'_j = \sum_k c_{jk}(x) F_k$, resp., $\omega'_i = \sum_k c'_{ik}(x)\omega_k$) for some holomorphic functions $c_{jk}(x), c'_{ik}(x)$. The forms and the fields defining the same distribution must be dual to each other, $\omega_i \cdot F_j = 0$ for all i, j.

A one-dimensional distribution is usually called a *line field*.

Clearly, any holomorphic foliation defines the corresponding tangent distribution of the same dimension. The converse in general is not true unless $n = 1$.

A holomorphic n-dimensional distribution is called *integrable* in U, if it is tangent to leaves of a nonsingular holomorphic foliation in U.

Theorem 2.9 (Frobenius integrability criteria)**.** *A distribution defined by a tuple of holomorphic vector fields is integrable, if and only if the commutator of any two vector fields belongs to the same distribution, i.e., if*

$$[F_i, F_j] = \sum_{k=1}^{n} c_{ijk}\, F_k, \qquad c_{ijk} \in \mathcal{O}(U). \tag{2.4}$$

A distribution defined by a tuple of holomorphic 1-forms is integrable, if and only if the ideal spanned by these forms in the exterior algebra $\Lambda^\bullet(U)$ over $\mathcal{O}(U)$, is closed by the exterior derivative, i.e., if

$$d\omega_i = \sum_{k=1}^{m} c'_{ik}\, \omega_k \wedge \eta_k, \qquad \eta_k \in \Lambda^1(U),\ \ c_{ik} \in \mathcal{O}(U). \tag{2.5}$$

We will not prove this theorem. Its proof can be derived from the local existence theorem for holomorphic vector fields in the same way as it is done, *mutatis mutandis*, in the C^∞-smooth case in [**War83**].

Remark 2.10. The Frobenius integrability condition trivially holds for $n = 1$. On the other hand, from the real point of view the holomorphic vector field F corresponds to a 2-dimensional distribution generated by *two* vector fields $F_1 = F$ and $F_2 = iF$, $i = \sqrt{-1}$, in $\mathbb{R}^{2n} \cong \mathbb{C}^n$. The Frobenius integrability condition for this distribution reduces, as one can easily verify, to the Cauchy–Riemann identities between the real and imaginary parts of the components of the holomorphic vector field F.

Remark 2.11. In the (complex) 2-dimensional case where $U \subseteq \mathbb{C}^2$ that will be our principal object of studies later, the only nontrivial possibility is a one-dimensional distribution that is automatically integrable. It can be defined either by one vector field $F \in \mathcal{D}(U)$ or by one Pfaffian form $\omega \in \Lambda^1(U)$. For many reasons the Pfaffian presentation is more convenient.

2C. Holonomy. The notion of holonomy intends to be a replacement of the flow of the vector fields in the case where the natural parametrization of the solutions is absent or ignored.

Definition 2.12. A (parameterized) *cross-section* to a leaf L of a foliation $\mathcal{F}$ of codimension m on U at a point $a \in U$ is a holomorphic map $\tau\colon (\mathbb{C}^m, 0) \to (U, a)$ transversal to L. Very often we identify the cross-section with the image of the map τ.

If $\mathcal{F}$ is a *standard* foliation and τ, τ' any two cross-sections (at different, in general) points a, a' of the leaf, say $L_0 = \{y = 0\}$, then any other leaf L_α sufficiently close to L_0 intersects each cross-section exactly once. This defines in a unique way the holomorphic *correspondence map* $\Delta_{\tau,\tau'}\colon (\tau, a) \to (\tau', a')$: points with the same y-components are mapped into each other. In the charts on τ, τ' defined by the parameterizations, the correspondence map becomes the germ of a holomorphic map from $\operatorname{Diff}(\mathbb{C}^m, 0)$.

The correspondence maps obviously satisfy the identity

$$\Delta_{\tau,\tau''} = \Delta_{\tau',\tau''} \circ \Delta_{\tau,\tau'} \tag{2.6}$$

for any three cross-sections τ, τ', τ'' to the same leaf of the standard foliation.

Taking a biholomorphic image of this construction, we arrive at the following conclusion. For any two cross-sections τ, τ' to two sufficiently close points on the same leaf, there exists a uniquely defined correspondence map $\Delta_{\tau,\tau'}$ between the cross-sections that satisfies the identity (2.6) for any third cross-section which is also sufficiently close.

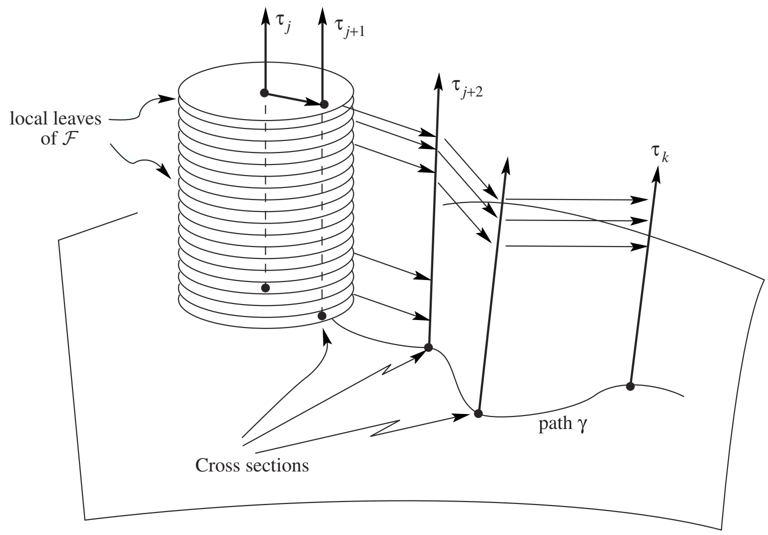

Figure I.2. Construction of the holonomy map for a foliation over a given path γ connecting two points on the leaf. The cross-sections τ_j are chosen close enough

Globalization of this construction associates the correspondence map not with just a pair of cross-sections to the same leaf, but rather with a *path* connecting the base points of these cross-sections. Let L be a leaf of a holomorphic foliation $\mathcal{F}$, τ, τ' two cross-sections cutting L at the points $a, a' \in L$, and $\gamma \colon [0,1] \to L$ an (oriented) path connecting $a = \gamma(0)$ with $a' = \gamma(1)$.

Since the segment $[0,1]$ and its image are compact, one can cover them by finitely many open sets U_j in such a way that in each set the foliation is locally trivial (biholomorphically equivalent to the standard foliation). One can insert between the cross-sections τ, τ' sufficiently many intermediate cross-sections τ_j, $j = 1, \dots, k$, $\tau_0 = \tau$, $\tau_k = \tau'$, at some intermediate points of the curve γ such that every two consecutive cross-sections τ_j, τ_{j+1} belong to the same domain U_j (for this purpose one has to choose $\tau_j \subset U_{j-1} \cap U_j$. Let $\Delta_{\tau_j, \tau_{j+1}}$ be the corresponding local correspondence maps as defined earlier. The composition

$$\Delta_\gamma = \Delta_{\tau_{k-1}, \tau_k} \circ \cdots \circ \Delta_{\tau_0, \tau_1} \colon (\tau, a) \to (\tau', a') \tag{2.7}$$

is a holomorphic map (more precisely, a germ) from $\mathrm{Diff}(\mathbb{C}^m, 0)$, also called the *correspondence map along the path* γ.

The identity (2.6) means that the correspondence map Δ_γ in fact does not depend on the choice of the intermediate cross-sections τ_j. Moreover, Δ_γ depends on the *homotopy class* of the path γ (with fixed endpoints) rather than on the path itself. Indeed, for another sufficiently close path

γ' connecting *the same endpoints*, we can choose cross-sections $\tau'_1, \dots, \tau'_{k-1}$ sufficiently close to the respective cross-sections τ_j for all $j = 1, \dots, k-1$ (the two extreme cross-sections coincide). Then one can use the identities (2.6) to show that the composition $\Delta_{\gamma'} = \Delta_{\tau'_{k-1},\tau'_k} \circ \cdots \circ \Delta'_{\tau'_0,\tau_1} \colon (\tau, a) \to (\tau', a')$ coincides with Δ_γ, since $\tau'_0 = \tau_0$ and $\tau'_k = \tau_k$.

Remark 2.13. The construction of holonomy maps corresponds to what in the classical parlance was called "continuation of solutions of differential equations over a path": a specific solution (corresponding to the leaf) was explicitly or implicitly singled out together with a certain path on it, and all nearby solutions were "continued over the path" on the selected solution.

Choosing another pair of cross-sections at the same endpoints (or another parametrization of the same cross-sections) results in composition of Δ_γ with two biholomorphisms from left and right, so using suitable charts, one can always bring any particular correspondence map Δ_γ to be the identity map. The situation changes completely if there is more than one homotopically distinct path connecting the same endpoints, or, what is the same, when one considers *closed paths*.

Let $a \in L$ be a point on the leaf L of a holomorphic foliation, $\tau \colon (\mathbb{C}^m, 0) \to (U, a)$ a cross-section at a, and $\gamma \in \pi_1(L, a)$ a closed loop considered modulo the homotopic equivalence.

Definition 2.14. The *holonomy self-map* $\Delta_\gamma \colon (\tau, a) \to (\tau, a)$ is the holomorphic holonomy correspondence map associated with a closed path $\gamma \in \pi_1(L, a)$.

The *holonomy group* of the foliation $\mathcal{F}$ along the leaf $L \in \mathcal{F}$ is the image of the fundamental group $\pi_1(L, a)$ in the group of germs of holomorphic self-maps $\operatorname{Diff}(\tau, a)$.

The holonomy group is defined as a subgroup in $\operatorname{Diff}(\mathbb{C}^m, 0)$ modulo a simultaneous conjugacy of all holonomy maps, independently of the choice of the cross-section τ or even the base point $a \in L$. It is an obvious invariant of a foliation which carries almost all information on behavior of leaves of the foliation, adjacent to L.

Proposition 2.15. *Assume that two holomorphic foliations $\mathcal{F}$, $\mathcal{F}'$ are topologically or holomorphically conjugate by a homeomorphism* (*resp., biholomorphism*) *H. If $L \in \mathcal{L}$ is a leaf mapped by H into a leaf $L' \in \mathcal{F}'$, then for any choice of the points $a \in L$, $a' \in L'$ and the corresponding cross-sections τ, τ' the corresponding holonomy groups $G \subset \operatorname{Diff}(\tau, a)$ and $G' \subset \operatorname{Diff}(\tau', a')$ are topologically* (*resp., holomorphically*) *conjugate*: *there exists the germ of a map $h \colon (\tau, a) \mapsto (\tau', a')$, holomorphic or continuous respectively, such* *that h conjugates each element of g with some element $g' \in G'$ and respects the group law.*

Proof. Let τ be a cross-section to L at a and $\tau' = H(\tau)$ (with the induced chart), then the assertion is a tautology: the restriction $h = H|_\tau$ realizes the required conjugacy between G and G'. Any other choice of a' and τ' results in replacing G' by a holomorphically conjugate group. □

However, the inverse statement is in general wrong (see Exercise 2.10).

Definition 2.16. Let $\mathcal{F}$ be a holomorphic foliation on a complex manifold U, and $B \subseteq U$ an arbitrary subset. The *saturation* of B by leaves of $\mathcal{F}$ is the union of all leaves that intersect B:

$$\operatorname{Sat}(B, \mathcal{F}) = \bigcup_{L \in \mathcal{F},\ L \cap B \neq \varnothing} L.$$

In general, saturations of even simple sets can be rather complicated. Yet some basic things can be guaranteed. The following can be considered as a generalization of the theorem on continuous dependance of solutions of differential equations on initial conditions.

Lemma 2.17. *Saturation of an open set is open. In particular, saturation of a neighborhood of any point on each leaf contains an open neighborhood of the leaf.* □

From this observation we can derive a corollary that will be used later. Let $G \subset \operatorname{Diff}(\tau, a)$ be a finitely generated subgroup. A germ of an analytic function $u \in \mathcal{O}(\tau, a)$ is called G-invariant, if $u \circ g = u$ for all germs of self-maps $g \in G$.

Lemma 2.18. *Any germ of a holomorphic function $u \in \mathcal{O}(\tau, a)$ which is invariant by the holonomy group $G \subseteq \operatorname{Diff}(\tau, a)$, uniquely extends as a holomorphic function defined in some open neighborhood V of the leaf L and constant along all leaves of the foliation $\mathcal{F}$ in V.*

Proof. Let $a' \in L$ be any point on L, connected by a path $\gamma\colon [0,1] \to L$ with the base point a. The holonomy map $\Delta_{a,a'}$ allows us to translate (analytically continue) the germ u, considered as a function from $\mathcal{O}(U, a)$ constant along the local plaques of $\mathcal{F}$, to the germ $u' \in \mathcal{O}(U, a')$, also constant along the local plaques. This extension depends on the choice of the path γ, yet for a different choice of this path γ' the result will differ by the continuation of the germ $u \circ g$, where g is the holonomy map associated with the loop $\gamma' \circ \gamma^{-1} \in \pi_1(L, a)$. Yet since u by assumption is G-invariant, the result will be the same and thus correctly defined for an arbitrary point $a' \in L$. □

Remark 2.19. Most holonomy groups do not admit nonconstant invariant functions. Exceptions correspond to *integrable foliations*; see §11.

2D. Singular foliations. The holonomy group may be nontrivial only for a leaf of the foliation which has a nontrivial fundamental group. Such leaves, in general difficult to find for arbitrary holomorphic foliations, can be easily found for *foliations with singularities*, or *singular foliations*. Starting from this moment, we consider only one-dimensional foliations unless explicitly stated otherwise.

A holomorphic vector field $F \in \mathcal{D}(U)$ defines a nonsingular holomorphic foliation on the complement to its singular locus $\Sigma = \Sigma_F = \{x \in U : F(x) = 0\}$ by Proposition 2.6. This singular locus can be an arbitrary analytic subset of U. However, very often the foliation can be extended from U on a bigger open subset eventually containing a part of Σ.

If $U \subset U'$ are two domains and $\mathcal{F}'$ a foliation on the larger domain, then $\mathcal{F}'$ can be restricted on U: by definition, this means the foliation whose leaves are *connected components* of the intersections $L'_\alpha \cap U$ for all leaves $L'_\alpha \in \mathcal{F}'$.

Theorem 2.20. *Let U be a connected open domain in $\mathbb{C}^n$ and $0 \not\equiv F \in \mathcal{D}(U)$ a holomorphic vector field with the singular locus $\Sigma \subset U$.*

Then there exists an analytic subset $\Sigma' \subseteq \Sigma$ of complex codimension $\geqslant 2$ in U and the foliation $\mathcal{F}'$ of $U \smallsetminus \Sigma'$ whose restriction on $U \smallsetminus \Sigma$ coincides with the foliation generated by the initial vector field F.

Proof. The assertion needs the proof only when Σ is an analytic hypersurface (a complex analytic set of codimension 1).

Consider an arbitrary *smooth point* $a \in \Sigma$ of the singular locus Σ: nonsmooth points already form an analytic subset $\Sigma_1 \subset \Sigma$ of codimension $\geqslant 2$ in U. Locally near this point Σ can be described by one equation $\{f = 0\}$ with f holomorphic and $df(a) \neq 0$. Let $\nu \geqslant 1$ be the maximal power such that all components $F_1, \dots, F_n$ of the vector field F are divisible by f^ν. By construction, the vector field $f^{-\nu} F$ extends analytically on Σ near a and its singular locus is a *proper* analytic subset $\Sigma_2 \subset \Sigma$ (locally near a). Since the germ of Σ at a is smooth hence irreducible, such a subset necessarily has codimension $\geqslant 2$ respective to the ambient space.

The union $\Sigma' = \Sigma_1 \cup \Sigma_2$ has codimension $\geqslant 2$ and in $U \smallsetminus \Sigma'$ the field locally represented as $f^{-\nu} F$ is nonsingular and thus defines a holomorphic foliation $\mathcal{F}'$ extending $\mathcal{F}$ on the neighborhood of all points of Σ. □

Remark 2.21. If U is two-dimensional, the holomorphic vector field F can be replaced by the distribution defined by an appropriate holomorphic 1-form $\omega \in \Lambda^1(U)$ with the singular locus Σ which consists of isolated points only (the singular locus of a holomorphic 1-form is the common zero of its coefficients).

Theorem 2.20 means that when speaking about holomorphic foliations with singularities, generated by holomorphic vector fields, one can always assume that the singular locus has codimension $\geqslant 2$; in particular, singularities of holomorphic foliations on the plane (and more generally, on holomorphic surfaces) are *isolated points*. The inverse statement is also true, as was observed in [**Ily72b**].

Theorem 2.22 ([**Ily72b**]). *Assume that $\Sigma \subset U \subseteq \mathbb{C}^n$ is an analytic subset of codimension $\geqslant 2$ and $\mathcal{F}$ a holomorphic nonsingular 1-dimensional foliation of $U \smallsetminus \Sigma$ which does not extend on any part of Σ.*

Then near each point $a \in \Sigma$ the foliation $\mathcal{F}$ is generated by a holomorphic vector field F with the singular locus Σ.

Proof. One can always assume that the local coordinates near a are chosen so that the line field tangent to leaves of $\mathcal{F}$, is not everywhere parallel to the coordinate x_1-plane. Then this line field is spanned by the *meromorphic* vector field $G = (1, G_2, \dots, G_n)$, where $G_j \in \mathcal{M}(U \smallsetminus \Sigma)$ are *meromorphic* functions in $U \smallsetminus \Sigma$. By E. Levi's theorem, any meromorphic function can be meromorphically extended on analytic subsets of codimension 1 [**GH78**, Chapter III, §2]. Therefore we may assume that G_j are in fact meromorphic in U. Decreasing if necessary the size of U, each G_j can be represented as the ratio $G_j = P_j/Q_j$, where $P_j, Q_j \in \mathcal{O}(U)$ are holomorphic in U and the representation is irreducible.

Let $\Sigma_j = \{P_j = Q_j = 0\}$, $j = 2, \dots, n$: by irreducibility, Σ_j is of codimension $\geqslant 2$, so $\bigcup_{j \geqslant 2} \Sigma_j$ is also of codimension $\geqslant 2$. Multiplying the field G by the product of denominators $Q_2 \cdots Q_n$, we obtain a holomorphic vector field tangent to the same foliation; cancelling a nontrivial common factor for the components of this field as in Theorem 2.20, we arrive at yet another holomorphic field F, also tangent to $\mathcal{F}$, such that the singular locus $\Sigma' = \operatorname{Sing}(F)$ of this field has codimension $\geqslant 2$.

It remains to show that the singular locus Σ' *coincides* with Σ locally in U. In one direction it is obvious: if Σ' is *smaller* than Σ, this means that $\mathcal{F}$ is analytically extended as a nonsingular holomorphic foliation to some parts of Σ, contrary to the assumption that Σ is the minimal singular locus. Assume that Σ' is *larger* than Σ, i.e., there exists a nonsingular point $b \in U \smallsetminus \Sigma$ of $\mathcal{F}$, at which F vanishes. Since the foliation $\mathcal{F}$ is biholomorphically equivalent to the standard foliation near b, in the suitable chart F is parallel to the first coordinate axis, so that singular points of F are zeros of its first component. On the other hand, by construction Σ' is of codimension $\geqslant 2$ and hence cannot be the zero locus of any holomorphic function. The contradiction proves that $\Sigma' \cap U$ cannot be larger than $\Sigma \cap U$. □

Example 2.23. The vector field $\frac{\partial}{\partial x} + e^{1/x} \frac{\partial}{\partial y}$ is analytic outside the line $\Sigma = \{x = 0\}$ of codimension 1 on the plane and defines a holomorphic foliation in $\mathbb{C}^2 \smallsetminus \Sigma$. This foliation cannot be defined by a vector field holomorphically extendable on Σ, which shows that the condition on the codimension in Theorem 2.22 cannot be relaxed.

Together Theorems 2.20 and 2.22 motivate the following concise definition. Since we will never consider in this book holomorphic foliations of dimension other than 1, this is explicitly included in the definition.

Definition 2.24. A *singular holomorphic foliation* in a domain U (or a complex analytic manifold) is a holomorphic foliation $\mathcal{F}$ with complex one-dimensional leaves in the complement $U \smallsetminus \Sigma$ to an analytic subset Σ of codimension $\geqslant 2$, called the *singular locus* of $\mathcal{F}$.

Usually we will assume that the singular locus Σ is maximal, i.e., the foliation cannot be analytically extended on any set larger than $U \smallsetminus \Sigma$.

The second part of Proposition 2.7 motivates the following important definition.

Definition 2.25. Two holomorphic vector fields $F \in \mathcal{D}(U)$, $F' \in \mathcal{D}(U')$ with singular loci Σ, Σ' of codimension $\geqslant 2$ are *holomorphically orbitally equivalent* if the singular foliations $\mathcal{F}, \mathcal{F}'$ they generate, are holomorphically equivalent, i.e., there exists a biholomorphism $H\colon U \to U'$ which maps Σ into Σ' and is a biholomorphism of foliations outside these loci.

Proposition 2.7 remains valid also for *singular* holomorphic foliations: if two such foliations are holomorphically equivalent, then the corresponding vector fields are orbitally equivalent, i.e., related by the identity (2.3) with the holomorphic function ρ nonvanishing in U.

Indeed, from Proposition 2.7 it follows that for the holomorphically orbitally equivalent fields there exists a holomorphic function ρ satisfying (2.3) and nonvanishing outside $\Sigma = \operatorname{Sing}(F)$. Since Σ has codimension $\geqslant 2$, ρ must be nonvanishing *everywhere* on U.

Changing only one adjective in Definition 2.25 (requiring that H be merely a homeomorphism), we obtain the definition of *topologically* orbitally equivalent vector fields. This weaker equivalence cannot be translated into a formula similar to (2.3), since homeomorphisms in general do not act on the vector fields.

2E. Complex separatrices. Foliations with isolated singularities may have *multiply-connected* leaves, i.e., leaves with a nontrivial holonomy group.

Recall that a (singular) analytic curve $S \subset U$ is a complex analytic set of complex dimension 1 at its smooth points. Intrinsic structure of *irreducible* components of analytic curves is relatively easy. This result can be found, e.g., in [**Chi89**, §6].

Theorem 2.26. *The germ of an irreducible analytic curve $S \subset (\mathbb{C}^n, 0)$ admits a holomorphic* injective *map*

$$\gamma\colon (\mathbb{C}^1, 0) \to (\mathbb{C}^n, 0), \qquad t \mapsto \gamma(t) \in S. \tag{2.8}$$

The map γ is called *local uniformization*, or *local parametrization* of analytic curves. It is obviously nonconstant, and without loss of generality one may assume that the derivative $\frac{d}{dt}\gamma(t)$ is nonvanishing outside the origin $t = 0$. The local parametrization is defined uniquely modulo a biholomorphism: for any other injective parametrization γ' there exists $h \in \operatorname{Diff}(\mathbb{C}^1, 0)$ such that $\gamma' = \gamma \circ h$ (cf. with Exercise 2.1).

Let $\mathcal{F}$ be a singular holomorphic foliation on an open domain U with the singular locus Σ.

Definition 2.27. A *complex separatrix* of a singular holomorphic foliation $\mathcal{F}$ at a singular point $a \in \operatorname{Sing}(\mathcal{F})$ is a local leaf $L \subset (U, a) \smallsetminus \Sigma$ whose closure $L \cup \{a\}$ is the germ of an analytic curve.

Since the leaves are by definition connected, the closure is irreducible (as a germ) at any it's point, hence (by the above uniformization arguments) the complex separatrix is topologically a punctured disk near the singularity. The fundamental group of the separatrix is nontrivial (infinite cyclic), thus the holomorphic map generating the *local holonomy group* is an invariant of the singular foliation. Note that the leaves are naturally oriented by their complex structure, so the loop generating the local fundamental group is uniquely defined modulo free homotopy.

In other words, every singular point that admits a complex separatrix, produces at least one holomorphic germ of a self-map that is an analytic invariant of the foliation. Later, in §14 we will show that *every* planar foliation (on a complex 2-dimensional manifold) has at least one separatrix through each singularity. Besides, by blow-up (desingularization) and Poincaré compactification, two related operations discussed in detail in §8 and §25**A** respectively, one can often *create* multiply-connected leaves of singularities *extending* a given singular foliation.

The rest of this section consists of a few examples important for future applications.

Example 2.28. Consider first the singular foliation spanned by a *diagonal* linear system

$$\dot{x} = Ax, \qquad A = \operatorname{diag}\{\lambda_1, \dots, \lambda_n\}, \quad \lambda_j \neq 0. \tag{2.9}$$

This foliation has an isolated singularity (of codimension n) at the origin, and all coordinate axes are complex separatrices.

Consider the first coordinate axis $S_1 = \{x_2 = \cdots = x_n = 0\}$ and the separatrix $L_1 = S_1 \smallsetminus \{0\}$. The loop $\gamma = \{|x_1| = 1\}$ parameterized counterclockwise is the canonical generator of L_1. Choose the affine hyperplane $\tau = \{x_1 = 1\} \subset \mathbb{C}^n$ as the cross-section to S_1 at the point $(1, 0, \ldots, 0) \in S_1$. A solution of the system (the parameterized leaf of the foliation) passing through the point $(1, b_2, \ldots, b_n) \in \tau$ is as follows:

$$\mathbb{C}^1 \ni t \mapsto x(t) = (\exp \lambda_1 t, b_2 \exp \lambda_2 t, \ldots, b_n \exp \lambda_n t) \in \mathbb{C}^n.$$

The image of the straight line segment $[0, 2\pi i/\lambda_1] \subset \mathbb{C}$ on the t-plane coincides with the loop γ when $b = 0$ (i.e., on the separatrix S_1) and is uniformly close to this loop on all leaves near S_1. The endpoints $x(2\pi i/\lambda_1)$ all belong to τ and hence the holonomy map $M_1 \colon \mathbb{C}^{n-1} \to \mathbb{C}^{n-1}$ is linear diagonal,

$$b \mapsto M_1 b, \qquad M_1 = \operatorname{diag}\{2\pi i \lambda_j/\lambda_1\}_{j=2}^{n}. \tag{2.10}$$

The other holonomy maps M_k for the canonical loops on the separatrices S_k parallel to the kth axis, are obtained by obvious relabelling of the indices.

Particular cases of this result are of special importance.

Example 2.29. Consider an *integrable* planar foliation given by the Pfaffian equation $\omega = 0$ with an *exact* form $\omega = du$, $u \in \mathcal{O}(\mathbb{C}^2, 0)$. If u has a Morse critical point, then in suitable analytic coordinates (x, y) the germ u takes the form $u = xy$, hence the foliation is given by the *linear* form $x\,dy + y\,dx = 0$ corresponding to the vector field $\dot y = y$, $\dot x = -x$. The holonomy operators corresponding to the two coordinate axes, are both *identical.*

Integrable foliations with more degenerate singularities will be treated in detail in §11.

Example 2.30. Let $n = 2$. Consider the vector field $F = (x + y)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ corresponding to a linear vector field with a nontrivial Jordan matrix. The corresponding singular foliation has only one complex separatrix, the punctured axis $S = \{y = 0\}$.

Consider the standard cross-section $\tau = \{x = 1\}$. Solutions of the differential equation with the initial condition (x_0, y_0) can be written explicitly,

$$x(t) = (x_0 + t y_0) \exp t, \quad y = y_0 \exp t.$$

Let $t(y_0)$ be another moment of complex time when the solution close to the separatrix again crosses τ after continuing along a path close to the standard loop on the separatrix; by definition, this means that we consider the initial point with $x_0 = 1$ and $x(t(y_0)) \equiv 1$, i.e., $1 + t(y_0) y_0 = 1/\exp t(y_0)$.

If the holonomy map is linear, then $y(t(y_0)) = \lambda y_0$ identically in y_0, i.e., $\exp t(y_0) = \lambda$ is a constant. Substituting this into the previous identity, we obtain $1 + t(y_0)y_0 = 1/\lambda$. This is impossible in the limit $y_0 \to 0$ unless $\lambda = 1$. On the other hand, $\lambda = 1$ is also impossible since $t(y_0) \not\equiv 0$.

Thus the holonomy map cannot be linear. The principal term of this map in a more general setting is computed in Proposition 27.14.

This example shows that a linear foliation may have nonlinear (and even nonlinearizable) holonomy.

2F. Suspension of a self-map. The construction of holonomy associates with any loop γ on a leaf $L \in \mathcal{F}$ of a holomorphic foliation $\mathcal{F}$ the holomorphic self-map Δ_γ. Very often the inverse problem appears: given an invertible holomorphic self-map f, construct a foliation for which this self-map would be the holonomy, associated with a loop on a leaf.

We will show that in absence of additional constraints on the phase space M and the leaf L, this problem is always trivially solvable. The construction is well known in the real analysis as *suspension* of a map to a flow.

Theorem 2.31. *Any biholomorphic germ $f \in \operatorname{Diff}(\mathbb{C}^n, 0)$ can be realized as the holonomy map along a loop on the leaf of a holomorphic foliation on an $(n+1)$-dimensional holomorphic manifold M^{n+1}.*

Construction of the foliation. For simplicity we discuss only the case $n = 1$: the general case requires only minimal modifications.

Consider the segment $[0,1] \subset \mathbb{C}$ and let U be its ε-neighborhood, $\varepsilon < \frac{1}{2}$. In the Cartesian product $\widetilde{M} = U \times (\mathbb{C}, 0)$ with the coordinates (z, w) consider the trivial foliation $\mathcal{F}_0$ by "horizontal lines" $\{w = \text{const}\}$.

Any self-map from $f \in \operatorname{Diff}(\mathbb{C}^1, 0)$ can be considered as a map $f\colon (\tau_0, 0) \to (\tau_1, 0)$, $w \mapsto f(w)$, between the cross-sections $\tau_0 = \{z = 0\}$ and $\tau_1 = \{z = 1\}$. The latter can be extended as a holomorphic invertible map $\mathbf{f}\colon (z, w) \mapsto (z+1, f(w))$ between the open sets $M_0 = \{|z| < \varepsilon\} \times (\mathbb{C}, 0) \subset \widetilde{M}$ and $M_1 = \{|z-1| < \varepsilon\} \times (\mathbb{C}, 0) \subset \widetilde{M}$. By construction, this map preserves the restriction of the foliation $\mathcal{F}_0$ on the open sets M_i.

The quotient space $M = \widetilde{M}/\mathbf{f}$ is defined as the topological space obtained from $\widetilde{M}$ by identification of all points a and $\mathbf{f}(a)$. This space inherits the structure of an (abstract) holomorphic manifold (the charts are inherited from those on M). Moreover, since $\mathbf{f}$ preserves the foliation, the quotient manifold M carries a well defined foliation $\mathcal{F}$. Two different cross-sections $\tau_0, \tau_1 \subset \widetilde{M}$ after identification become a single cross-section τ to the leaf L of the foliation $\mathcal{F}$ obtained from the zero leaf $\{w = 0\} \in \mathcal{F}_0$, and the segment $[0,1]$ on this leaf becomes a closed loop on L. The holonomy of the foliation

$\mathcal{F}$, associated with the loop $\gamma \subset L$, by construction coincides with the map f which is transformed into the self-map. □

The construction can be modified by a number of ways, while keeping the principal idea the same. If $\widetilde{M}$ is a manifold with a foliation $\mathcal{F}_0$ on it, and $\mathbf{f}\colon M_0 \to M_1$ is a biholomorphic map between open subsets of $\widetilde{M}$, which is an automorphism of the foliation $\mathcal{F}_0$, then the quotient space $M = \widetilde{M}/\mathbf{f}$ is a new manifold with a different topology, which carries a holomorphic foliation on it.

Exercises and Problems for §2.

Exercise 2.1. Let $S \subset (\mathbb{C}^n, 0)$ be the germ of an irreducible analytic curve and γ an injective analytic parametrization. Prove that any other holomorphic map $\gamma'\colon (\mathbb{C}^1, 0) \to (\mathbb{C}^n, 0)$ with the range in S differs from γ by a holomorphic map $h\colon (\mathbb{C}^1, 0) \to (\mathbb{C}^1, 0)$ so that $\gamma' = \gamma \circ h$.

Problems 2.2–2.7 together constitute a proof of the Frobenius Theorem 2.9.

Problem 2.2. Prove that vector fields generating an integrable distribution, are *in involution*, i.e., always satisfying condition (2.4).

Prove that Pfaffian forms generating an integrable distribution, are in involution, i.e., satisfy the conditions (2.5).

Problem 2.3. Prove that two holomorphic vector fields $F, F' \in \mathcal{D}(M)$ on a holomorphic manifold M, have identically zero commutator, $[F, F'] \equiv 0$, if and only if their flows $\exp(tF), \exp(t'F') \in \operatorname{Diff}(M)$ commute for all complex values of $t, t' \in \mathbb{C}$.

Formulate and prove an analog of this result for *incomplete* vector fields (i.e., when the flows are not globally defined for all values of t, t', as in the case where $U \subseteq \mathbb{C}^2$ is a noninvariant planar domain).

Problem 2.4. Prove that any tuple of everywhere linearly independent commuting vector fields generates an integrable distribution tangent to leaves of a holomorphic foliation.

Problem 2.5. Let $F_1, \dots, F_k$ be holomorphic everywhere linearly independent vector fields in involution (i.e., satisfying condition (2.4)).

Construct another tuple of holomorphic vector fields $F_1', \dots, F_k'$ spanning the same distribution, such that the fields $[F_i', F_j'] \equiv 0$ for all $1 \leqslant i, j \leqslant k$.

Prove that vector fields in involution generate an integrable distribution.

Problem 2.6. Prove that for any differential 1-form ω and two vector fields F, G on a manifold M,

$$d\omega(F, G) = F\,\omega(G) - G\,\omega(F) - \omega([F, G]) \tag{2.11}$$

(the right hand side contains the evaluation of ω on the fields F, G and $[F, G]$ and their derivatives along G and F).

Problem 2.7. Prove that a tuple of everywhere linearly independent 1-forms satisfying (2.5), defines an integrable distribution.

Exercise 2.8. Prove that a nonvanishing Pfaffian form ω in $\mathbb{C}^3$ defines an integrable distribution, if and only if $\omega \wedge d\omega = 0$.

Problem 2.9. Prove that each holonomy operator g corresponding to any separatrix of an integrable foliation $du = 0$ with an analytic potential $u \in \mathcal{O}(x, y)$, is periodic: some iterated power of g is identity.

Exercise 2.10. Construct two foliations having leaves with holomorphically conjugated holonomy groups, which are themselves not holomorphically conjugate in neighborhoods of the leaves.

Exercise 2.11. Is it always possible to rectify *simultaneously* two nonsingular vector fields? Two *commuting* nonsingular vector fields? Give a simple sufficient condition guaranteeing such simultaneous rectification.

Exercise 2.12. Consider the foliation $\{\omega = 0\}$ on $\mathbb{C}^2 = \{(z, t)\}$ defined by a *meromorphic* Pfaffian 1-form

$$\omega = \frac{dz}{z} - \sum_{j=0}^{n} \frac{\lambda_j\, dt}{t - a_j}, \qquad \lambda_j \in \mathbb{C},\ \sum_{0}^{n} \lambda_j = 0,$$

and its extension on $\mathbb{C} \times \mathbb{P}^1$.

Prove that the projective line $L = \{0\} \times \mathbb{P}^1$ is a separatrix of this foliation carrying singular points $(0, a_j)$, $j = 0, \dots, n$. Compute the holonomy group of the leaf $L \smallsetminus$ (singular points).

Exercise 2.13. The same question about the foliation on $\mathbb{C}^m \times \mathbb{P}^1$ defined by the vector Pfaffian form

$$dz - \Omega z = 0, \qquad \Omega = \sum_{0}^{n} \frac{A_j\, dt}{t - a_j},$$

where $A_j \in \mathrm{Mat}(m, \mathbb{C})$ are *commuting* matrix residues of the meromorphic matrix 1-form Ω.

Problem 2.14. Consider the Riccati equation

$$\frac{dz}{dt} = a(t)\, z^2 + b(t)\, z + c(t), \qquad a, b, c \in \mathfrak{M}(\mathbb{P}) \cong \mathbb{C}(t), \tag{2.12}$$

with meromorphic coefficients a, b, c having poles only on the finite point set $\Sigma \subseteq \mathbb{P}$. Is it true that solutions of this equation can be continued along any path on the t-plane, avoiding the singular locus Σ?

Prove that equation (2.12) defines a singular holomorphic foliation $\mathcal{F}$ on the compactified phase space $\mathbb{P}^1 \times \mathbb{P}^1$, which is transversal to any "vertical" projective line $\{t = a\}$, $a \notin \Sigma$. Show that each leaf of $\mathcal{F}$ can be continued over any path in the t-sphere, avoiding the singular locus. Prove that the induced transformation between any two cross-sections $\{t = a\} \times \mathbb{P}^1$ and $\{t = b\} \times \mathbb{P}^1$, $a, b \notin U$, is a well-defined Möbius transformation (fractional linear map $z \mapsto \frac{\alpha z + \beta}{\gamma z + \delta}$ with $\alpha\delta - \beta\gamma \neq 0$). Does $\mathcal{F}$ always possess a separatrix?

Exercise 2.15. How many separatrices a *homogeneous* vector field of degree r on $\mathbb{C}^2$ may have? How many separatrices a *generic* homogeneous vector field has?

3. Formal flows and embedding theorem

The assumption on convergence of Taylor series for the right hand sides of differential equations and their respective solutions is a very serious restriction: if it holds, then one can use various geometric tools as described in §2. However, considerable information can be gained without the convergence assumption, on the level of *formal power (Taylor) series.* For natural reasons, the corresponding results have more algebraic flavor.

In this section we introduce the class of formal vector fields and formal morphisms (self-maps), and prove that the flow of any such formal field can be correctly defined as a formal automorphism. The correspondence "field $\mapsto$ flow" can be inverted for maps with unipotent linearization: as was shown by F. Takens in 1974, any such formal map can be embedded in a unique formal flow [**Tak01**]. In §4 we establish classification of formal vector fields by the natural action of formal changes of variables.

3A. Formal vector fields and formal self-maps. For convenience, we will always assume that all Taylor series are centered at the origin, and use the standard multi-index notation: for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$ we denote $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\alpha! = \alpha_1! \cdots \alpha_n!$.

Definition 3.1. A *formal* (Taylor) series at the origin in $\mathbb{C}^n$ is the expression

$$f = \sum_\alpha c_\alpha x^\alpha, \qquad \alpha \in \mathbb{Z}_+^n, \quad c_\alpha \in \mathbb{C}. \tag{3.1}$$

The minimal degree $|\alpha|$ corresponding to a nonzero coefficient c_α, is called the *order* of f.

The set of all formal series is denoted by $\mathbb{C}[[x]] = \mathbb{C}[[x_1, \dots, x_n]]$. It is a commutative *infinite-dimensional* algebra over $\mathbb{C}$ which contains as a proper subset the algebra of germs of holomorphic functions, isomorphic to the algebra $\mathbb{C}\{x_1, \dots, x_n\}$ of *converging* series.

Definition 3.2. The *canonical basis* of $\mathbb{C}[[x]]$ is the collection of all monomials x^α, $\alpha \in \mathbb{Z}_+^n$, ordered in the following way: (i) all monomials of lower degree $|\alpha|$ precede all monomials of higher degree, and (ii) all monomials of the same degree are ordered lexicographically. This order will be denoted `deglex`-order.

Since the series may diverge, evaluation of $f(x_0)$ at any point $x_0 \in \mathbb{C}^n$ other than $x_0 = 0$, makes no sense. However, without risk of confusion we will denote the free term of a series $f \in \mathbb{C}[[x]]$ by $f(0)$ and the coefficient c_α by $\frac{1}{\alpha!} \frac{\partial^\alpha f}{\partial x^\alpha}(0)$. Under these agreements the Taylor formula becomes a *definition* of the Taylor series $f = \sum_{\alpha \geqslant 0} \frac{1}{\alpha!} \frac{\partial^\alpha f}{\partial x^\alpha}(0)\, x^\alpha$. Sometimes we write

$f(x)$ as an indication of the formal variables $x = (x_1, \dots, x_n)$ in which the series f depends.

All formal partial derivatives $\partial^\alpha f/\partial x^\alpha$ of a formal series f are well defined in the class $\mathbb{C}[[x]]$ as termwise derivatives.

The subset of $\mathbb{C}[[x]]$ which consists of formal series without the free term, is (as one can easily verify) a maximal ideal of the commutative ring $\mathbb{C}[[x]]$; it will be denoted by

$$\mathfrak{m} = \{f \in \mathbb{C}[[x]] \colon f(0) = 0\} = \Big\{ \sum_{|\alpha|\geqslant 1} c_\alpha x^\alpha \Big\}.$$

The maximal ideal is *unique* (again a simple exercise). In other words, the ring $\mathbb{C}[[x]]$ is a *local ring*.

For any finite $k \in \mathbb{N}$ the space of kth order jets can be described as the quotient

$$J^k(\mathbb{C}^n, 0) = \mathbb{C}[[x_1, \dots, x_n]]/\mathfrak{m}^{k+1}.$$

As a quotient ring, the affine finite-dimensional $\mathbb{C}$-space $J^k(\mathbb{C}^n, 0)$ inherits the structure of a commutative $\mathbb{C}$-algebra.

Definition 3.3. The *truncation* of formal series to a finite order k is the canonical projection map $j^k \colon \mathbb{C}[[x]] \to J^k(\mathbb{C}^n, 0)$, $f \mapsto f \bmod \mathfrak{m}^{k+1}$.

The name comes from the natural identification of $J^k(\mathbb{C}^n, 0)$ with polynomials of degree $\leqslant k$ in the variables $x_1, \dots, x_n$. If $l > k$ is a higher order, then $\mathfrak{m}^{l+1} \subset \mathfrak{m}^{k+1}$ so that the truncation operator j^k naturally "descends" as the projection $J^l(\mathbb{C}^n, 0) \to J^k(\mathbb{C}^n, 0)$ which will also be denoted by j^k.

In other words, a formal Taylor series $f \in \mathbb{C}[[x]]$ uniquely defines the k-jet $j^k f$ of any finite order k so that $\mathbb{C}[[x_1, \dots, x_n]]$ is in a sense the limit of the jet spaces $J^k(\mathbb{C}^n, 0)$ as $k \to \infty$. We will sometimes refer to formal series as *infinite* jets and write $\mathbb{C}[[x_1, \dots, x_n]] = J^\infty(\mathbb{C}^n, 0)$.

The canonical monomial basis in $\mathbb{C}[[x]]$ projects into canonically `deglex`-ordered monomial bases in all jet spaces $J^k(\mathbb{C}^n, 0)$. Convergence in $\mathbb{C}[[x]]$ is defined via finite truncations.

Definition 3.4. A sequence $\{f_j\}_{j=1}^\infty \subset \mathbb{C}[[x]]$ is said to be convergent, if and only if all its truncations $j^k f_j$ converge in the respective finite-dimensional k-jet space $J^k(\mathbb{C}^n, 0)$ for any finite $k \geqslant 0$.

Remark 3.5 (important)**.** All formal algebraic constructions described above can be implemented over the field $\mathbb{R}$ rather than $\mathbb{C}$ as the ground field. Moreover, for future purposes we will need the algebra $\mathfrak{A}[[x]]$ of formal power series in the indeterminates $x = (x_1, \dots, x_n)$ with the coefficients belonging to more general $\mathbb{C}$- or $\mathbb{R}$-algebras $\mathfrak{A}$. The principal examples are the algebras $\mathfrak{A} = \mathbb{C}[\lambda_1, \dots, \lambda_m]$ of polynomials in auxiliary indeterminates or the algebra $\mathcal{A} = \mathcal{O}(U)$ of holomorphic functions of additional variables $\lambda_1, \dots, \lambda_m$.

After introducing the algebra of "formal functions" we can define formal vector fields and formal maps via their algebraic (functorial) properties as in §1**G**.

With any *vector formal series* $F = (F_1, \dots, F_n)$ (n-tuple of elements from $\mathbb{C}[[x]]$) one can associate a derivation $\mathbf{F} = \sum_1^n F_j \partial/\partial x_j \in \operatorname{Der} \mathbb{C}[[x]]$ of the algebra $\mathbb{C}[[x]]$, a $\mathbb{C}$-linear application satisfying the Leibnitz rule (cf. with (1.27)),

$$\mathbf{F}\colon \mathbb{C}[[x]] \to \mathbb{C}[[x]], \qquad \mathbf{F}(gh) = g\,(\mathbf{F}h) + h\,(\mathbf{F}g).$$

Conversely, any derivation $\mathbf{F} \in \operatorname{Der} \mathbb{C}[[x]]$ is of the form $\mathbf{F} = \sum_1^n F_j \partial/\partial x_j$ with the components $F_j = \mathbf{F}x_j$. By *formal vector fields*, we mean both realizations, $F \in \mathbb{C}[[x]]^n$ or $\mathbf{F} \in \operatorname{Der} \mathbb{C}[[x]]$. The field F is said to have *singularity* (at the origin), if all these series are without free terms, $F_j(0) = 0$, $j = 1, \dots, n$.

The collection of formal vector fields will be denoted $\mathcal{D}[[\mathbb{C}^n, 0]]$. It is a $\mathbb{C}$-linear (infinite dimensional) space which possesses additional algebraic structures of the *module* over the ring $\mathbb{C}[[x]]$. The *commutator* (Lie bracket) of formal fields is defined in the usual way as $[\mathbf{F}, \mathbf{G}] = \mathbf{FG} - \mathbf{GF}$.

In a parallel way, a vector formal series $H = (h_1, \dots, h_n) \in \mathbb{C}[[x]]^n$ can be identified with an *automorphism* $\mathbf{H} \in \operatorname{Aut} \mathbb{C}[[x]]$ of the algebra $\mathbb{C}[[x]]$ if $H(0) = 0$, i.e., $h_j \in \mathfrak{m}$. Under this assumption, for any formal series $f = \sum_\alpha c_\alpha x^\alpha \in \mathbb{C}[[x]]$ one can correctly define the *substitution*

$$\mathbf{H}f(x) = f(H(x)) = \sum_{\alpha \geqslant 0} c_\alpha h^\alpha = \sum_{\alpha \geqslant 0} c_\alpha h_1^{\alpha_1}(x) \cdots h_n^{\alpha_n}(x). \qquad (3.2)$$

Indeed, any k-truncation of $f(H(x))$ is completely determined by the k-truncations of f and H. We will say that $\mathbf{H}$ is *tangent to identity*, if $j^1\mathbf{H} = \mathrm{id}$.

The operator $\mathbf{H}$ defined by (3.2), is an *automorphism* of the algebra $\mathbb{C}[[x]]$, a $\mathbb{C}$-linear map respecting the multiplication,

$$\mathbf{H}\colon \mathbb{C}[[x]] \to \mathbb{C}[[x]], \qquad \mathbf{H}(fg) = \mathbf{H}f \cdot \mathbf{H}f.$$

Conversely, any homomorphism preserving convergence in $\mathbb{C}[[x]]$ is of the form $f \mapsto f \circ H$ for an appropriate vector series $H \in \mathbb{C}[[x]]^n$ with the *components* $h_j = \mathbf{H}x_j \in \mathbb{C}[[x]]$. By a *formal map* we mean either H or $\mathbf{H}$, depending on the context. If $\mathbf{H}$ is an homomorphism, then it must map the maximal ideal $\mathfrak{m} \subset \mathbb{C}[[x]]$ into itself and hence $h_j(0) = 0$, $j = 1, \dots, n$, which can be abbreviated to $H(0) = 0$.

If $\mathbf{H}$ is invertible (an isomorphism of the algebra $\mathbb{C}[[x]]$), we say it is a *formal isomorphism* of $\mathbb{C}^n$ at the origin. The collection of such isomorphisms forms a *group* denoted by $\mathrm{Diff}[[\mathbb{C}^n, 0]]$ with the operation of composition. The latter can be defined either via substitution of the series, or as the composition of the operators acting on $\mathbb{C}[[x]]$.

Since the maximal ideal $\mathfrak{m}$ is preserved by any formal map $\mathbf{H} \in \mathrm{Diff}[[\mathbb{C}^n, 0]]$ and any *singular* formal vector field $\mathbf{F} \in \mathcal{D}[[\mathbb{C}^n, 0]]$, $F(0) = 0$,

$$\mathbf{H}(\mathfrak{m}) = \mathfrak{m}, \qquad \mathbf{F}(\mathfrak{m}) \subseteq \mathfrak{m},$$

truncation of the series at the level of k-jets commutes with the action of $\mathbf{H}$ and $\mathbf{F}$, therefore defining correctly the isomorphism $j^k\mathbf{H}\colon J^k(\mathbb{C}^n, 0) \to J^k(\mathbb{C}^n, 0)$ and derivation $j^k\mathbf{F}\colon J^k(\mathbb{C}^n, 0) \to J^k(\mathbb{C}^n, 0)$ respectively, which can be identified with the k-jets of the formal map H and the formal vector field F. We wish to stress that $j^k\mathbf{F}$ is defined as an automorphism of the finite-dimensional jet space only if $F(0) = 0$.

3B. Inverse function theorem. For future purposes we will need the formal inverse function theorem.

Theorem 3.6. *Let H be a formal map with the linearization matrix $A = \left(\frac{\partial H}{\partial x}\right)(0)$ which is nondegenerate. Then H is invertible in* $\mathrm{Diff}[[\mathbb{C}^n, 0]]$.

If $A = E$ is the identity matrix and $H = (h_1, \dots, h_n)$, $h_i(x) = x_i + v_i(x) \bmod \mathfrak{m}^{k+1}$, where v_i are homogeneous polynomials of degree $k \geqslant 2$, then the formal inverse map $H^{-1} = (h'_1, \dots, h'_n)$ has the components $h'_i(x) = x_i - v_i(x) \bmod \mathfrak{m}^{k+1}$.

Clearly, the first assertion of the theorem follows from the second assertion applied to the formal map $A^{-1}H$.

Recall that a finite-dimensional linear operator $A\colon \mathbb{C}^n \to \mathbb{C}^n$ is *unipotent*, if $A - E$ is nilpotent, $(A - E)^n = 0$.

Lemma 3.7. *If $H \in \mathrm{Diff}[[\mathbb{C}^n, 0]]$ is a formal map with the identical linearization matrix $(\frac{\partial H}{\partial x})$, then its truncation $j^k\mathbf{H}$ considered as an automorphism of the finite-dimensional jet algebras $J^k(\mathbb{C}^n, 0)$, is a unipotent map for any finite order k.*

Proof. For any monomial x^α from the canonical basis, $\mathbf{H}x^\alpha = x^\alpha +$(higher order terms)$=$ $x^\alpha +$(linear combination of monomials of higher `deglex`-order). □

Proof of Theorem 3.6. Consider the homomorphism $\mathbf{H} \in \mathrm{Aut}\,\mathbb{C}[[x]]$ and denote $\mathbf{N} = \mathbf{H} - \mathbf{E}$ the formal "finite difference" operator ($\mathbf{E} = \mathrm{id}$ denotes the identical operator), $\mathbf{N}f = f \circ H - f$ (in the sense of the substitution of formal series). By Lemma 3.7, all finite truncations $j^k\mathbf{N}$ are nilpotent.

Define the operator H^{-1} as the series

$$\mathbf{H}^{-1} = \mathbf{E} - \mathbf{N} + \mathbf{N}^2 - \mathbf{N}^3 \pm \cdots . \tag{3.3}$$

This series converges (in fact, stabilizes) after truncation to any finite order because of the above nilpotency, hence by definition converges to an operator on $\mathbb{C}[[x]]$ satisfying the identities $\mathbf{H}\circ\mathbf{H}^{-1}=\mathbf{H}^{-1}\circ\mathbf{H}=\mathbf{E}$. It is an homomorphism of algebra(s), since for any $a,b\in\mathbb{C}[[x]]$ and their images $a'=\mathbf{H}a$, $b'=\mathbf{H}b$ which also can be chosen arbitrarily, we have $\mathbf{H}(ab)=a'b'$ and therefore

$$\mathbf{H}^{-1}(a'b')=\mathbf{H}^{-1}\mathbf{H}(ab)=ab=(\mathbf{H}^{-1}a')(\mathbf{H}^{-1}b').$$

Direct computation of the components of the inverse map yields

$$h_i'=\mathbf{H}^{-1}x_i=x_i-\mathbf{N}x_i+\cdots=x_i-(h_i(x)-x_i)+\cdots=x_i-v_i(x)+\cdots$$

as asserted by the theorem. □

The above formal construction is the algebraization of the recursive computation of the Taylor coefficients of the formal inverse map $H^{-1}(x)$. Note that stabilization of truncations of the series (3.3) means that computation of the terms of any finite degree k of the components h_i' of the inverse map is achieved in a finite (depending on k) number of steps.

3C. Integration and formal flow of formal vector fields. Consider an (autonomous) formal ordinary differential equation

$$\dot{x}=F(x),\qquad F=(F_1,\dots,F_n)\in\mathcal{D}[[\mathbb{C}^n,0]]\cong\mathbb{C}[[x]]^n \tag{3.4}$$

with a *formal* right hand side part F. Since evaluation of a formal series at any point other than the origin makes no sense, the “standard” definition of solutions can at best be applied to constructing a solution with the initial condition $x(0)=0$. Yet in the most interesting case where $F(0)=0$, this solution is trivial, $x(t)\equiv 0$.

The alternative, suggested by Remark 1.20, is to define a *one-parametric subgroup of formal self-maps* $\{H^t\colon t\in\mathbb{C}\}\subset\operatorname{Diff}[[\mathbb{C}^n,0]]$ satisfying the condition

$$H^t\circ H^s=H^{t+s}\qquad\forall t,s\in\mathbb{C},\qquad H^0=E. \tag{3.5}$$

Together with the group $\{H^t\}$ of self-maps we always consider the corresponding one-parameter group of automorphisms $\{\mathbf{H}^t\}\subset\operatorname{Aut}\mathbb{C}[[x]]$.

This subgroup is said to be *holomorphic*, if all finite truncations j^kH^t depend holomorphically on t. For a holomorphic subgroup the derivative

$$\mathbf{F}=\left.\frac{d\mathbf{H}^t}{dt}\right|_{t=0}=\lim_{t\to 0}t^{-1}(\mathbf{H}^t-\mathbf{E})\colon\mathbb{C}[[x]]\to\mathbb{C}[[x]] \tag{3.6}$$

is a formal vector field,

$$\begin{aligned}\mathbf{F}(fg) &= \frac{d}{dt}\bigg|_{t=0}\mathbf{H}^t(fg) = \frac{d}{dt}\bigg|_{t=0}\big[(\mathbf{H}^t f)(\mathbf{H}^t g)\big] \\ &= \left[\frac{d}{dt}\bigg|_{t=0}(\mathbf{H}^t f)\right](\mathbf{H}^0 g) + (\mathbf{H}^0 f)\left[\frac{d}{dt}\bigg|_{t=0}(\mathbf{H}^t g)\right] \\ &= g\,\mathbf{F}f + f\,\mathbf{F}g.\end{aligned}$$

Definition 3.8. A holomorphic one-parametric subgroup of formal self-maps $\{H^t\} \subseteq \operatorname{Diff}[[\mathbb{C}^n, 0]]$ is a *formal flow* of the formal vector field F corresponding to the derivation $\mathbf{F} \in \operatorname{Der}\mathbb{C}[[x]]$, if the corresponding group of automorphisms $\{\mathbf{H}^t\}$ satisfies the identity

$$\mathbf{F} = \frac{d\mathbf{H}^t}{dt}\bigg|_{t=0} \in \operatorname{Der}\mathbb{C}[[x]]. \tag{3.7}$$

The formal field F is called the *generator* of the subgroup $\{H^t\}$.

The above observation means that *any analytic one-parametric subgroup* of formal maps is always a formal flow of some formal field F (3.7). The following theorem is a formal analog of Proposition 1.19 showing that, conversely, *any* formal vector field F generates an holomorphic one-parametric subgroup of formal self-maps $\{H^t\} \subset \operatorname{Diff}[[\mathbb{C}, 0]]$.

Denote by $\mathbf{F}^m$ the iterated composition $\mathbf{F} \circ \cdots \circ \mathbf{F} \colon \mathbb{C}[[x]] \to \mathbb{C}[[x]]$ (m times) and consider the exponential series

$$\mathbf{H}^t = \exp t\mathbf{F} = \mathbf{E} + t\mathbf{F} + \frac{t^2}{2!}\,\mathbf{F}^2 + \cdots + \frac{t^m}{m!}\,\mathbf{F}^m + \cdots. \tag{3.8}$$

Theorem 3.9. *Any singular formal vector field F admits a formal flow $\{H^t\}$. This flow is defined by the series* (3.8) *which converges for all values of $t \in \mathbb{C}$ and depends analytically on t.*

Proof. We have to show that this series converges and its sum is an isomorphism of the algebra $\mathbb{C}[[x]]$ for any $t \in \mathbb{C}$. Then the identity (3.7) will follow automatically by the termwise differentiation of the series (3.8).

Convergence of the series (3.8) can be seen from the following argument. Let k be any finite order. Truncating the series (3.8), i.e., substituting $j^k\mathbf{F}$ instead of $\mathbf{F}$, we obtain a matrix formal power series. This series is always convergent: for an arbitrary choice of the norm $|\cdot|$ on the finite-dimensional space $J^k(\mathbb{C}^n, 0)$ the norm of the operator $j^k\mathbf{F}$ is finite, $|j^k\mathbf{F}| = r < +\infty$, and hence the series (3.8) is majorized by the convergent scalar series $1 + |t|r + |t|^2r^2/2! + \cdots = \exp|t|r < +\infty$ for any finite $t \in \mathbb{C}$; cf. with Definition 1.7. Denote its sum by $\exp j^k\mathbf{F} \colon J^k(\mathbb{C}^n, 0) \to J^k(\mathbb{C}^n, 0)$.

Truncations $\exp j^k\mathbf{F}$ for different orders k agree in common terms: if $l > k$, then $j^k(\exp t\, j^l\mathbf{F}) = \exp t\, j^k\mathbf{F}$. This allows us to define the sum of the series $\exp t\mathbf{F}$ as a linear operator $\mathbf{H}^t\colon \mathbb{C}[[x]] \to \mathbb{C}[[x]]$ via its finite truncations of all orders.

The group property $\mathbf{H}^{t+s} = \mathbf{H}^t \circ \mathbf{H}^s$ equivalent to the group property (3.5), follows from the formal identity $\exp(t+s) = \exp t \cdot \exp s$, since $t\mathbf{F}$ and $s\mathbf{F}$ obviously commute. It remains to show that $\mathbf{H}^t$ is an algebra *homomorphism*, i.e., $\mathbf{H}^t(fg) = \mathbf{H}^t f\, \mathbf{H}^t g$ for any two series $f, g \in \mathbb{C}[[x]]$.

By the iterated Leibnitz rule, for any $f, g \in \mathbb{C}[[x]]$,

$$\mathbf{F}^k(fg) = \sum_{p+q=k} \tfrac{(p+q)!}{p!q!}\, \mathbf{F}^p f \cdot \mathbf{F}^q g.$$

Substituting this identity into the exponential series, we have

$$\begin{aligned}\mathbf{H}^t(fg) &= \sum_k \tfrac{t^k}{k!}\, \mathbf{F}^k(fg) = \sum_k \sum_{p+q=k} \tfrac{t^{p+q}}{p!q!}\, \mathbf{F}^p f \cdot \mathbf{F}^q g \\ &= \Bigl(\sum_p \tfrac{t^p}{p!}\, \mathbf{F}^p f\Bigr) \cdot \Bigl(\sum_q \tfrac{t^q}{q!}\, \mathbf{F}^q g\Bigr) = \mathbf{H}^t f \cdot \mathbf{H}^t g. \quad \square\end{aligned}$$

Motivated by the series (3.8), we will often use the exponential notation: if F is a formal or analytic vector field with a singular point at the origin, we will denote by $\exp tF$ the time t flow (formal or analytic) of this field.

3D. Embedding in the flow and matrix logarithms.

Definition 3.10. A holomorphic germ $H \in \operatorname{Diff}(\mathbb{C}^n, 0)$ or a formal self-map $H \in \operatorname{Diff}[[\mathbb{C}^n, 0]]$ is said to be *embeddable*, if there exists a holomorphic germ of a vector field F (resp., a formal vector field $F \in \mathcal{D}[[\mathbb{C}^n, 0]]$) such that H is a time one (resp., formal time one) flow map of F, i.e., $H = \exp F$.

For a linear system $\dot x = Ax$ with constant coefficients, the flow consists of *linear* maps $x \mapsto (\exp tA)x$; see (1.12). Therefore for a *linear* map $x \mapsto Mx$, $M \in \mathrm{GL}(n, \mathbb{C})$, it is natural to consider the embedding problem in the class of linear vector fields $F(x) = Ax$. Then the problem reduces to finding a *matrix logarithm*, a matrix solution of the equation

$$\exp A = M, \qquad A, M \in \operatorname{Mat}(n, \mathbb{C}). \tag{3.9}$$

Clearly, the necessary condition for solvability of this equation is *nondegeneracy* of M. It also turns out to be sufficient for matrices over the field $\mathbb{C}$.

Lemma 3.11. *For any nondegenerate matrix $M \in \operatorname{Mat}(n, \mathbb{C})$, $\det M \neq 0$, there exists the matrix logarithm $A = \ln M$, a complex matrix satisfying the equation* (3.9)

Proof. We give two constructions of matrix logarithms for nondegenerate matrices.

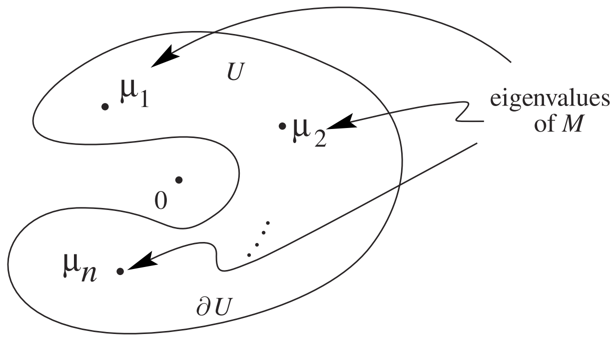

Figure I.3. Construction of the integral representation of the matrix logarithm for a nondegenerate matrix with the given spectrum

1. First, for a scalar matrix $M = \lambda E$, $0 \neq \lambda \in \mathbb{C}$, the logarithm can be defined as $\ln M = (\ln \lambda)\, E$, for any choice of $\ln \lambda$. A matrix having a single nonzero eigenvalue of high multiplicity has the form $M = \lambda(E + N)$, where N is a nilpotent (upper-triangular) matrix. Its logarithm can be defined using the formal series for the scalar logarithm as follows:

$$\ln M = \ln(\lambda E) + \ln(E + N) = (\ln \lambda)\, E + N - \tfrac{1}{2}N^2 + \tfrac{1}{3}N^3 - \cdots \qquad (3.10)$$

(the sum is finite). This formula gives a well-defined answer by virtue of the formal identity $\exp(x - \frac{x^2}{2} + \frac{x^3}{3} \pm \dots) = 1 + x$, since the matrices E and N commute.

An arbitrary matrix M can be reduced to a block diagonal form with each block having a single eigenvalue. The block diagonal matrix formed by logarithms of individual blocks solves the problem of computing the matrix logarithm in the general case.

2. The second proof uses the integral representation for analytic matrix functions. For any function $f(x)$ complex analytic in a domain $U \subset \mathbb{C}$ bounded by a simple curve ∂U and any matrix M with all eigenvalues in U, the value $f(M)$ can be defined by the contour integral

$$f(M) = \frac{1}{2\pi i} \oint_{\partial U} f(\lambda)(\lambda E - M)^{-1}\, d\lambda \qquad (3.11)$$

[**Gan59**, Ch. V, §4]. In application to $f(x) = \ln x$ we have to choose a simple loop ∂U containing all eigenvalues of M inside U but the origin $\lambda = 0$ outside (cf. with Fig. I.3). Then in the domain U one can unambiguously select a branch of complex logarithm $\ln \lambda$ which can be substituted into the integral representation.

To prove that the integral representation gives the same answer as before, it is sufficient to verify it only for the diagonal matrices, when the inverse can be computed explicitly. The advantage of this formula is the possibility of bounding the norm $|\ln M|$ defined by the above integral, in terms of $|M|$ and $|M^{-1}|$. □

Remark 3.12. The matrix logarithm is *by no means unique.* In the first construction one has the freedom to choose branches of logarithms of eigenvalues arbitrarily and independently for different Jordan blocks. In the second construction besides choosing the branch of the logarithm, there exists a freedom to choose the domain U (i.e., the loop ∂U encircling all the eigenvalues of M but not the origin).

Remark 3.13. There is a natural obstruction for extracting the matrix logarithm in the class of *real* matrices. If $\exp A = M$ for some real matrix A, then M can be connected with the identity E by a path of nondegenerate matrices $\exp tA$, in particular, M should be orientation-preserving. If M is nondegenerate but orientation-reverting, it has no real matrix logarithm.

However, there are more subtle obstructions. Consider the real matrix $M = \left(\begin{smallmatrix} -1 & 1 \\ & -1 \end{smallmatrix}\right)$ with determinant 1. If $M = \exp A$, then by (1.16) $\exp \operatorname{tr} A = 1$ so that for a real matrix necessarily $\operatorname{tr} A = 0$. The two eigenvalues cannot be simultaneously zero, since then the exponent will have the eigenvalues both equal to 1. Therefore the eigenvalues must be different, in which case the matrix A and hence its exponent M must be diagonalizable. The contradiction shows impossibility of solving the equation $\exp A = M$ in the class of real matrices.

3E. Logarithms and derivations. Inspired by the construction of the matrix exponential, one can attempt to prove that for any formal map $H \in \operatorname{Diff}[[\mathbb{C}^n, 0]]$ there exists a formal vector field F whose formal time one flow coincides with H, as follows.

Consider an arbitrary finite order k and the k-jet $\mathbf{H}_k = j^k\mathbf{H}$ considered as an isomorphism of the finite-dimensional $\mathbb{C}$-algebra $\mathfrak{F}^k = J^k(\mathbb{C}^n, 0)$. By Lemma 3.11, there exists a linear map $\mathbf{F}_k \colon \mathfrak{F}^k \to \mathfrak{F}^k$ such that $\exp \mathbf{F}_k = \mathbf{H}_k$.

Assume that for some reasons

(i) jets of the logarithms $\mathbf{F}_k$ of different orders agree after truncation, i.e., $j^k\mathbf{F}_l = \mathbf{F}_k$ for $l > k$, and

(ii) each $\mathbf{F}_k$ is a *derivation* of the commutative algebra $\mathfrak{F}^k$, thus a k-jet of a vector field.

Then together these jets would define a derivation $\mathbf{F}$ of the algebra $\mathfrak{F} = \mathbb{C}[[x]]$.

The first objective can be achieved if $\mathbf{F}_k$ are truncations of some polynomial or infinite series. There is only one such candidate, the *logarithmic series* $\ln \mathbf{H} \colon \mathbb{C}[[x]] \to \mathbb{C}[[x]]$, obtained from the formal series for

$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 \mp \cdots$ by substitution,

$$\ln \mathbf{H} = (\mathbf{H} - \mathbf{E}) - \frac{1}{2}(\mathbf{H} - \mathbf{E})^2 + \frac{1}{3}(\mathbf{H} - \mathbf{E})^3 \mp \cdots \tag{3.12}$$

(cf. with (3.10)). Until the end of this section we use the notation $\ln \mathbf{H}$ only in the sense of the series (3.12).

The series for $\ln \mathbf{H}$ does not converge everywhere even in the finite-dimensional case: the domain of convergence contains the ball $|\mathbf{H} - \mathbf{E}| < 1$ and all unipotent finite-dimensional matrices, but most certainly *not* the matrix $-\mathbf{E}$. Besides that difficulty, it is absolutely not clear why the formal logarithm of an isomorphism of the commutative algebra $\mathbb{C}[[x]]$, even if it converges, must be a derivation: no simple arguments similar to the one used in the proof of Theorem 3.9, exist (sometimes this circumstance is overlooked).

Let $\mathfrak{F}$ be a commutative $\mathbb{C}$-algebra of finite dimension n over $\mathbb{C}$ and $\mathbf{H}$ an automorphism of $\mathfrak{F}$.

Theorem 3.14. *The series* (3.12) *converges for all unipotent automorphisms* $\mathbf{H}$ *of a finite dimensional algebra* $\mathfrak{F}$ *and its sum* $\mathbf{F} = \ln \mathbf{H}$ *in this case is a derivation of this algebra.*

Proof using the Lie group tools. Consider the matrix Lie group $\mathfrak{T} \subset \mathrm{GL}(n, \mathbb{C})$ of upper-triangular matrices with units on the principal diagonal and the corresponding Lie algebra $\mathfrak{t} \subset \mathrm{Mat}(n, \mathbb{C})$ of *strictly* upper-triangular matrices.

The exponential series (3.8) and the matrix logarithm (3.12) restricted on $\mathfrak{t}$ and $\mathfrak{T}$ respectively, are *polynomial* maps defined everywhere. They are mutually inverse: for any $\mathbf{F} \in \mathfrak{t}$ and $\mathbf{H} \in \mathfrak{T}$ we have $\ln \exp \mathbf{F} = \mathbf{F}$ and $\exp \ln \mathbf{H} = \mathbf{H}$. This follows from the identities $\ln e^z = z$, $e^{\ln w} = w$ expanded in the series. In particular, exp is surjective.

For any Lie subalgebra $\mathfrak{g} \subseteq \mathfrak{t}$ and the corresponding Lie subgroup $\mathfrak{G} \subseteq \mathfrak{T}$ the exponential map $\exp \colon \mathfrak{g} \to \mathfrak{G}$ is the restriction of (3.8) on $\mathfrak{g}$.

By [**Var84**, Theorem 3.6.2], the exponential map remains surjective also on $\mathfrak{G}$, i.e., $\exp \mathfrak{g} = \mathfrak{G}$. We claim that in this case the logarithm maps $\mathfrak{G}$ *into* $\mathfrak{g}$. Indeed, if $\mathbf{H} \in \mathfrak{G}$ and $\mathbf{H} = \exp \mathbf{F}$ for some $\mathbf{F} \in \mathfrak{g}$, then $\ln \mathbf{H} = \ln \exp \mathbf{F} = \mathbf{F} \in \mathfrak{g}$.

The assertion of the theorem arises if we take $\mathfrak{G} = \mathfrak{T} \cap \mathrm{Aut}(\mathfrak{F})$ to be the Lie subgroup of *triangular automorphisms* of $\mathfrak{F} \cong \mathbb{C}^n$ and $\mathfrak{g} = \mathfrak{t} \cap \mathrm{Der}(\mathfrak{F})$ of *triangular derivations* of the commutative algebra $\mathfrak{F}$. □

Remark 3.15. Surjectivity of the exponential map for a subgroup of the triangular group $\mathfrak{T}$ is a delicate fact that follows from the nilpotency of the Lie algebra $\mathfrak{t}$. Indeed, by the Campbell–Hausdorff formula, $\exp \mathbf{F} \cdot \exp \mathbf{G} =$

$\exp \mathbf{K}$, where $\mathbf{K} = \mathbf{K}(\mathbf{F}, \mathbf{G})$ is a series which in the nilpotent case is a polynomial map $\mathfrak{t} \times \mathfrak{t} \to \mathfrak{t}$ defined everywhere. Thus the image $\exp \mathfrak{g}$ is a *Lie subgroup* in $\mathfrak{G} \subseteq \mathfrak{T}$ for *any* subalgebra $\mathfrak{g}$, containing a small neighborhood of the unit $\mathbf{E}$. It is well known that any such neighborhood generates (by the group operation) the whole connected component of the unit, so that $\exp \mathfrak{g}$ coincides with this component. If $\mathfrak{G}$ is simply connected, then $\exp \mathfrak{g} = \mathfrak{G}$ as asserted.

Without nilpotency the answer may be different: as follows from Remark 3.13, for two Lie algebras $\mathfrak{gl}(n, \mathbb{R}) \subset \mathfrak{gl}(n, \mathbb{C})$ and the respective Lie groups $\mathrm{GL}(n, \mathbb{R}) \subset \mathrm{GL}(n, \mathbb{C})$, the exponent is surjective on the ambient (bigger) group but *not* on the subgroup.

Remark 3.16. Using similar arguments, one can prove that for an arbitrary automorphism $\mathbf{H} \in \mathrm{Aut}(\mathfrak{F})$ *sufficiently close to the unit* $\mathbf{E}$, the logarithm $\ln \mathbf{H}$ given by the series (3.12) is a derivation, $\ln \mathbf{H} \in \mathrm{Der}(\mathfrak{F})$. This follows from the fact that ln and exp are mutually inverse on sufficiently small neighborhoods of $\mathbf{E}$ and 0 respectively. However, the size of this neighborhood depends on $\mathfrak{F}$.

3F. Embedding in the formal flow. Based on Theorem 3.14, one can prove the following general result obtained by F. Takens in 1974; see [**Tak01**].

Theorem 3.17. *Let $H \in \mathrm{Diff}[[\mathbb{C}^n, 0]]$ be a formal map whose linearization matrix $A = \frac{\partial H}{\partial x}(0)$ is unipotent, $(A - E)^n = 0$.*

Then there exists a formal vector field $F \in \mathcal{D}[[\mathbb{C}^n, 0]]$ whose linearization is a nilpotent matrix N, such that H is the formal time 1 map of F.

Proof. As usual, we identify the formal map with an automorphism $\mathbf{H}$ of the algebra $\mathfrak{F} = \mathbb{C}[[x_1, \dots, x_n]]$ so that its finite k-jets $j^k\mathbf{H}$ become automorphisms of the finite dimensional algebras $\mathfrak{F}^k = J^k(\mathbb{C}^n, 0)$. Without loss of generality we may assume that the matrix A is upper-triangular so that the isomorphism $\mathbf{H}$ and all its truncations $j^k\mathbf{H}$ in the canonical `deglex`-ordered basis becomes upper-triangular with units on the diagonal: the jets $j^k\mathbf{H}$ are finite-dimensional upper-triangular (unipotent) automorphisms of the algebras $\mathfrak{F}^k$.

Consider the infinite series (3.12) together with its finite-dimensional truncations obtained by applying the operation j^k to all terms. Each such truncation is a logarithmic series for $\ln j^k\mathbf{H}$ which converges (actually, stabilizes after finitely many steps) and its sum is a derivation $j^k\mathbf{F}$ of $\mathfrak{F}^k$ by Theorem 3.14. Clearly, different truncations agree on the lower order terms, thus $\ln \mathbf{H}$ converges in the sense of Definition 3.4 to a derivation $\mathbf{F}$ of $\mathfrak{F}$. This derivation corresponds to the formal vector field F as required. □

Exercises and Problems for §3.

Problem 3.1. Let $F \in \mathcal{D}[[\mathbb{C}^n, 0]]$ be a formal vector field corresponding to the derivation $\mathbf{F} \in \operatorname{Der} \mathbb{C}[[x]]$, and $\{H^t\} \subset \operatorname{Diff}[[\mathbb{C}^n, 0]]$ its formal flow corresponding to the one-parametric group of automorphisms $\{\mathbf{H}^t\} \subset \operatorname{Aut} \mathbb{C}[[x]]$ related by the identity (3.7).

Prove that in this case $\frac{d}{dt} H^t = F \circ H^t$ for any t on the level of vector formal series.

Exercise 3.2. Consider the derivation $\mathbf{F} = \frac{\partial}{\partial x}$ on the algebra $\mathbb{C}[x]$ of univariate polynomials. Prove that the exponential series $\exp t\mathbf{F}$ is well defined for all values of $t \in \mathbb{C}$ as an automorphism of $\mathbb{C}[x]$, but is not defined if the algebra $\mathbb{C}[x]$ is replaced by the algebras $\mathbb{C}[[x]]$ or $\mathcal{O}(\mathbb{D})$, where $\mathbb{D} = \{|x| < 1\}$ is the unit disk.

Problem 3.3. Prove that the integral representation (3.11) coincides with the standard definition of a matrix function $f(M)$ in the case where f is a (scalar) polynomial.

Exercise 3.4. Find *all* complex logarithms of the matrix $M = \begin{pmatrix} -1 & 1 \\ & -1 \end{pmatrix}$ (i.e., solutions of the equation $\exp A = M$).

4. Formal normal forms

In the same way as holomorphic maps act on holomorphic vector fields by conjugacy (1.26), formal maps act on formal vector fields. In this section we investigate the *formal normal forms*, to which a formal vector field can be brought by a suitable formal isomorphism.

Definition 4.1. Two formal vector fields F, F' are *formally equivalent*, if there exists an invertible formal self-map H such that the identity (1.26) holds on the level of formal series.

The fields are formally equivalent if and only if the corresponding derivations $\mathbf{F}, \mathbf{F}'$ of the algebra $\mathbb{C}[[x]]$ are conjugated by a suitable isomorphism $\mathbf{H} \in \operatorname{Diff}[[\mathbb{C}^n, 0]]$ of the formal algebra: $\mathbf{H} \circ \mathbf{F}' = \mathbf{F} \circ \mathbf{H}$.

Obviously, two holomorphically equivalent (holomorphic) germs of vector fields are formally equivalent. The converse is in general not true, as the formal self-maps may be divergent.

4A. Formal classification theorem. Formal classification of formal vector fields strongly depends on its principal part, in particular, on properties of the linearization matrix $A = \left(\frac{\partial F}{\partial x}\right)(0)$ when the latter is nonzero (cases with $A = 0$ are hopelessly complicated if the dimension is greater than one).

We start with the most important example and introduce the definition of a resonance as a certain arithmetic (i.e., involving integer coefficients) relation between complex numbers.

Definition 4.2. An ordered tuple of complex numbers $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ is called *resonant*, or, more precisely, *additive resonance* if there exist nonnegative integers $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$ such that $|\alpha| \geqslant 2$ and the *resonance* identity occurs,

$$\lambda_j = \langle \alpha, \lambda \rangle \,, \qquad |\alpha| \geqslant 2. \tag{4.1}$$

Here $\langle \alpha, \lambda \rangle = \alpha_1\lambda_1 + \cdots + \alpha_n\lambda_n$. The natural number $|\alpha|$ is the *order* of the resonance.

A square matrix is resonant, if the collection of its eigenvalues (with repetitions if they are multiple) is resonant. A formal vector field $F = (F_1, \dots, F_n)$ at the origin is resonant if its linearization matrix $A = \left(\frac{\partial F}{\partial x}\right)(0)$ is resonant.

Though resonant tuples $(\lambda_1, \dots, \lambda_n)$ can be dense in some parts of $\mathbb{C}^n$ (see §5**A**), their measure is zero.

Theorem 4.3 (Poincaré linearization theorem)**.** *A nonresonant formal vector field $F(x) = Ax + \cdots$ is formally equivalent to its linearization $F'(x) = Ax$.*

The proof of this theorem is given in the sections §4**B**–§4**C**. In fact, it is the simplest particular case of a more general statement valid for resonant formal vector fields that appears in §4**D**.

4B. Induction step: homological equation. The proof of Theorem 4.3 goes by induction. Assume that the formal vector field F is already partially normalized, and contains no terms of order less than some $m \geqslant 2$:

$$F(x) = Ax + V_m(x) + V_{m+1}(x) + \cdots \,,$$

where $V_m, V_{m+1}, \dots$ are arbitrary homogeneous vector fields of degrees $m, m+1$, *etc.*

We show that in the assumptions of the Poincaré theorem, the term V_m can be removed from the expansion of F, i.e., that F is formally equivalent to the formal field $F'(x) = Ax + V'_{m+1} + \cdots$. Moreover, the corresponding conjugacy can be in fact chosen as a polynomial of the form $H(x) = x + P_m(x)$, where P_m is a homogeneous vector polynomial of degree m. The Jacobian matrix of this self-map is $E + \left(\frac{\partial P_m}{\partial x}\right)$.

The conjugacy H with these properties must satisfy the equation (1.26) on the formal level. Keeping only terms of order $\leqslant m$ from this equation and using dots to denote the rest, we obtain

$$\left(E + \frac{\partial P_m}{\partial x}\right)(Ax + V_m + \cdots) = A(x + P_m(x)) + V'_m(x + P_m(x)) + \cdots \,.$$

The homogeneous terms of order 1 on both sides coincide. The next non-trivial terms appear in the order m. Collecting them, we see that in order to meet the condition $V'_m = 0$, the vector of homogeneous terms P_m must satisfy the commutator identity

$$[\mathbf{A}, P_m] = -V_m, \qquad \mathbf{A}(x) = Ax, \tag{4.2}$$

where $\mathbf{A} = Ax$ is the linear vector field, the principal part of F, and the homogeneous vector polynomials P_m and V_m are considered as vector fields on $\mathbb{C}^n$. The left hand side of (4.2) is the commutator, $[\mathbf{A}, P](x) = \bigl(\frac{\partial P}{\partial x}\bigr)\,Ax - AP(x)$.

Conversely, if the condition (4.2) is satisfied by P_m, the polynomial map $H(x) = x + P_m(x)$ conjugates $F = \mathbf{A} + V_m + \cdots$ with the (formal) vector field $F'(x) = \mathbf{A} + \cdots$ having no terms of degree m.

Definition 4.4. The identity (4.2), considered as an equation on the unknown homogeneous vector field P_m, is called the *homological equation.*

4C. Solvability of the homological equation. Solvability of the homological equation depends on invertibility of the operator ad_A of commutation with the linear vector field $\mathbf{A}$.

Let $\mathcal{D}_m$ be the linear space of all homogeneous vector fields of degree m (we will be interested only in the case $m \geqslant 2$). This linear space has the *standard monomial basis* consisting of the fields

$$F_{k\alpha} = x^\alpha \tfrac{\partial}{\partial x_k}, \qquad k = 1, \dots, n,\ |\alpha| = m. \tag{4.3}$$

We shall order elements of this basis lexicographically so that x_i precedes x_j if $i < j$, but $\frac{\partial}{\partial x_j}$ precedes $\frac{\partial}{\partial x_i}$. To that end, we assign to each formal variable $x_1, \dots, x_n$ pairwise different positive weights $w_1 > \cdots > w_n$ that are *rationally independent*. This assignment extends on all monomials and monomial vector fields if the symbol $\frac{\partial}{\partial x_j}$ is assigned the weight $-w_j$. Now the monomial vector fields can be arranged in the decreasing order of their weights: the independence condition guarantees that the only different vector monomials having the same weight can be $x^\alpha \cdot x_j \frac{\partial}{\partial x_j}$ and $x^\alpha \cdot x_k \frac{\partial}{\partial x_k}$ with the same α and $j \neq k$. The order between these monomials is not essential for future exposition.

The operator

$$\operatorname{ad}_A \colon P \mapsto [\mathbf{A}, P], \qquad (\operatorname{ad}_A P)(x) = \left(\frac{\partial P}{\partial x}\right) \cdot Ax - AP(x), \tag{4.4}$$

preserves the space $\mathcal{D}_m$ for any $m \in \mathbb{N}$.

Lemma 4.5. *If A is nonresonant, then the operator* ad_A *is invertible. More precisely, if the coordinates $x_1, \dots, x_n$ are chosen such that A has the upper-triangular Jordan form, then* ad_A *is lower-triangular in the respective standard monomial basis ordered in the decreasing weight order.*

Proof. The assertion of the lemma is completely transparent when A is a diagonal matrix $\Lambda = \operatorname{diag}\{\lambda_1, \dots, \lambda_n\}$. In this case each $F_{k\alpha} \in \mathcal{D}_m$ is an eigenvector for $\operatorname{ad}_\Lambda$ with the eigenvalue $\langle \lambda, \alpha \rangle - \lambda_k$. Indeed, by the Euler identity,

$$F_{k\alpha} = x^\alpha \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \qquad \left(\frac{\partial F_{k\alpha}}{\partial x} \right) = x^\alpha \begin{pmatrix} & 0 & \\ & \vdots & \\ \frac{\alpha_1}{x_1} & \cdots & \frac{\alpha_n}{x_n} \\ & \vdots & \\ & 0 & \end{pmatrix},$$

so that in the diagonal case $\Lambda F_{k\alpha} = \lambda_k F_{k\alpha}$, and $\left(\frac{\partial F_{k\alpha}}{\partial x} \right) \Lambda x = \langle \lambda, \alpha \rangle F_{k\alpha}$. Being diagonal with nonzero eigenvalues, $\operatorname{ad}_\Lambda$ is invertible.

To prove the lemma in the general case where A is in the upper-triangular Jordan form, we consider the weight introduced above.

The operator $\operatorname{ad}_\Lambda$ with the diagonal matrix Λ preserves the weights, since all vector monomials are eigenvectors for it.

On the other hand, the monomial vector field $\mathbf{J}_j = x_j \frac{\partial}{\partial x_{j+1}}$ with the upper-diagonal constant matrix J_j acts by increasing weight. Indeed,

$$\left[x^\alpha \frac{\partial}{\partial x_k}, x_j \frac{\partial}{\partial x_{j+1}} \right] = x^\alpha \left[\frac{\partial}{\partial x_k}, x_j \frac{\partial}{\partial x_{j+1}} \right] + \alpha_{j+1} x^\alpha \frac{x_j}{x_{j+1}} \cdot \frac{\partial}{\partial x_k}.$$

The second term, if present, has higher weight than $F_{k\alpha} = x^\alpha \frac{\partial}{\partial x_k}$, since $w_j > w_{j+1}$. The first term is nonzero only if $j = k$, and in this case reduces to $x^\alpha \frac{\partial}{\partial x_{k+1}}$, which also has higher weight than $F_{k\alpha}$.

It remains to notice that an arbitrary matrix A in the upper-triangular Jordan normal form is the sum of the diagonal part Λ and a linear combination of matrices $J_1, \dots, J_{n-1}$. The operator ad_A linearly depends on A, so ad_A is equal to $\operatorname{ad}_\Lambda$ modulo a linear combination of the weight-increasing operators ad_{J_j}. Therefore, if the monomial fields $F_{k\alpha}$ are ordered in the decreasing order of their weights, as in the standard basis, then the operator ad_A is lower-triangular with the diagonal part $\operatorname{ad}_\Lambda$. □

Proof of Theorem 4.3. Now we can prove the Poincaré linearization theorem. By Lemma 4.5, the operator ad_A is invertible and therefore the homological equation (4.2) is always solvable no matter what the term

$V = V_m$ is. Repeating this process inductively, we can construct an infinite sequence of polynomial maps $H_1, H_2, \dots, H_m, \dots$ and the formal fields $F_1 = F$, $F_2, \dots, F_m, \dots$ such that $F_m = Ax +$ (terms of order m and more), and the transformation $H_m = \mathrm{id} +$(terms of order m and more) conjugates F_m with F_{m+1}. Thus the composition $H^{(m)} = H_m \circ \cdots \circ H_1$ conjugates the initial field F_1 with the field F_{m+1} without nonlinear terms up to order m.

It remains to notice that by construction of H_{m+1} the composition $H^{(m+1)} = H_{m+1} \circ H^{(m)}$ has the same terms of order $\leqslant m$ as $H^{(m)}$ itself. Thus the limit

$$H = H^{(\infty)} = \lim_{m\to\infty} H^{(m)}$$

(the infinite composition) exists in the class of formal morphisms. By construction, $H_* F$ cannot contain any nonlinear terms and hence is linear, as required. □

Remark 4.6. The formal map linearizing a nonresonant formal vector field and tangent to the identity, is unique. Indeed, otherwise there would exist a *nontrivial* formal map $\mathrm{id} + h$ which conjugates the linear field with itself,

$$\left(\frac{\partial h}{\partial x}\right) Ax = Ah(x), \qquad \text{i.e.,} \qquad \mathrm{ad}_A\, h = 0.$$

But in the nonresonant case the commutator ad_A is injective, hence $h = 0$.

Thus the only formal maps conjugating a linear field with itself, are linear maps $x \mapsto Bx$, with the matrix B commuting with A, $[A, B] = 0$.

4D. Resonant normal forms: Poincaré–Dulac paradigm. The inductive construction linearizing nonresonant vector fields, can be used to *simplify* the resonant ones.

In this *resonant* case the operator $\mathrm{ad}_A = [\mathbf{A}, \cdot]$ of commutation with the linear part may be no longer surjective and in general the condition $V'_m = 0$, meaning absence of terms of order m after the transformation, cannot be achieved.

In the presence of resonances one can choose in each linear space $\mathcal{D}_m$ a complementary (transversal) subspace $\mathcal{N}_m$ to the image of the operator ad_A, so that

$$\mathcal{D}_m = \mathcal{N}_m + \mathrm{ad}_A(\mathcal{D}_m) \tag{4.5}$$

(the sum should not necessarily be direct).

Theorem 4.7 (Poincaré–Dulac paradigm)**.** *If the subspaces $\mathcal{N}_m \subset \mathcal{D}_m$ are transversal to the image of the commutator ad_A as in* (4.5)*, then any formal vector field $F(x) = Ax + \cdots$ with the linearization matrix A is formally conjugated to some formal vector field whose all nonlinear terms of degree m belong to the subspace $\mathcal{N}_m$.*

Proof. If the transformation $H_m(x) = x + P_m$ conjugates the field $F(x) = Ax + \cdots + V_m(x) + \cdots$ with another field $F'(x) = Ax + \cdots + V'_m(x) + \cdots$ *with the same* $(m-1)$*-jet* on the level of terms of order m, then instead of the homological equation (4.2) in the case $V'_m \neq 0$, the correction term P_m must satisfy the equation

$$\operatorname{ad}_A P_m = V'_m - V_m. \tag{4.6}$$

If $\mathcal{N}_m$ satisfies (4.5), then (4.6) can always be solved with respect to P_m for any V_m provided that V'_m is suitably chosen from the subspace $\mathcal{N}_m$.

The transform H_m does not affect the lower order terms and hence the process can be iterated for larger values of m exactly as in the nonresonant case. As a result, one can prove that any formal vector field F is formally equivalent to a formal field containing only terms belonging to the "complementary" parts $\mathcal{N}_m$ for all $m = 2, 3, \ldots$.

The rest of the proof of Theorem 4.7 is the same as that of the Poincaré–Dulac theorem. □

The choice of the transversal subspaces $\mathcal{N}_m$ depends on ad_A, hence on the matrix A itself.

Example 4.8. Assume that the matrix $A = \Lambda = \operatorname{diag}\{\lambda_1, \ldots, \lambda_n\}$ is diagonal. In this case the operator $\operatorname{ad}_\Lambda$ was already shown to be diagonal in the vector monomial basis, eventually with some zeros among the eigenvalues. For diagonal operators on finite-dimensional space the kernel and the image are complementary subspaces, so one may choose $\mathcal{N}_m = \ker \operatorname{ad}_L \subset \mathcal{D}_m$. The kernel of the diagonal operator $\operatorname{ad}_\Lambda$ can be immediately described.

Definition 4.9. A *resonant vector monomial* corresponding to the resonance $\lambda_k - \langle \lambda, \alpha \rangle = 0$, is the monomial vector field $F_{k\alpha} = x^\alpha \frac{\partial}{\partial x_k}$; see (4.3).

The kernel $\ker \operatorname{ad}_\Lambda$ consists of linear combinations of resonant monomials. From the discussion above it follows immediately that a formal vector field with diagonal linear part Λx is formally equivalent to the vector field with the same linear part and only resonant monomials among the nonlinear terms.

Actually, the assumption on diagonalizability is redundant. The following statement is one of the most popular formal classification results.

Theorem 4.10 (Poincaré–Dulac theorem)**.** *A formal vector field is formally equivalent to a vector field with the linear part in the Jordan normal form and only resonant monomials in the nonlinear part.*

Proof. Assume that the coordinates are already chosen so that the linearization matrix A is Jordan upper-triangular.

Choose the subspace $\mathcal{N}_m$ as the linear span of all resonant monomials, $\mathcal{N}_m = \bigoplus_{\langle\lambda,\alpha\rangle-\lambda_k=0} \mathbb{C}\cdot F_{k\alpha}$.

By Lemma 4.5, the operator $L_m = \operatorname{ad}_A|_{\mathcal{D}_m}$ is upper triangular with the expressions $\langle\lambda,\alpha\rangle - \lambda_k = 0$ on the diagonal. By the choice of $\mathcal{N}_m$, whenever zero occurs on the diagonal of L, the corresponding basis vector was included in $\mathcal{N}_m$. This obviously means (4.5). The rest is the Poincaré–Dulac paradigm. □

4E. Belitskii theorem. The choice of the "resonant normal form" (i.e., of the family of subspaces $\mathcal{N}_m$) in Theorem 4.10, is excessive in the sense that the *dimension* of these spaces (the number of parameters in the normal form) is not minimal. For example, if A is a nonzero nilpotent Jordan matrix, then *all* monomials are resonant in the sense of Definition 4.9, whereas the image of ad_A is clearly nontrivial. We describe now one possible *minimal* choice, introduced by G. Belitskii [**Bel79**, Ch. II, §7].

Consider the standard Hermitian structure on the space $\mathbb{C}^n$, so that the basis vectors $e_j = \frac{\partial}{\partial x_j}$ form an orthonormal basis.

For any natural $m \geqslant 1$ denote by $\mathcal{H}_m$ the complex linear space of all homogeneous polynomials of degree m. We introduce the *standard Hermitian structure* in $\mathcal{H}_m$ in such a way that the normalized monomials $\varphi_\alpha = \frac{1}{\sqrt{\alpha!}}\,x^\alpha$ form an orthonormal basis,

$$(\varphi_\alpha, \varphi_\beta) = \delta_{\alpha\beta}, \qquad \varphi_\alpha = \frac{1}{\sqrt{\alpha!}}\,x^\alpha, \quad \alpha,\beta \in \mathbb{Z}^n_+,\ |\alpha| = |\beta| = m. \tag{4.7}$$

Here, as usual, $\alpha! = \alpha_1!\cdots\alpha_n!$ for $\alpha = (\alpha_1,\dots,\alpha_n)$, $0! = 1$ and $\delta_{\alpha\beta}$ is the standard Kronecker symbol.

Then the linear space $\mathcal{D}_m$ of homogeneous vector fields of degree m can be naturally identified with the tensor product $\mathcal{D}_m = \mathcal{H}_m \otimes_{\mathbb{C}} \mathbb{C}^n$ and inherits the standard Hermitian structure for which the monomials $\varphi_\alpha \otimes e_k = \frac{1}{\sqrt{\alpha!}}\,F_{\alpha k}$ form an orthonormal basis.

Given a matrix $A \in \operatorname{Mat}(n,\mathbb{C})$, denote by A^* the *adjoint matrix* obtained from A by transposition and complex conjugacy: $a^*_{ij} = \bar a_{ji}$. If $\mathbf{A}(x) = Ax$ is the corresponding linear vector field on $\mathbb{C}^n$ and, respectively, $\mathbf{A}^*(x) = A^*x$, then both $\mathbf{A}, \mathbf{A}^*$ act as linear differential operators, $\mathbf{A} = \sum a_{ij}x_i\frac{\partial}{\partial x_j}$ and $\mathbf{A}^* = \sum \bar a_{ji}x_i\frac{\partial}{\partial x_j}$, on $\mathcal{H}_m$. Furthermore, the commutation operators $\operatorname{ad}_A = [\mathbf{A},\cdot]$ and $\operatorname{ad}_{A^*} = [\mathbf{A}^*,\cdot]$ are linear operators on $\mathcal{D}_m$.

The following statement claims that the operators in each pair are mutually adjoint (dual to each other) with respect to the standard Hermitian structures on the respective spaces.

Lemma 4.11.

1. The derivation $\mathbf{A}^*\colon \mathcal{H}_m \to \mathcal{H}_m$ *is adjoint to the derivation* $\mathbf{A}$ (*with respect to the standard Hermitian structure*) *and vice versa.*

2. The commutator $\operatorname{ad}_{A^*} = [\mathbf{A}^*, \cdot]\colon \mathcal{D}_m \to \mathcal{D}_m$ *is adjoint to the commutator* $\operatorname{ad}_A = [\mathbf{A}, \cdot]$ (*with respect to the standard Hermitian structure*) *and vice versa.*

Proof. 1. The identity $(\mathbf{A}f, g) = (f, \mathbf{A}^*g)$ for any pair of polynomials $f, g \in \mathcal{H}_m$ "linearly" depends on the matrix A: if it holds for two matrices $A, A' \in \operatorname{Mat}(n, \mathbb{C})$, then it also holds for their combination $\lambda A + \lambda' A'$ with any two complex numbers $\lambda, \lambda' \in \mathbb{C}$.

Thus it is sufficient to verify the assertion for the monomial derivations $\mathbf{A} = x_i \frac{\partial}{\partial x_j}$ and $\mathbf{A}^* = x_j \frac{\partial}{\partial x_i}$.

If $i = j$, then $\mathbf{A} = \mathbf{A}^* = x_i \frac{\partial}{\partial x_i}$ is diagonal in the orthonormal basis $\{\varphi_\alpha\}$ with the real eigenvalues $\lambda_\alpha = \alpha_i = \alpha_j \in \mathbf{Z}_+$, and hence is self-adjoint.

Otherwise both $\mathbf{A}$ and $\mathbf{A}^*$ can be represented as permutations of the basic vectors composed with the diagonal operators. If β is the multi-index obtained from α by the operation

$$\beta_k = \begin{cases} \alpha_k, & k \neq i, j, \\ \alpha_i + 1, & k = i, \\ \alpha_j - 1, & k = j, \end{cases} \qquad \alpha_k = \begin{cases} \beta_k, & k \neq i, j, \\ \beta_i - 1, & k = i, \\ \beta_j + 1, & k = j, \end{cases}$$

then $\beta!/\alpha! = (\alpha_i + 1)/\alpha_j = \beta_i/\alpha_j$ and

$$\mathbf{A}\varphi_\alpha = \frac{\alpha_j}{\sqrt{\alpha!}}\, x^\beta = \alpha_j \frac{\sqrt{\beta!}}{\sqrt{\alpha!}}\, \varphi_\beta = \alpha_j \frac{\sqrt{\beta_i}}{\sqrt{\alpha_j}}\, \varphi_\beta = \sqrt{\alpha_j \beta_i}\, \varphi_\beta.$$

Reciprocally, $\mathbf{A}^*\varphi_\beta = \beta_i\, x^\alpha/\sqrt{\beta!} = \cdots = \sqrt{\beta_i \alpha_j}\, \varphi_\alpha$. But since the vectors φ_α form an orthonormal basis,

$$(\mathbf{A}\varphi_\alpha, \varphi_\beta) = (\varphi_\alpha, \mathbf{A}^*\varphi_\beta) = \sqrt{\beta_i \alpha_j} \in \mathbb{R}$$

and all other matrix entries in the basis $\{\varphi_\alpha\}$ are zeros. Therefore the derivations $\mathbf{A}$ and $\mathbf{A}^*$ are mutually adjoint on $\mathcal{H}_m$.

2. Using the structure of the tensor product $\mathcal{D}_m = \mathcal{H}_m \otimes \mathbb{C}^n$, one can represent the commutators as follows:

$$\operatorname{ad}_A = \mathbf{A} \otimes E - \operatorname{id} \otimes A.$$

Indeed, for any element φv, where $\varphi \in \mathcal{H}_m$ is a polynomial and $v \in \mathbb{C}^n$ a vector considered as a constant vector field on $\mathbb{C}^n$, by the Leibnitz rule

$$[\mathbf{A}, \varphi v] = (\mathbf{A}\varphi)v + \varphi[\mathbf{A}, v] = (\mathbf{A}\varphi)v - \varphi\, Av.$$

Obviously, because of the choice of the Hermitian structure on $\mathcal{H}_m \otimes \mathbb{C}^n$, the operator $\operatorname{id} \otimes A$ is adjoint to $\operatorname{id} \otimes A^*$ whereas the adjoint to $\mathbf{A} \otimes E$ is the tensor product of the adjoint to $\mathbf{A}$ by the identity. By the first statement of the lemma, the former is equal to $\mathbf{A}^*$, so that the adjoint to $[\mathbf{A}, \cdot]$ is $\mathbf{A}^* \otimes E - \operatorname{id} \otimes A^*$ which coincides with $[\mathbf{A}^*, \cdot] = \operatorname{ad}_{A^*}$. □

Theorem 4.12 (G. Belitskii [**Bel79**]; see also [**Dum93, Van89**]). *A formal vector field $F(x) = Ax + V_2(x) + \cdots$ with the linearization matrix A is formally equivalent to a vector field $F'(x) = Ax + V_2'(x) + \cdots$ whose nonlinear part commutes with the linear vector field $\mathbf{A}^*(x) = A^*x$:*

$$[F' - \mathbf{A}, \mathbf{A}^*] = 0. \tag{4.8}$$

If the vector field F is real (i.e., has only real Taylor coefficients, in particular, A is real), then both the formal normal form and the conjugating transformation can be chosen real.

Proof. The proof is based on the following well-known observation: if L is a linear endomorphism of a complex Hermitian or real Euclidean space H into itself, then the image of L and the kernel of its Hermitian (resp., Euclidean) adjoint L^* are orthogonal complements to each other:

$$(\operatorname{img} L)^\perp = \ker L^*.$$

It follows then that $\ker L^*$ is complementary to $\operatorname{img} L$ in H.

Indeed, $\xi \in (\operatorname{img} L)^\perp$ if and only if $(\xi, Lv) = 0$ for all $v \in H$, which means that any vector v is orthogonal to $L^*\xi$. This is possible if and only if $L^*\xi = 0$.

Applying this observation to the operator $L_m = \operatorname{ad}_A$ restricted on any space $\mathcal{D}_m$ and using Lemma 4.11, we see that the subspaces $\mathcal{N}_m = \ker \operatorname{ad}_{A^*}|_{\mathcal{D}_m}$ are orthogonal (hence complementary) to the image of L_m and therefore satisfy the assumption (4.5) of Theorem 4.7. Therefore all *nonlinear* terms $V_2, V_3, \ldots$ can be chosen to commute with $\mathbf{A}^*(x) = A^*x$, which is in turn possible if and only if their formal sum, equal to $F - \mathbf{A}$, commutes with $\mathbf{A}^*$.

In the real case one has to replace the Hermitian spaces $\mathcal{H}_m$, $\mathbb{C}^n$ and $\mathcal{D}_m = \mathcal{H}_m \otimes_{\mathbb{C}} \mathbb{C}^n$ by their real (Euclidean) counterparts ${}^{\mathbb{R}}H_m$, $\mathbb{R}^n$ and ${}^{\mathbb{R}}\mathcal{D}_m = {}^{\mathbb{R}}\mathcal{H}_m \otimes_{\mathbb{R}} \mathbb{R}^n$. Then for any real matrix A the image of the commutator ad_A and the kernel of ad_{A^*}, where A^* is a transposed matrix, are orthogonal and hence complementary. Then the homological equation $\operatorname{ad}_A P_m = V_m' - V_m$ can be solved with respect to $P_m \in {}^{\mathbb{R}}\mathcal{D}_m$ and $V_m' \in \ker \operatorname{ad}_{A^*} \cap {}^{\mathbb{R}}\mathcal{D}_m$ when $V_m \in {}^{\mathbb{R}}\mathcal{D}_m$. The Poincaré–Dulac paradigm does the rest of the proof. □

This general statement immediately implies a number of corollaries.

Example 4.13. If A is a diagonal matrix with the spectrum $\{\lambda_1,\dots,\lambda_n\}$, then A^* is also diagonal with the conjugate eigenvalues $\{\bar\lambda_1,\dots,\bar\lambda_n\}$. As was already noted, restriction of ad_{A^*} on $\mathcal{D}_m$ is diagonal with the eigenvalues $\langle\bar\lambda,\alpha\rangle-\bar\lambda_k=\overline{\langle\lambda,\alpha\rangle-\lambda_k}$. Its kernel consists of the same resonant monomials as defined previously, so in this case Theorem 4.12 yields the usual Poincaré–Dulac form.

Sometimes, diagonalization of the linear part is nonconvenient (especially for real vector fields). In such a case Theorem 4.12 may yield a simple real normal form.

Example 4.14. If $I=\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)=-I^*$ is the matrix of rotation on the *real* plane $\mathbb{R}^2$ with the coordinates (x,y), then $\ker\operatorname{ad}_{I^*}=\ker\operatorname{ad}_I$ and the *entire* formal normal form, including the linear part, commutes with the rotation vector field $\mathbf{I}=x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}$. Any such rotationally symmetric real vector field must necessarily be of the form

$$f(x^2+y^2)\bigl(x\tfrac{\partial}{\partial x}+y\tfrac{\partial}{\partial y}\bigr)+g(x^2+y^2)\bigl(x\tfrac{\partial}{\partial y}-y\tfrac{\partial}{\partial x}\bigr),\qquad(4.9)$$

where $f(r),g(r)\in\mathbb{R}[[r]]$ are two real formal series in one variable. Indeed, A commutes with itself and the radial (Euler) vector field $\mathbf{E}=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}$ which form a basis at all nonsingular points; a linear combination $f\mathbf{E}+g\mathbf{I}$ with f,g scalar coefficients, commutes with $\mathbf{I}$ if and only if $\mathbf{I}f=\mathbf{I}g=0$, that is, if f and g are constants on all circles $x^2+y^2=r^2$.

The linear part is of the prescribed form if $f(0)=0$, $g(0)=1$. Since g is formally invertible, the normal form (4.9) is formally orbitally equivalent to the formal vector field

$$\begin{aligned}F'&=\mathbf{I}+f(x^2+y^2)\mathbf{E},\qquad f\in\mathbb{R}[[u]],\quad f(0)=0,\\ \mathbf{I}&=x\tfrac{\partial}{\partial y}-y\tfrac{\partial}{\partial x},\qquad \mathbf{E}=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y},\end{aligned}\qquad(4.10)$$

with a formal series $f(u)$ in the resonant monomial $u=x^2+y^2$.

Note that the “standard” demonstration of this result via preliminary diagonalization of A requires that all subsequent Poincaré–Dulac transformations be preserving the complex conjugacy, which is an additional independent condition.

The same observation explains why the normal form is so often explicitly integrable.

Corollary 4.15. *Assume that the matrix $A\neq0$ is* normal, *i.e., it commutes with the adjoint matrix A^*. Then the vector field can be formally transformed to a field which commutes with the (nontrivial) linear vector field $\mathbf{A}^*$.* □

Indeed, in this case from (4.8) and $[\mathbf{A}, \mathbf{A}^*] = 0$ it follows that $[F, \mathbf{A}^*] = 0$. This observation allows us to decrease the dimension of the system; cf. with §4**J**.

Remark 4.16. We wish to stress that *there is no distinguished Hermitian structure on* $\mathbb{C}^n$. One can choose this structure arbitrarily and only then the standard Hermitian structure appears on $\mathcal{H}_m$ and $\mathcal{D}_m$. Thus the assumption of this corollary is not restrictive, in particular, it always holds whenever A is diagonalizable.

4F. Parametric case. The Poincaré–Dulac method of normalization of any finite jet or the entire Taylor series, involves only the *polynomial* (*ring*) operations (additions, subtractions and multiplications) with the Taylor coefficients of the original field, *except for inversion of the operator* ad_A. This allows us to construct formal normal forms depending on parameters.

Definition 4.17. A formal series $f \in \mathbb{C}[[x]]$ is said to depend *polynomially* on finitely many parameters $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{C}^n$, if each coefficient depends polynomially on λ,

$$f = \sum_\alpha c_\alpha x^\alpha, \qquad c_\alpha \in \mathbb{C}[\lambda].$$

No assumption on the degrees $\deg c_\alpha$ is made.

The formal series $f = \sum c_\alpha x^\alpha \in \mathbb{C}[[x]]$ depends (strongly) *analytically* on the parameters *in a domain* $\lambda \in U$, if each coefficient c_α of this series depends on the parameters analytically in the *common domain* $U \subseteq \mathbb{C}^m$, $c_\alpha \in \mathcal{O}(U)$.

The formal series $f = \sum c_\alpha x^\alpha$ *weakly analytically* depends on the parameters $\lambda \in (\mathbb{C}^m, 0)$, if each coefficient c_α is a germ of analytic function, $c_\alpha \in \mathcal{O}(\mathbb{C}^m, 0)$. In contrast to the previously defined analytic dependence, intersection of all domains where representatives of the germs c_α are defined, can reduce to the single point $\lambda = 0$.

We will use the common name *semiformal series* to denote elements from the algebras $\mathfrak{A}[[x]]$ in the above three cases when $\mathfrak{A} = \mathbb{C}[\lambda]$, $\mathfrak{A} = \mathcal{O}(U)$ and $\mathfrak{A} = \mathcal{O}(\mathbb{C}^n, 0)$ respectively.

Theorem 4.18 (Formal normal form with parameters)**.**

1. *If the vector field* (*holomorphic or formal*) $F = F(\cdot, \lambda) = \mathbf{A}(\lambda) + F_2(\lambda) + \cdots$ *depends weakly analytically on parameters* $\lambda \in (\mathbb{C}^m, 0)$*, then by a formal transformation one can bring the field to the formal normal form* F' *satisfying the condition*

$$[F' - \mathbf{A}, \mathbf{A}^*(0)] = 0, \tag{4.11}$$

where $\mathbf{A}(0)$ *is the linear vector field corresponding to* $\lambda = 0$*, and* $\mathbf{A}^*(0)$ *its adjoint linear field. Both the formal normal form* F' *and the transformation* H *reducing* F *to* F' *can be chosen weakly analytically depending on the parameters* $\lambda \in (\mathbb{C}^m, 0)$ *in the sense of Definition 4.17. If* F *was real, then also* F' *and* H *can be chosen real.*

2. *If the linear part* $\mathbf{A}(\lambda) \equiv \mathbf{A}(0) \equiv \mathbf{A}$ *is constant* (*does not depend on* λ) *and the field itself depends polynomially or strongly analytically on the parameters* $\lambda \in U$*, then both the normal form* (4.11) *and the corresponding normalizing transformation can be chosen polynomially (resp., strongly analytically) depending on the parameters in the same domain.*

Proof. We start with a very general observation, basically, a geometrical reformulation of the Implicit Function theorem.

If $L\colon X \to Y$ is a linear map between vector spaces, which is *transversal* to a subspace $Z \subseteq Y$, then for any analytic or polynomial map $y\colon \lambda \mapsto y(\lambda)$, $\lambda \in U$ or $\lambda \in \mathbb{C}^n$, one can find two maps $x\colon \lambda \mapsto x(\lambda) \in X$ and $z\colon \lambda \mapsto z(\lambda) \in Z$, such that $Lx(\lambda) + z(\lambda) = y(\lambda)$. If in addition L also depends on λ and is transversal to Z for $\lambda = 0$, then the solutions still can be found, but only locally for the parameter values $\lambda \in (\mathbb{C}^m, 0)$ sufficiently close to the origin. In this case analyticity of $x(\lambda), z(\lambda)$ in the larger domain U or polynomiality in general may fail.

This observation can be applied to the homological operator $L = \mathrm{ad}_A$ acting in the space $X = \mathcal{D}_m$, and the subspace $Y = \mathcal{N}_m$ of homogeneous vector fields commuting with $\mathbf{A}^*(0)$. Holomorphic (polynomial) solvability of the homological equation on each step guarantees the possibility of transforming the field to the normal form with the required properties. $\square$

Remark 4.19 (Warning)**.** The difference between constant and nonconstant linearization matrices is rather essential in what concerns the size of the common domain of analyticity of all Taylor coefficients of the normal form and/or conjugating transformation.

Suppose that all coefficients of the analytic family $F(\lambda)$ of formal vector fields are defined and holomorphic in some *common* domain U (e.g., the field is analytic in $D \times U$, where D is a small polydisk).

If the linearization matrix of $F(\lambda)$ does not depend on the parameters, then by the second assertion of Theorem 4.18, one may remove from the expansion of F all terms that are nonresonant (i.e., the terms that do not commute with the linear field $\mathbf{A}^*$ which is independent of the parameters). All coefficients of all series (the normal form and the conjugacy) will be holomorphic in the maximal natural domain U.

All the way around, if the linearized field $\mathbf{A}(\lambda)$ depends on parameters, then by a formal transformation one can eliminate all terms that are resonant with respect to $\mathbf{A}(0)$. The coefficients of the normal form and the transformation will still be analytically dependent on λ, but their domains should be expected to shrink as the degree of the corresponding terms grow.

Indeed, assume that the linear field $\mathbf{A}(0)$ is nonresonant. Then the formal normal form guaranteed by the first assertion of Theorem 4.18 is *linear*, $F' = \mathbf{A}(\lambda)$. Yet clearly for some values of the parameter λ which are arbitrarily close to $\lambda = 0$, the spectrum of the matrix $A(\lambda)$ can become resonant, hence it will be impossible to eliminate completely all terms of the corresponding order. The apparent contradiction is easily explained: the domain of analyticity of the coefficient of a high order cannot be so large as to include values of the parameter corresponding to resonances of that order. Note that if $A(0)$ is nonresonant, then the possible order of resonances occurring for $A(\lambda)$ necessarily grows to infinity as $\lambda \to 0$.

4G. Formal classification of self-maps. Besides formal vector fields, formal isomorphisms act also on themselves by conjugacy: if

$$G(x) = Mx + V_2(x) + \cdots \in \operatorname{Diff}[[\mathbb{C}^n, 0]], \qquad \det M \neq 0, \tag{4.12}$$

is a formal self-map, then another formal self-map $H \in \operatorname{Diff}[[\mathbb{C}^n, 0]]$ transforms G to $G' = H \circ G \circ H^{-1}$. In the same way as before, one may ask if all nonlinear terms $V_2, V_3, \ldots$ can be removed from the expansion by applying a suitable formal conjugacy.

The strategy is the same as described in §4**B**. The polynomial transformation $H(x) = x + P_m(x)$ with a vector homogeneous nonlinearity P_m of degree m conjugates $G(x)$ as in (4.12) with a self-map $G'(x) = G(x) + R_m(x) + \cdots$, in which R_m is a homogeneous vector polynomial of order m, implicitly defined by the identity

$$G(x) + P_m(G(x)) = G(x + P_m(x)) + R_m(x + P_m(x)) + \cdots . \tag{4.13}$$

After collection of terms of order m this yields the equation

$$P(Mx) - MP(x) = R(x), \qquad P = P_m,\ R = R_m, \tag{4.14}$$

which we can attempt to solve with respect to P. This is the multiplicative analog of the homological equation (4.2). The operator

$$S_M \colon \mathcal{D}_m \to \mathcal{D}_m, \qquad P(x) \mapsto MP(x) - P(Mx), \tag{4.15}$$

can be studied by methods similar to the operator ad_A. If M is a diagonal matrix with the diagonal entries $\mu_1, \ldots, \mu_n$, then all monomials $F_{k\alpha}$ of the standard basis in $\mathcal{D}_m$ are eigenvectors for S_M with the eigenvalues $\mu_j - \mu^\alpha = \mu_j - \mu_1^{\alpha_1} \cdots \mu_n^{\alpha_n}$. If all these expressions are nonzero, the operators S_M will always be invertible and hence the formal self-map G will be formally linearizable. If some of the expressions $\mu_j - \mu^\alpha$ are zeros, then one can transform G to a nonlinear normal form. All these results can be obtained in exactly the same way as for the formal vector fields.

Definition 4.20. A *multiplicative* resonance between nonzero complex numbers $\mu = (\mu_1, \ldots, \mu_n) \in (\mathbb{C}^*)^n$ is an identity of the form

$$\mu_j - \mu^\alpha = 0, \qquad |\alpha| \geqslant 2,\ j = 1, \ldots, n. \tag{4.16}$$

A nondegenerate matrix $M \in \operatorname{GL}(n, \mathbb{C})$ and a formal self-map $G(x) = Mx + \cdots \in \operatorname{Diff}[[\mathbb{C}^n, 0]]$ are nonresonant if there are no multiplicative resonances between the eigenvalues of M. A *multiplicative resonant monomial* corresponding to the resonance (4.16), is the vector whose jth component is x^α and all others are zeros.

Theorem 4.21 (Poincaré–Dulac theorem for self-maps)**.** *Any invertible formal self-map is formally equivalent to a formal self-map whose linear part is in the Jordan normal form, and the nonlinear part contains only resonant monomials with complex coefficients. In particular, a nonresonant formal self-map is formally conjugated to the linear map $G'(x) = Mx$.* □

Rather obviously, Theorem 4.21 can be further elaborated and an analog of Belitskii Theorem 4.12 established. However, we will not deal with these matters and concentrate from now on on vector fields and automorphisms in low dimension (2 for fields, 1 for self-maps) which will be the principal tool in the rest of the book.

* * *

4H. Cuspidal points. One important case where Theorem 4.12 is considerably stronger than the Poincaré–Dulac Theorem 4.10 is that of vector fields with *nilpotent* linear parts, which are "maximally nondiagonalizable". In this case *all* monomials will be resonant and Theorem 4.10 is void. We will only consider the planar case where the linear part is the vector field $J = y\frac{\partial}{\partial x} \in \operatorname{Mat}(2, \mathbb{R})$ (the linearization matrix is a nilpotent Jordan cell of size 2). From Theorem 4.12 we can immediately derive the following corollary.

Theorem 4.22. *A vector field on the plane with the linear part $J = y\frac{\partial}{\partial x}$ is formally equivalent to the vector field*

$$J + b(x)E + a(x)\frac{\partial}{\partial y}, \qquad a, b \in \mathbb{C}[[x]],\ E = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \tag{4.17}$$

with the formal series $a, b \in \mathbb{C}[[x]]$ in one variable x starting with terms of order 2 and 1 respectively.

Proof. We need only to describe the kernel of the operator ad_{J^*}, where $J^* = x\frac{\partial}{\partial y}$ is the "adjoint" vector field. The kernel of the operator $\operatorname{ad}_{J^*} = [x\frac{\partial}{\partial y}, \cdot\,]$ restricted on $\mathcal{D}_m$ can be immediately computed. Indeed,

$$[x\tfrac{\partial}{\partial y}, u\tfrac{\partial}{\partial x} + v\tfrac{\partial}{\partial y}] = xu_y\tfrac{\partial}{\partial x} + (xv_y - u)\tfrac{\partial}{\partial y},$$

and the commutator vanishes only if both u and hence v_y depend only on x. Since both u, v must be homogeneous of degree m, we conclude that

$$\ker \operatorname{ad}_{J^*}\big|_{\mathcal{D}_m} = \beta(x^m\tfrac{\partial}{\partial x} + x^{m-1}y\tfrac{\partial}{\partial y}) + \alpha x^m\tfrac{\partial}{\partial y} = \beta x^m\,(x\tfrac{\partial}{\partial x} + y\tfrac{\partial}{\partial y}) + \alpha x^m\tfrac{\partial}{\partial y}$$

for some constants $\alpha = \alpha_m$ and $\beta = \beta_m$ which will be the coefficients of the respective series a, b. □

Yet the complementary subspaces $\mathcal{N}_m$ may be chosen in a different way, not necessary as prescribed by Theorem 4.12. This may be more convenient for some applications.

Theorem 4.23. *The planar formal vector field with the linear part* $J = y\frac{\partial}{\partial x}$, *is formally equivalent to the vector field*

$$J + [yb(x) + a(x)]\tfrac{\partial}{\partial y}, \qquad (4.18)$$

where $a(x)$ *and* $b(x)$ *are two formal series of orders* 2 *and* 1 *respectively.*

Proof. We reduce this assertion directly to the general Poincaré–Dulac paradigm. The image of ad_J in $\mathcal{D}_m$ can be complemented by the 2-dimensional space $\mathcal{N}'_m$ of vector fields $(\alpha x^m + \beta x^{m-1}y)\frac{\partial}{\partial x}$, as noted in [**Arn83**, §35 D]. Indeed, the condition $[y\frac{\partial}{\partial x}, f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}] = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$ takes the form of the system of linear partial differential equations

$$yf_x - g = u, \qquad yg_x = v.$$

While it can be not solvable for some u, v, the system of equations

$$yf_x - g = u, \quad yg_x + \alpha x^m + \beta x^{m-1}y = v \qquad (4.19)$$

can be always resolved for any pair of homogeneous polynomials $u, v \in \mathbb{C}[x, y]$ of degree m and the constants α, β. To see this, apply $y\frac{\partial}{\partial x}$ to the first equation:

$$y^2 f_{xx} = yu_x + v - \alpha x^m - \beta x^{m-1}y.$$

The equation $y^2 f_{xx} = w$ is uniquely solvable for any monomial w divisible by y^2. On the other hand, the constants α, β can be found to guarantee that the terms proportional to x^m and $x^{m-1}y$ in the right hand side of this equation vanish. This choice automatically guarantees solvability of the second equation in (4.19) as well. The constants found in this way, appear as coefficients of the respective series a, b. □

4I. Vector fields with zero linear parts. If the formal vector field F starts with kth order terms, $F(x) = V_k(x) + V_{k+1}(x) + \cdots$, $k \geqslant 2$, then application of the formal transformation $H(x) = x + P_2(x)$ conjugates F with the vector field $F'(x) = V_k + V'_{k+1} + \cdots$ with the same (nonlinear) principal part V_k, if

$$V_k(x)+V_{k+1}(x)+\left(\frac{\partial P_2}{\partial x}\right)V_k(x)+\cdots = V_k(x+P_2(x))+V'_{k+1}(x+P_2(x))+\cdots$$

which after collecting the homogeneous terms of order $k+1$ yields

$$[V_k, P_2] = V_{k+1} - V'_{k+1}.$$

If this equation is resolved for a suitably chosen V'_{k+1} (e.g., equal to zero if that is possible), one can pass to terms of order $k+2$ by applying a transform of the form $H(x) = x + P_3(x)$ which does not affect the terms of order V_k and V_{k+1} and so on. As a result, one has to resolve in each order the homological equation

$$\operatorname{ad}_{V_k} P_m = V_{m+k-1} - V'_{m+k-1} \tag{4.20}$$

with respect to the homogeneous vector field P_m of degree m. As before, complete elimination of all nonprincipal terms of orders $k+1$ and more, is possible if the operator ad_{V_k} is surjective, otherwise it will be necessary to introduce the "normal subspaces" $\mathcal{N}_{m+k-1} \subset \mathcal{D}_{m+k-1}$ complementary to the image $\operatorname{ad}_{V_k}(\mathcal{D}_m) \subseteq \mathcal{D}_{m+k-1}$ and choose the components V'_{m+k-1} of the formal normal form from these subspaces.

In contrast to the case $k = 1$ discussed earlier, the operator ad_{V_k} *increases the degrees*, i.e., acts between *different* spaces, the dimension of the target space in general being higher than that of the source space. Thus the number of parameters in the normal form will be infinite. A notable exception is the one-dimensional case $\dim x = 1$.

Theorem 4.24. *A nonzero formal vector field from* $\mathcal{D}[[\mathbb{C}, 0]]$ *is formally equivalent to one of the vector fields of the form*

$$\left(x^{k+1} + ax^{2k+1}\right)\tfrac{\partial}{\partial x}, \qquad k \in \mathbb{N},\ a \in \mathbb{C}. \tag{4.21}$$

Proof. Any nonzero formal vector field on $\mathbb{C}^1$ starts with the term $a_{k+1}x^{k+1}\frac{\partial}{\partial x}$, $a_{k+1} \neq 0$. One can make a_{k+1} equal to 1 by a linear transformation $x \mapsto cx$, if the ground field is $\mathbb{C}$.

In this case all spaces $\mathcal{D}_m$ are one-dimensional, and the commutator with the principal term $x^{k+1}\frac{\partial}{\partial x}$ can be immediately computed:

$$\left[x^{k+1}\tfrac{\partial}{\partial x}, x^m\tfrac{\partial}{\partial x}\right] = (k-m+1)x^{k+m}\tfrac{\partial}{\partial x}. \tag{4.22}$$

This operator is surjective for all $m \neq k+1$. Thus only the term $x^{2k+1}\frac{\partial}{\partial x}$ cannot be eliminated. □

Note that over the field of reals $\mathbb{R}$ the normal form is different: if k is *even*, then by the *real* homothety one can make the principal coefficient only ± 1,

$$\left(\pm\, x^{k+1} + ax^{2k+1}\right)\tfrac{\partial}{\partial x}, \qquad k \in \mathbb{N},\ a \in \mathbb{R}.$$

For odd k the fields with different signs are equivalent (transformed into each other by the symmetry $x \mapsto -x$).

Remark 4.25. In fact, the above arguments show that *any two formal vector fields on the line having a zero of multiplicity* $k+1$ *at the origin and common* $(2k+1)$*-jet, are formally equivalent.*

It is sometimes more convenient instead of the polynomial normal form (4.21) to use the *rational* formal normal form

$$\frac{x^{k+1}}{1-ax^k}\cdot\frac{\partial}{\partial x},\qquad k\in\mathbb{N},\ a\in\mathbb{C}. \tag{4.23}$$

This (rational) field is *analytically* equivalent to the field (4.21) with the same a. On the other hand, two vector fields in the normal form (4.23) with different values of a cannot be equivalent, as will be shown in §6$\mathbf{B}_2$.

Theorem 4.26. *Any self-map $x\mapsto x+x^{k+1}+\dots$, $k\in\mathbb{N}$, tangent to identity, is formally equivalent to*:

(1) *the time one map of the polynomial vector field* (4.21),
(2) *the time one map of the rational vector field* (4.23),
(3) *the polynomial map $x\mapsto x+x^{k+1}+ax^{2k+1}$*,

with the same complex parameter $a\in\mathbb{C}$ which is the formal invariant of the classification together with the order $k+1$.

Proof. One can prove this result in exactly the same way as Theorem 4.24, namely, modifying the Poincaré–Dulac paradigm for the equation (4.13) and using the computation from Proposition 6.11 below.

Yet one can circumvent this parallel construction by reference to the formal embedding Theorem 3.17. Indeed, any formal self-maps from $\operatorname{Diff}[[\mathbb{C},0]]$ tangent to the identity with some order $k+1$ can be represented as a time one formal flow of a formal vector field from $\mathcal{D}[[\mathbb{C},0]]$. This field in turn can be brought to one of the two formal normal forms or to the formal (nonpolynomial!) field generating the polynomial normal form. □

4J. Formal normal forms of elementary singular points on the real plane. In this section we summarize the (orbital) formal normal forms for all planar (i.e., for $n=2$) *real* vector fields with nonzero linear parts. Recall that two formal vector fields $F,F'\in\mathcal{D}[[\mathbb{R}^2,0]]$ are called *orbitally formally equivalent*, if there exist an invertible real formal series $\varphi\in\mathbb{R}[[x,y]]$, $\varphi(0,0)\neq 0$, such that F is formally equivalent to $\varphi\cdot F'$, and the corresponding formal self-map has all real coefficients, i.e., belongs to the group $\operatorname{Diff}[[\mathbb{R}^2,0]]$. We use everywhere the term *singularity* to denote jets or germs of analytic vector fields or formal vector fields at the origin, depending on the context.

Definition 4.27. A singularity of the planar vector field is *elementary*, if at least one of the eigenvalues $\lambda_{1,2}$ of its linearization matrix is nonzero.

The only nonelementary singularity that has nonzero linearization matrix with both zero eigenvalues, is called *cuspidal*, or *nilpotent* singularity.

Real elementary points can be of several types that exhibit essentially different properties.

Definition 4.28. An elementary singularity is a *resonant node*, if the ratio of its eigenvalues is a natural or inverse natural number. The singularity is a *resonant saddle*, if both eigenvalues are real and their ratio is negative rational. A singularity is *elliptic*, if $\lambda_{1,2} = \pm i\omega$, $\omega > 0$. Finally, the singularity is a *saddle-node*, if exactly one eigenvalue is zero.

Proposition 4.29 (Formal normal forms of planar singularities)**.** *By a real orbital formal transformation from the group* $\operatorname{Diff}[[\mathbb{R}^1, 0]] \times \operatorname{Diff}[[\mathbb{R}^2, 0]]$ *any real formal vector field* $\mathcal{D}[[\mathbb{R}^2, 0]]$ *appearing in Table I.1, can be brought to the normal form from the right column of this table.*

Proof. Most of these results are particular cases of the general results proved earlier for the ground field $\mathbb{C}$, modulo the following obvious remark. If the linear part of the vector field can be brought into its Jordan normal form by a *real* linear transformation, then all results of the formal classification remain valid if the ground field is replaced by $\mathbb{R}$. The only nontrivial case where a real matrix cannot be normalized over $\mathbb{R}$ is that of the *elliptic* singular points whose linear part is linear rotation $\omega x \frac{\partial}{\partial y} - \omega y \frac{\partial}{\partial x}$, with the eigenvalues $\pm i\omega$. From the complex point of view this is a resonant saddle, yet diagonalization of this matrix requires enlarging the ground field. The alternative treatment of the elliptic case is explained in Example 4.14.

The assertion concerning saddle-nodes is a combination of the Poincaré–Dulac theorem and Theorem 4.24. While the condition $\lambda_2 = 0$ is not a resonance, it implies infinitely many resonances $\lambda_j = \lambda_j + m$ for any $m \in \mathbb{N}$. By the Poincaré–Dulac theorem, the field is formally equivalent to the field $xf(y)\frac{\partial}{\partial x} + yg(y)\frac{\partial}{\partial y}$ with $f(0) \neq 0$ and $g(0) = 0$ (otherwise the singular point cannot be elementary degenerate). Dividing by the invertible series $f(y)$ one can assume that $f \equiv 1$ and the variables (formally) separate. It remains to make the formal change of the variable y which puts the one-dimensional vector field $g(y)\frac{\partial}{\partial y}$ into the normal form (4.21).

The saddle case is analyzed similarly: the identity $\langle \lambda, m \rangle = 0$ itself is not a resonance, but its integer multiple can be added to the right hand side of each of the identities $\lambda_1 = \lambda_1$ or $\lambda_2 = \lambda_2$, thus producing infinitely many resonances. Without loss of generality we assume that $\lambda_1 = -p$, $\lambda_2 = q$, $p, q \in \mathbb{N}$. Clearly, there are no other resonances and the Poincaré–Dulac normal form looks like $-px\, f(u)\frac{\partial}{\partial x} + qy\, g(u)\frac{\partial}{\partial y}$, $f(0) = g(0) = 1$, where $u = x^p y^q$ is the resonant monomial. Passing to an orbitally equivalent system, one can assume that $f \equiv 1$.

Type	**Conditions**	**Formal normal form**
Nonresonant	$[\lambda_1 : \lambda_2] \notin \mathbb{Q}$ or $\lambda_1 = \lambda_2 \neq 0$	Linear
Resonant node	$[\lambda_1 : \lambda_2] = [r : 1]$, $r \in \mathbb{N}$, $r \geqslant 2$	$\dot{x} = rx + ay^r$, $\dot{y} = y$ $a \in \mathbb{C}$ formal invariant.
Resonant saddle (orbital)	$[\lambda_1 : \lambda_2] = -[p : q]$, $p, q \in \mathbb{N}$, not formally orbitally linearizable	$\dot{x} = -px$, $\dot{y} = \quad qy(1 \pm u^r + au^{2r})$, $u = x^q y^p$, $r \in \mathbb{N}$, $a \in \mathbb{R}$ formal orbital invariants
Elliptic points (orbital)	$\lambda_{1,2} = \pm i\omega$, not formally orbitally linearizable	$\dot{x} = \quad y \pm x(u^r + au^{2r})$, $\dot{y} = -x \pm y(u^r + au^{2r})$, $u = x^2 + y^2$, $a \in \mathbb{R}$ formal orbital invariant
Saddle-node (orbital classification)	$\lambda_1 \neq 0$, $\lambda_2 = 0$, formally isolated singularity	$\dot{x} = x$, $\dot{y} = \pm y^{r+1} + ay^{2r+1}$, $r \in \mathbb{N}$, $a \in \mathbb{R}$ formal orbital invariants
Cuspidal (nilpotent) point (nonelementary)	Nonvanishing linearization matrix with two zero eigenvalues	$\dot{x} = y$, $\dot{y} = a(x) + yb(x)$, $a, b \in \mathbb{R}[[x]]$ two formal series, $\operatorname{ord} a \geqslant 2$, $\operatorname{ord} b \geqslant 1$.
One-dimensional degenerate vector field	$\lambda = 0$, formally isolated singularity	$\dot{x} = \pm x^{r+1} + ax^{2r+1}$, or $\dot{x} = \pm \dfrac{x^{r+1}}{1 - ax^r}$, $r \in \mathbb{N}$, $a \in \mathbb{C}$ formal invariants

Table I.1. Formal normal forms for real vector fields. All rows of the table, except the last one, refer to planar formal vector fields and give orbital formal normal forms.

The field in the Poincaré–Dulac normal form admits the projection $\mathbb{R}^2 \to \mathbb{R}^1$, $(x, y) \mapsto u = x^p y^q \in \mathbb{R}^1$. The projected system has the form

$$\dot{u} = uF(u), \qquad F(u) = g(u) - 1, \tag{4.24}$$

called the *quotient equation*. By a suitable formal transformation $u \mapsto u' = u(1 + h(u))$, $h(0) = 0$, the quotient vector field can be brought to the form (4.21), corresponding to $g(u) = 1 + u^{k-1} + au^{2k-1}$. It remains to observe that any formal transformation of the variable $u \mapsto u[1 + h(u)]$, $h(0) = 0$, can be "covered" by the transformation $(x, y) \mapsto (x', y'(x, y))$, where

$$x' = x, \qquad y' = y[1 + h(x^p y^q)]^{1/q} \in \mathbb{R}[[x, y]],$$

re-expanding the invertible series in square brackets into the binomial series. This transformation brings the initial field into the required formal normal form.

The same construction almost literally applies to the elliptic case: the infinite formal normal form (4.10) admits projection onto the u-axis with $u = x^2 + y^2$, and the quotient equation takes the form $\dot{u} = 2uf(u)$. We leave it as an exercise to prove that any formal line transformation $u \mapsto u[1 + h(u)]$, $h(0) = 0$, can also be "covered" by a suitable *real* plane formal transformation. □

Remark 4.30. If necessary, the polynomial normal forms from Table I.1 can be replaced by rational normal forms involving the rational normal form for one-dimensional quotient vector fields.

Note also that all normal forms of *elementary* singularities from this table are integrable: the quotient equation can be explicitly integrated in quadratures (especially easily if it has the rational normal form (4.23)). After this integration the variables x and y always separate. This integrability will be repeatedly used in the rest of the book to produce explicit computations with normal forms.

The cuspidal normal form is the famous Liénard system, corresponding to one of the simplest nonlinear and nonintegrable vector fields for which questions on the number of *limit cycles* is highly nontrivial. The Liénard system is sometimes written under the form

$$\dot{x} = y - f(x), \qquad \dot{y} = -g(x),$$

or as a second order scalar differential equation.

Remark 4.31. The dynamic (full, nonorbital) formal normal form contains more parameters than indicated in Table I.1. For instance, for the saddle-node the formal normal form is

$$\begin{cases} \dot{x} = x(\lambda_1 + b_1 y + \cdots + b_k y^k), \\ \dot{y} = y^{k+1}k + ay^{2k+1}, \qquad \lambda_1, b_1, \dots, b_k, a \in \mathbb{C}. \end{cases} \tag{4.25}$$

To prove this formula, we reduce the vector field to the form $xf(y)\frac{\partial}{\partial x} + g(y)\frac{\partial}{\partial y}$ as above and then by a suitable change of the variable y only put g into the standard form $g(y) = y^{k+1} + ay^{2k+1}$. The function $f(x)$ can be further simplified by transformations of the form $(x, y) \mapsto (h(y)x, y)$, $h(0) \neq 0$, preserving the second component: one immediately verifies that such a transformation results in replacing the series $f = f(y) \in \mathbb{C}[[y]]$ by another series

$$f' = f + \frac{g}{y} \cdot \frac{dh}{dy} = f + (y^{k+1} + ay^{2k+1}) \frac{d}{dy} \ln h.$$

Since g begins with terms of order $k+1$, the difference between f and f' is necessarily k-flat (the logarithmic derivative $\frac{d}{dy} \ln h$ in the above formula is a well defined formal series from $\mathbb{C}[[y]]$ since $h(0)$ is nonvanishing). On the other hand, if the difference $f - f'$ is divisible by y^{k+1}, the quotient can be represented as the logarithmic derivative of a suitable series $h \in \mathbb{C}[[y]]$. Thus all terms of order $k+1$ and above can be eliminated from f by the formal transformation.

A similar result can be formulated for resonant saddles and elliptic singularities.

Exercises and Problems for §4.

A complex tuple $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ is called single-resonant, if all resonances between the components of this tuple follow from a single integer identity

$$\langle \alpha, \lambda \rangle = 0, \qquad \alpha \in \mathbb{Z}^n_+, \ \alpha \neq 0. \tag{4.26}$$

Problem 4.1. Describe the formal normal form of a vector field with a single-resonant spectrum of the linearization matrix. Show that this normal form is integrable in quadratures.

Problem 4.2. Describe all linear maps that preserve the formal normal form in Problem 4.1.

Problem 4.3. Describe the real formal normal forms for vector fields in $\mathbb{R}^3$ with the spectrum $0, \pm i\omega$.

Problem 4.4. The same question for fields in $\mathbb{R}^4$ with the spectrum $\pm i\omega_1$, $\pm i\omega_2$, if the ratio ω_1/ω_2 is irrational.

Problem 4.5. Describe symmetries of the formal normal forms in the Problems 4.3 and 4.4.

Exercise 4.6. Prove that if F is a resonant formal vector field, then $\exp tF$ is a multiplicative resonant formal self-map for any $t \neq 0$. Is the inverse true?

Problem 4.7. Construct a formal normal form for vector fields in $\mathbb{C}^3$ with the nilpotent Jordan linear part $J = y\frac{\partial}{\partial x} + z\frac{\partial}{\partial y}$.

Answer: $J + a(x,u)E + b(x,u)F + c(x,u)F'$, where $E = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$ is the Euler field in three dimensions, $F = x\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}$, $F' = \frac{\partial}{\partial z}$, and $u = u(x,y,z) = 2xz - y^2$.

Exercise 4.8. Find a formal normal form for a saddle-nodal self-map with the spectrum $(1, \mu)$, $|\mu| \neq 1$, in two dimensions.

Problem 4.9. Give a complete proof of the Poincaré–Dulac theorem for self-maps (Theorem 4.21).

Problem 4.10. Prove that the formal normal form of any vector field in the Poincaré domain is integrable in quadratures.

5. Holomorphic normal forms

5A. Poincaré and Siegel domains. To linearize a given (say, nonresonant) vector field, on each step of the Poincaré–Dulac process one has to compute the inverse of the operator $\operatorname{ad}_A = [\mathbf{A}, \cdot\,]$ on the spaces of homogeneous vector fields. To that end, one has to divide the Taylor coefficients by the *denominators*, expressions of the form $\lambda_j - \langle \alpha, \lambda \rangle \in \mathbb{C}$ with $\alpha \in \mathbb{Z}_+^n$, $|\alpha| \geqslant 2$, that may a priori be *small* even in the nonresonant case where ad_A is invertible. These denominators associated with the spectrum λ of the linearization matrix A, behave differently as $|\alpha|$ grows to infinity, in the following two different cases.

Definition 5.1. The *Poincaré domain* $\mathfrak{P} \subset \mathbb{C}^n$ is the collection of all tuples $\lambda = (\lambda_1, \dots, \lambda_n)$ such that the convex hull of the point set $\{\lambda_1, \dots, \lambda_n\} \subset \mathbb{C}$ does not contain the origin inside or on the boundary.

The Siegel domain $\mathfrak{S}$ is the complement to the Poincaré domain in $\mathbb{C}^n$.

The *strict Siegel domain* is the set of tuples for which the convex hull contains the origin strictly inside.

Sometimes we say about *tuples*, *spectra* or even germs of vector fields at singular points as being of Poincaré (resp., Siegel) type.

Proposition 5.2. *If λ is of Poincaré type, then only finitely many denominators $\lambda_j - \langle \alpha, \lambda \rangle$, $\alpha \in \mathbb{Z}_+^n$, $|\alpha| \geqslant 2$, may actually vanish.*

Moreover, nonzero *denominators are bounded away from the origin: the latter is an isolated point of the set of all denominators $\{\lambda_j - \langle \alpha, \lambda \rangle \mid j = 1, \dots, n,\ |\alpha| \geqslant 2\}$.*

On the contrary, if λ is of Siegel type, then either there are infinitely many vanishing denominators, or the origin $0 \in \mathbb{C}$ is their accumulation point (these two possibilities are not mutually exclusive).

Proof. If the convex hull of $\{\lambda_1, \dots, \lambda_n\} \subset \mathbb{C}$ does not contain the origin, by the convex separation theorem there exists a real linear functional $\ell \colon \mathbb{C}^2 \to \mathbb{R}$ such that $\ell(\lambda_j) \leqslant -r < 0$ for all λ_j, and hence $\ell(\langle \alpha, \lambda \rangle) \leqslant -r|\alpha|$. But then for any denominator we have

$$\ell(\lambda_j - \langle \alpha, \lambda \rangle) \geqslant \ell(\lambda_j) + |\alpha| r \to +\infty \qquad \text{as } |\alpha| \to \infty.$$

Since ℓ is bounded on any small neighborhood of the origin $0 \in \mathbb{C}$, the first two assertions are proved.

To prove the last assertion, notice that in the Siegel case there are either two or three numbers, whose linear combination with positive (real) coefficients is zero, depending on whether the origin lies on the boundary or in the interior of the convex hull. We give the proof in the second case, more difficult and more generic (the proof for the first case is simpler).

If the origin lies inside a triangle formed by the eigenvalues, then modulo their re-enumeration and a (nonconformal) affine transformation of the complex plane $\mathbb{R}^2 \cong \mathbb{C}$, we may assume without loss of generality that $\lambda_1 = 1$, $\lambda_2 = +i$ and $-\lambda_3 \in \mathbb{R}^2_+ = \mathbb{R}_+ + i\mathbb{R}_+$. In this case all "fractional parts" $-\mathbb{N}\lambda_3 \bmod \mathbb{Z} + i\mathbb{Z}$ of natural multiples of $-\lambda_3$ either form a finite subset of the 2-torus $\mathbb{R}^2/\mathbb{Z}^2$ (in which case all points of this set correspond to infinitely many vanishing denominators), or are uniformly distributed along some 1-torus, or dense. In both latter cases the point $(0,0) \in \mathbb{R}^2/\mathbb{Z}^2$ is the accumulation point of the "fractional parts" which are affine images of the denominators. □

Corollary 5.3. *If the spectrum of the linearization matrix A of a formal vector field belongs to the Poincaré domain, then the resonant formal normal form for this field established in Theorem 4.10, is polynomial.* □

Remark 5.4. Resonant tuples $\lambda \in \mathbb{C}^n$ are dense in the Siegel domain $\mathfrak{S}$ and not dense in the Poincaré domain $\mathfrak{P}$. The proof of this fact can be found in [**Arn83**].

5B. Holomorphic classification in the Poincaré domain. In the Poincaré domain, the normalizing series reducing vector fields or holomorphic maps to their Poincaré–Dulac normal forms, always converge.

Theorem 5.5 (Poincaré normalization theorem)**.** *A holomorphic vector field with the linear part of Poincaré type is holomorphically equivalent to its polynomial Poincaré–Dulac formal normal form.*

In particular, if the field is nonresonant, then it can be linearized by a holomorphic transformation.

We prove this theorem first for vector fields with a diagonal nonresonant linear part $\Lambda = \operatorname{diag}\{\lambda_1, \dots, \lambda_n\}$. The resonant case will be addressed later in §5**C**. The classical proof by Poincaré was achieved by the so-called *majorant method.* In the modern language, it takes a more convenient form of the contracting map principle in an appropriate functional space, the *majorant space.*

Definition 5.6. The *majorant operator* is the nonlinear operator acting on formal series by replacing all Taylor coefficients by their absolute values,

$$\mathbf{M}\colon \sum_{\alpha\in\mathbb{Z}^+_n} c_\alpha\, z^\alpha \mapsto \sum_{\alpha\in\mathbb{Z}^+_n} |c_\alpha|\, z^\alpha.$$

The action of the majorant operator naturally extends on all formal objects (vector formal series, formal vector fields, formal transformations, *etc.*).

Definition 5.7. The *majorant ρ-norm* is the functional on the space of formal power series $\mathbb{C}[[z_1, \dots, z_n]]$, defined as

$$\llbracket f \rrbracket_\rho = \sup_{|z|<\rho} |\mathbf{M}\, f(z)| = |\mathbf{M} f(\rho, \dots, \rho)| \leqslant +\infty. \tag{5.1}$$

For a formal vector function $F = (F_1, \dots, F_n)$ the majorant norm is

$$\llbracket F \rrbracket_\rho = \llbracket F_1 \rrbracket_\rho + \cdots + \llbracket F_n \rrbracket_\rho. \tag{5.2}$$

The majorant space $\mathcal{B}_\rho$ is the subspace of formal (vector) functions from $\mathbb{C}[[x]]$ having finite majorant ρ-norm.

Proposition 5.8. *The space $\mathcal{B}_\rho$ with the majorant norm $\llbracket \cdot \rrbracket_\rho$ is complete.*

Proof. If $\rho = 1$, this is obvious: $\mathcal{B}_1$ is the space of infinite absolutely converging sequences $\{c_\alpha\}$, and hence is isomorphic to the standard Lebesgue space ℓ^1 which is complete. The general case of an arbitrary ρ follows from the fact that the correspondence $f(\rho x) \leftrightarrow f(x)$ is an isomorphism between $\mathcal{B}_\rho$ and $\mathcal{B}_1$. □

Remark 5.9. The space $\mathcal{B}_\rho$ is closely related but not coinciding with the space $\mathcal{A}_\rho = \mathcal{A}(D_\rho)$ of functions, holomorphic in the polydisk $D_\rho = \{|z| < \rho\}$, continuous on its closure and equipped with the usual sup-norm $\|f\|_\rho = \max_{|z|<\rho} |f(z)|$.

Clearly, $\mathcal{B}_\rho \subset \mathcal{A}_\rho$, since a series from $\mathcal{B}_\rho$ is absolutely convergent on the closed polydisk $\overline{D_\rho}$. Conversely, if f is holomorphic in D_ρ and continuous on the boundary, then by the Cauchy estimates, the Taylor coefficients c_α of f satisfy the inequality

$$|c_\alpha| \leqslant \|f\|_\rho \cdot \rho^{-|\alpha|}, \quad \alpha \in \mathbb{Z}^n_+.$$

Though the series $\llbracket f \rrbracket_\rho = \sum |c_\alpha|\, \rho^{|\alpha|}$ may still diverge, any other norm $\llbracket f \rrbracket_{\rho'}$ with $\rho' < \rho$, will already be finite:

$$\llbracket f \rrbracket_{\rho'} \leqslant \|f\|_\rho \cdot \sum_{\alpha \in \mathbb{Z}^n_+} \delta^{|\alpha|} < C\, \|f\|_\rho, \qquad C = C(\delta, n), \quad \delta = \rho'/\rho < 1.$$

To construct a counterexample showing that indeed $\mathcal{A}_\rho \supsetneqq \mathcal{B}_\rho$, consider a convergent but not absolutely convergent Fourier series $\sum_{k\in\mathbb{Z}_+} c_k e^{ikt}$ in one real variable t and let $f(z) = \sum c_k z^k$. Such a series converges at all points of the boundary $|z| = 1$ and represents a function from $\mathcal{A}(D_1)$, but by construction its 1-norm is infinite. Details can be found in [**Edw79**, §10.6]

The important properties of the majorant spaces and norms concern operations on functions. We will use the notation $f \ll g$ for two vector series from $\mathbb{C}^n[[x]]$ with positive coefficients, if each coefficient of f is no greater than the corresponding coefficient of g. In a similar way the notation $x \ll y$ will be used to denote the componentwise set of inequalities between two vectors $x, y \in \mathbb{R}^n$. If $f \in \mathbb{R}^n[[x]]$ is a (vector) series with *nonnegative* coefficients, then it is monotonous: $f(x) \ll f(y)$ if $x \ll y$.

Lemma 5.10. *1. For any two series $f, g \in \mathbb{C}[[x]]$ and any ρ,*

$$\llbracket fg \rrbracket_\rho \leqslant \llbracket f \rrbracket_\rho \cdot \llbracket g \rrbracket_\rho, \tag{5.3}$$

provided that all norms are finite.

2. If $G \ll G'$, are two formal series from $\mathbb{R}^n[[x]]$ and F is a series with nonnegative coefficients, then $F \circ G \ll F \circ G'$.

3. If $F, G \in \mathbb{C}^n[[z_1, \dots, z_n]]$ are two formal vector series, $F(0) = G(0) = 0$, then for their composition we have

$$\llbracket F \circ G \rrbracket_\rho \leqslant \llbracket F \rrbracket_\sigma, \qquad \sigma = \llbracket G \rrbracket_\rho. \tag{5.4}$$

Proof. The first two statements are obvious: all Taylor coefficients of the product or composition are obtained from the coefficients of entering terms by operations of addition and multiplication only. In particular, $\mathbf{M}(fg) \ll \mathbf{M}\, f \cdot \mathbf{M}\, g$. Evaluating both parts at $\boldsymbol{\rho} = (\rho, \dots, \rho)$ proves the first statement.

Since all binomial coefficients are nonnegative (in fact, natural numbers), we have $\mathbf{M}(F \circ G) \ll (\mathbf{M}\, F) \circ (\mathbf{M}\, G)$. Evaluating at $\boldsymbol{\rho} = (\rho, \dots, \rho)$ yields $\mathbf{M}\, G(\boldsymbol{\rho}) = y \ll \boldsymbol{\sigma} = (\sigma, \dots, \sigma)$, where $\sigma = \llbracket G \rrbracket_\rho$. By monotonicity, $\|F \circ G\|_\rho = \big((\mathbf{M}\, F) \circ (\mathbf{M}\, G)\big)(\boldsymbol{\rho}) \ll \mathbf{M}\, F(y) \ll \mathbf{M}\, F(\boldsymbol{\sigma}) = \llbracket F \rrbracket_\sigma$. The last statement is proved. □

Lemma 5.11. *If $\Lambda \in \mathrm{Mat}(n, \mathbb{C})$ is a nonresonant diagonal matrix of Poincaré type, then the operator ad_Λ has a bounded inverse in the space of vector fields equipped with the majorant norm.*

Proof. The formal inverse operator ad_Λ^{-1} is diagonal,

$$\mathrm{ad}_\Lambda^{-1} \colon \sum_{k,\alpha} c_{k\alpha} x^\alpha \frac{\partial}{\partial x_k} \longmapsto \sum_{k,\alpha} \frac{c_{k\alpha}}{\lambda_k - \langle \alpha, \lambda \rangle}\, x^\alpha \frac{\partial}{\partial x_k}.$$

In the Poincaré domain the absolute values of all denominators are bounded from below by a positive constant $\varepsilon > 0$, therefore *any* majorant ρ-norm is increased by no more than ε^{-1}:

$$\llbracket \mathrm{ad}_\Lambda^{-1} \rrbracket_\rho \leqslant \left(\inf_{j,\alpha} |\lambda_j - \langle \alpha, \lambda \rangle\, | \right)^{-1} < +\infty.$$

This proves that ad_Λ has the bounded inverse. □

Remark 5.12. A diagonal operator of the form $\sum_\alpha c_\alpha z^\alpha \mapsto \sum \mu_\alpha c_\alpha x^\alpha$ with bounded entries, $\sup_\alpha |\mu_\alpha| < +\infty$, which is always defined and bounded in the majorant norm, may be *not defined* or defined but unbounded on the holomorphic space $\mathcal{A}(D_\rho)$; see Remark 5.9. The "real" counterexample is even simpler: the operator which multiplies odd coefficients by -1, sends the series $1 - x^2 + x^4 - \cdots$, converging and bounded on $[-1, 1]$, into an unbounded function.

Let $F = (F_1, \ldots, F_n) \in \mathcal{D}(\mathbb{C}^n, 0)$ be a holomorphic vector function defined in some polydisk near the origin. The *operator of argument shift* is the operator

$$S_F \colon h(x) \mapsto F(x + h(x)), \tag{5.5}$$

acting on holomorphic vector fields $h \in \mathcal{D}(\mathbb{C}^n, 0)$ without the free term, $h(0) = 0$. We want to show that S_F is in some sense strongly contracting. The formal statement looks as follows.

Consider the one-parameter family of majorant Banach spaces $\mathcal{B}_\rho$ as in Definition 5.7 indexed by the real parameter $\rho \in (\mathbb{R}_+, 0)$. We consider $\mathcal{B}_{\rho'}$ as a subspace in $\mathcal{B}_\rho$ for all $0 < \rho < \rho'$ (the natural embedding $\mathrm{id}_{\rho',\rho} \colon \mathcal{B}_{\rho'} \to \mathcal{B}_\rho$ is continuous).

Let S be an operator defined on all of these spaces for all sufficiently small values of ρ, considered as a family of operators $S_\rho \colon \mathcal{B}_\rho \to \mathcal{B}_\rho$ which commute with the "restriction operators" $\mathrm{id}_{\rho',\rho}$ for any $\rho < \rho'$, but we will omit the subscript in the notation of $S_\rho = S$.

Definition 5.13. The operator $S \cong \{S_\rho\}$ is *strongly contracting*, if

(1) $[\![S(0)]\!]_\rho = O(\rho^2)$ and
(2) S is Lipschitz on the ball $B_\rho = \{[\![h]\!]_\rho \leqslant \rho\} \subset \mathcal{B}_\rho$ of the majorant ρ-norm (with the same ρ), with the Lipschitz constant no greater than $O(\rho)$ as $\rho \to 0$.

Note that any strongly contracting operator takes the balls B_ρ strictly into themselves, since the center of the ball is shifted by $O(\rho^2)$ and the diameter of the image $S(B_\rho)$ does not exceed $2\rho\, O(\rho) = O(\rho^2)$.

The involved definition of strong contraction intends to make the formulation of the following claim easy.

Lemma 5.14. *Assume that the germ* $F \colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ *is holomorphic and its linearization is zero,* $\left(\frac{\partial F}{\partial x}\right)(0) = 0$.

Then the operator of argument shift (5.5) *is strongly contracting.*

Proof. First note that S_F takes $h = 0$ into $F(x)$; the latter function has ρ-norm $O(\rho^2)$ for all sufficiently small ρ, since F begins with quadratic terms.

Next we compute the Lipschitz constant for $S = S_F$ restricted on the ball $B_\rho \subseteq \mathcal{B}_\rho$. If $h, h' \in \mathbb{C}^n[[x_1, \ldots, x_n]]$ are two vector fields, then the difference

$$g = Sh - Sh' = F \circ (\mathrm{id} + h) - F \circ (\mathrm{id} + h')$$

can be represented as the integral

$$g(x) = \int_0^1 \left(\frac{\partial F}{\partial x}\right)\big(x + \tau h(x) + (1-\tau)h'(x)\big)\cdot (h(x) - h'(x))\,d\tau.$$

By Lemma 5.10, since $\tau \in [0,1]$, we have

$$[\![g]\!]_\rho \leqslant [\![\tfrac{\partial F}{\partial x}]\!]_\sigma \cdot [\![h - h']\!]_\rho, \qquad \sigma = [\![x + \tau h(x) + (1-\tau)h'(x)]\!]_\rho .$$

The norm σ is no greater than $[\![x]\!]_\rho + \max([\![h]\!]_\rho, [\![h']\!]_\rho) = (n+1)\rho$ if both h, h' are from the ρ-ball B_ρ. On the other hand, if F is a holomorphic vector function without free and linear terms, its Jacobian matrix is holomorphic without free terms and hence its σ-norm is no greater than $C\sigma$ for all sufficiently small $\sigma > 0$. Collecting everything together, we see that S_F is Lipschitz on the ρ-ball B_ρ, with the Lipschitz constant (contraction rate) not exceeding $(n+1)C\rho$, so S_F is strongly contracting. □

Proof of Theorem 5.5 (nonresonant case). Now we can prove that a holomorphic vector field with diagonal nonresonant linearization matrix Λ of Poincaré type is holomorphically linearizable in a sufficiently small neighborhood of the origin. The proof serves as a paradigm for a more technically involved proof required for the resonant case.

A holomorphic transformation $H = \mathrm{id} + h$ conjugates the linear vector field Λx (the normal form) with the initial nonlinear field $\Lambda x + F(x)$, if and only if

$$\Lambda h(x) - \left(\tfrac{\partial h}{\partial x}\right)\Lambda x = F\big(x + h(x)\big). \tag{5.6}$$

Using the operators introduced earlier, this can be rewritten as the identity

$$\mathrm{ad}_\Lambda h = S_F h, \qquad S_F\, h = F \circ (\mathrm{id} + h), \quad \mathrm{ad}_\Lambda = [\Lambda, \cdot\,]. \tag{5.7}$$

We will show in an instant that the operator $\mathrm{ad}_\Lambda^{-1} \circ S_F$ restricted on the space $\mathcal{B}_\rho$ has a fixed point h, if $\rho > 0$ is sufficiently small,

$$h = \big(\mathrm{ad}_\Lambda^{-1} \circ S_F\big)(h), \qquad h \in \mathcal{B}_\rho. \tag{5.8}$$

Applying to both parts the operator ad_Λ, we conclude that h solves (5.7) and therefore $\mathrm{id} + h$ conjugates the linear field Λx with the nonlinear field $\Lambda x + F(x)$ in the polydisk $\{|x| < \rho\}$.

Consider this operator $\mathrm{ad}_\Lambda^{-1} \circ S_F$ in the space $\mathcal{B}_\rho$ with sufficiently small ρ. The operator ad_Λ^{-1} is bounded by Lemma 5.11; its norm is the reciprocal to the smallest small divisor and is independent of ρ. On the other hand, the argument shift operator S_F is strongly contracting with the contraction rate (Lipschitz constant) going to zero with ρ as $O(\rho)$. Thus the composition will be contracting on the ρ-ball B_ρ in the ρ-majorant norm with the contraction rate $O(1) \cdot O(\rho) = O(\rho) \to 0$. By the contracting map principle, there exists a unique fixed point of the operator equation (5.8) in the space $\mathcal{B}_\rho$

which is therefore a holomorphic vector function. The corresponding map $H = (\mathrm{id} + h)^{-1}$ linearizes the holomorphic vector field. □

5C. Resonant case: polynomial normal form. Modification of the previous construction allows us to prove that a resonant holomorphic vector field in the Poincaré domain can be brought into a *polynomial* normal form.

Consider a holomorphic vector field $F(x) = Ax + V(x)$ with the linearization matrix A having eigenvalues in the Poincaré domain, and nonlinear part V of order $\geqslant 2$ (i.e., 1-flat) at the origin. Without loss of generality (passing, if necessary, to an orbitally equivalent field cF, $0 \neq c \in \mathbb{C}$), one may assume that the eigenvalues of A satisfy the condition

$$1 < \operatorname{Re} \lambda_j < r \qquad \forall j = 1, \dots, n \tag{5.9}$$

with some natural $r \in \mathbb{N}$.

Theorem 5.15 (A. M. Lyapunov, H. Dulac)**.** *If the eigenvalues of the linearization matrix A of a holomorphic vector field $F(x) = Ax + V(x)$ satisfy the condition* (5.9) *with some integer $r \in \mathbb{N}$, then the holomorphic vector field $F(x)$ is locally holomorphically equivalent to any holomorphic vector field with the same r-jet.*

Proof. A holomorphic conjugacy $H = \mathrm{id} + h$ between the fields F and $F + g$ satisfies the functional equation $\left(\frac{\partial H}{\partial x}\right) F = (F + g) \circ H$ which can be expanded to

$$\left(\frac{\partial h}{\partial x}\right) Ax - Ah = (V \circ (\mathrm{id} + h) - V) + g \circ (\mathrm{id} + h) - \left(\frac{\partial h}{\partial x}\right) V. \tag{5.10}$$

Consider the three operators,

$$T_V \colon h \mapsto V \circ (\mathrm{id} + h) - V, \quad S_g \colon h \mapsto g \circ (\mathrm{id} + h), \qquad \Psi \colon h \mapsto \left(\frac{\partial h}{\partial x}\right) V.$$

Using these three operators, the differential equation (5.10) can be written in the form

$$\mathrm{ad}_A h = Th + Sh + \Psi h, \tag{5.11}$$

where $T = T_V$, $S = S_g$ and, as before in (5.7), ad_A is the commutator with the *linear* field $\mathbf{A}(x) = Ax$. The key difference with the previous case is two-fold: first, because of the resonances, the operator ad_A is *not invertible* anymore, and second, since the field F is nonlinear, the additional operator Ψ occurs in the right hand side. Note that this operator is a derivation of h, thus is unbounded in *any* majorant norm $[\![\cdot]\!]_\rho$.

Let $\mathcal{B}_{m,\rho} = \{f \colon j^m f = 0\} \cap \mathcal{B}_\rho$ be a subspace of m-flat series in the Banach space $\mathcal{B}_\rho$, equipped with the same majorant norm $[\![\cdot]\!]_\rho$. Since V is 1-flat, all three operators T, S, Ψ map the subspace $\mathcal{B}_{m,\rho}$ into itself for any $m > 1$.

Moreover, by Lemma 5.14, the argument shift operator S is strongly contracting, regardless of the choice of m. The "finite difference" operator T_V differs from the argument shift, S_V by the constant operator $V = T(0)$ which does not affect the Lipschitz constant. Since $\|V\|_\rho = O(\rho^2)$, the operator T is also strongly contracting.

The operator ad_A preserves the order of all monomial terms and hence also maps $\mathcal{B}_{m,\rho}$ into itself for all m, ρ, and is *invertible* on these spaces if m is sufficiently large. Indeed, if $|\alpha| > r+1$, then by (5.9) $\operatorname{Re}(\langle\alpha,\lambda\rangle - \lambda_j) > 0$, and all denominators in the formula

$$\operatorname{ad}_A^{-1}\big|_{\mathcal{B}_{m,\rho}} : \sum_{|\alpha|\geqslant m} c_{k\alpha}\, x^\alpha \frac{\partial}{\partial x_j} \longmapsto \sum_{|\alpha|\geqslant m} \frac{c_{k\alpha}}{\langle\alpha,\lambda\rangle - \lambda_j}\, x^\alpha \frac{\partial}{\partial x_j} \tag{5.12}$$

are nonzero if $m \geqslant r+1$, and the restriction of ad_A^{-1} on $\mathcal{B}_{m,\rho}$ is bounded. Moreover,

$$\|\operatorname{ad}_A^{-1} h\|_\rho \leqslant O(1/m)\, \|h\|_\rho \tag{5.13}$$

uniformly over all $h \in \mathcal{B}_{m,\rho}$ of order $m \geqslant r+1$.

Thus the two compositions, $\operatorname{ad}_A^{-1} \circ S$ and $\operatorname{ad}_A^{-1} \circ T$, are strongly contracting. To prove the theorem, it remains to prove that the *linear* operator $\operatorname{ad}_A^{-1} \circ \Psi \colon \mathcal{B}_{m,\rho} \to \mathcal{B}_{m,\rho}$ is strongly contracting when m is larger than $r+1$.

Consider the $\|\cdot\|_\rho$-normalized vectors $h_{k\beta} = \rho^{-|\beta|} x^\beta \frac{\partial}{\partial x_k}$ for all $k = 1,\dots,m$ and all $|\beta| \geqslant m$ spanning the entire space $\mathcal{B}_{m,\rho}$. We prove that

$$\|\operatorname{ad}_A^{-1} \Psi h_{k\beta}\|_\rho = O(\rho) \qquad \text{as } \rho \to 0 \tag{5.14}$$

uniformly over $|\beta| \geqslant m$ and all k. Since $\operatorname{ad}_A^{-1} \circ \Psi$ is linear, this would imply that $\operatorname{ad}_A^{-1} \circ \Psi$ is strongly contracting.

The direct computation yields

$$\Psi h_{k\beta} = \sum_{i=1}^n \rho^{-|\beta|} \frac{\beta_i}{x_i}\, x^\beta\, V_i \frac{\partial}{\partial x_k}.$$

Since V is 1-flat, $\|V_i\|_\rho = O(\rho^2)$; substituting this into the definition of the majorant norm, we obtain

$$\|\Psi h_{k\beta}\|_\rho \leqslant \sum_i \beta_i\, \rho^{-1} O(\rho^2) = \beta_i\, O(\rho),$$

where $O(\rho)$ is uniform over all β. Since the order of the products $\frac{x^\beta}{x_i} V_i$ is at least $|\beta|+1$, by (5.13) we have

$$\|\operatorname{ad}_A^{-1} \Psi h_{k\beta}\|_\rho \leqslant \frac{\beta_i}{|\beta|}\, O(\rho) = O(\rho)$$

uniformly over all β with $|\beta| \geqslant m \geqslant r+1$. Thus the last remaining composition $\operatorname{ad}_A^{-1} \circ \Psi$ is also strongly contracting, which implies existence of a solution for the fixed point equation

$$h = \mathrm{ad}^{-1} \circ (T + S + \Psi)h$$

equivalent to (5.11), in a sufficiently small polydisk $\{|x| < \rho\}$. □

Now one can easily complete the proof of the holomorphic normalization theorem in the Poincaré domain in the resonant case.

Proof of Theorem 5.5 (resonant case). By the Poincaré–Dulac normalization process, one can eliminate all nonresonant terms up to any finite order m by a polynomial transformation. By Theorem 5.15, m-flat holomorphic terms can be eliminated by a holomorphic transformation if m is large enough (depending on the spectrum of the linearization matrix). □

Remark 5.16. In the Poincaré domain one can prove an even stronger claim: if a holomorphic vector field depends analytically on finitely many additional parameters $\lambda \in (\mathbb{C}^m, 0)$ and belongs to the Poincaré domain for $\lambda = 0$, then by a holomorphic change of variables holomorphically depending on parameters, the field can be brought to a polynomial normal form involving only resonant terms. In such a form this assertion is formulated in [**Bru71**]. The proof can be achieved by minor adjustment of the arguments used in the demonstration of Theorem 5.15.

5D. Holomorphic normal forms for self-maps. In the same way as the formal theory for vector fields $\mathcal{D}[[\mathbb{C}^n, 0]]$ and maps $\mathrm{Diff}[[\mathbb{C}^n, 0]]$ are largely parallel (see §4**G**), the analytic theory of vector fields and biholomorphisms are also parallel.

The additive resonance conditions $\lambda_j - \langle \alpha, \lambda \rangle \neq 0$ correspond to the multiplicative resonance conditions $\mu_j^{-1} \mu^\alpha \neq 1$. The additive Poincaré condition (Definition 5.1) requires that (eventually after a rotation) all eigenvalues λ_j of the vector field lie to one side of the imaginary axis. Its multiplicative counterpart requires that all eigenvalues μ_j of the map must be to one side of the unit circle. Such maps are automatically contracting or expanding, and admit at most finitely many multiplicative resonance relations between the eigenvalues.

The result parallel to the Poincaré Theorem 5.5 takes the following form. Let $M \in \mathrm{GL}(n, \mathbb{C})$ be a matrix in the upper triangular Jordan normal form with the eigenvalues $\mu_1, \ldots, \mu_n \in \mathbb{C}^*$. The Poincaré–Dulac normal form is a map

$$f\colon \mathbb{C}^n \to \mathbb{C}^n, \qquad x \mapsto f(x) = Mx + \sum_{\substack{\alpha \in \mathbb{Z}_+,\ |\alpha| \geqslant 2 \\ \mu_j = \mu^\alpha}} x^\alpha \mathbf{e}_j, \tag{5.15}$$

where $\mathbf{e}_j \in \mathbb{C}^n$ is the jth basis vector. If M is in the *multiplicative Poincaré domain*, i.e., if the eigenvalues are all of modulus less than one or all of modulus greater than one, then the normal form (5.15) is polynomial (contains finitely many terms).

The general result for holomorphic self-maps in the Poincaré domain has the following form.

Theorem 5.17. *A holomorphic invertible map $f \in \operatorname{Diff}(\mathbb{C}^n, 0)$ with the spectrum $\mu_1, \dots, \mu_n$ inside the unit disk, $0 < |\mu_j| < 1$, $j = 1, \dots, n$, is analytically equivalent to its polynomial Poincaré–Dulac formal normal form* (5.15). □

In the important particular case of one-dimensional maps, the multiplicative Poincaré condition holds automatically if the map is *hyperbolic*, i.e., if its multiplicator μ has modulus different from one. This automatically guarantees that resonances are impossible, and hence the Poincaré–Dulac normal form (5.15) is linear. The corresponding result was proved by E. Schröder (1870) and A. Kœnigs (1884).

Theorem 5.18. *A holomorphic germ $f \colon (\mathbb{C}, 0) \to (\mathbb{C}, 0)$, $f(x) = \mu x + O(x^2)$, is analytically linearizable if $|\mu| \neq 1$.*

If $f = f_t$ depends analytically on additional parameter $t \in U \subseteq \mathbb{C}^p$, the linearizing chart can also be chosen analytically depending on this parameter as soon as the respective multiplier μ_t remains off the unit circle.

Because of its importance, we will give an independent proof of this result by the path method in §5**F** below. Yet another (shortest known) proof is outlined in Problem 5.6.

5E. Linearization in the Siegel domain: Siegel, Brjuno and Yoccoz theorems (micro-survey). In the Siegel domain the denominators $\lambda_j - \langle \alpha, \lambda \rangle$ are not separated from zero, hence even in the nonresonant case the operator $\operatorname{ad}_A = [\mathbf{A}, \cdot\,]$ of commutation with the linear part of the field has unbounded inverse ad_a^{-1}. Yet since the operator S_F is *strongly contracting*, the equation (5.7) can be solved with respect to h by Newton-type iterations, provided that the small denominators $|\lambda_j - \langle \alpha, \lambda \rangle\,|$ do not approach zero too fast as $|\alpha| \to \infty$.

The corresponding technique is known under the general name of *KAM theory* (after A. Kolmogorov, V. Arnold and J. Moser). The issue is very classical; accurate formulations and proofs can be found in many excellent sources, e.g., [**CG93, Arn83**]. We formulate only the basic results.

Definition 5.19. A tuple of complex numbers $\lambda \in \mathbb{C}^n$ from the Siegel domain $\mathfrak{S}$ is called *Diophantine*, if the small denominators decay no faster than polynomially with $|\alpha|$, i.e.,

$$\exists C, N < +\infty \text{ such that } \quad \forall \alpha \in \mathbb{Z}_+^n, \quad |\lambda_j - \langle \alpha, \lambda \rangle|^{-1} \leqslant C\,|\alpha|^N. \qquad (5.16)$$

Otherwise the tuple (vector, collection) is called *Liouvillean*.

Liouvillean vectors are scarce: the set of points $\lambda \in \mathbb{C}^n$ satisfying violating the condition 5.16 with a given N, has Lebesgue measure zero in $\mathfrak{S} \subset \mathbb{C}^n$ if $N > (n-2)/2$; see [**Arn83**].

Theorem 5.20 (Siegel theorem)**.** *If the linearization matrix Λ of a holomorphic vector field is nonresonant of Siegel type and has Diophantine spectrum, then the field is holomorphically linearizable.*

Thus the majority (in the sense of Lebesgue measure) of germs of holomorphic vector fields are analytically linearizable. Yet one may further relax sufficient conditions for convergence of linearizing series in the Siegel domain.

Definition 5.21. A nonresonant collection $\lambda \in \mathbb{C}^n$ is said to satisfy the *Brjuno condition*, if the small denominators decrease *sub-exponentially*,

$$|\lambda_j - \langle \alpha, \lambda \rangle|^{-1} \leqslant C e^{|\alpha|^{1-\varepsilon}}, \qquad \text{as } |\alpha| \to \infty, \qquad (5.17)$$

for some finite C and positive $\varepsilon > 0$.

Theorem 5.22 (Brjuno theorem)**.** *A holomorphic vector field with nonresonant linearization matrix of Siegel type satisfying the Brjuno condition, is holomorphically linearizable.*

On the other hand, if the denominators $|\lambda_j - \langle \alpha, \lambda \rangle|$ accumulate to zero too fast, e.g., *super-exponentially*, then the corresponding germs are in general nonlinearizable (cf. with Remark 5.33 below).

Analogs of the Siegel and Brjuno theorems hold for holomorphic germs. The most important case is that of one-dimensional *conformal germs* from the group $\mathrm{Diff}(\mathbb{C}^1, 0)$. Such germs belong to the Siegel domain if and only if their multiplicator μ belongs to the unit circle, $\mu = \exp 2\pi i l$, with some $l \in \mathbb{R}$; they are nonresonant if l is an irrational number. The Diophantine and Brjuno conditions translate for this case as assumptions that this irrational number $l \in \mathbb{R} \smallsetminus \mathbb{Q}$ does not admit abnormally accurate rational approximations.

For instance, if the complex number $\mu = \exp 2\pi i l$, $l \in \mathbb{R}$, satisfies the *multiplicative Brjuno condition*

$$|\mu^k - 1|^{-1} < C e^{k^{1-\varepsilon}}, \qquad C < +\infty,\ \varepsilon > 0, \qquad (5.18)$$

then any holomorphic map $(\mathbb{C}, 0) \to (\mathbb{C}, 0)$, $z \mapsto \mu z + z^2 + \cdots$, is holomorphically linearizable. The sufficient arithmetic condition (5.18) turns out to also be *necessary* in the following sense.

Theorem 5.23 (J.-C. Yoccoz [**Yoc88, Yoc95**]). *If the complex number* $\mu = \exp 2\pi i l$, $l \in \mathbb{R}$, *violates the multiplicative Brjuno condition* (5.18), *then there exists a nonlinearizable holomorphic germ* $(\mathbb{C}, 0) \to (\mathbb{C}, 0)$, $z \mapsto \mu z + f(z)$, $f(z) = z^2 + \cdots$.

In fact, in the assumptions of this theorem for almost all complex numbers $w \in \mathbb{C}$ the germ $f_w(z) = \mu z + w f(z)$ is analytically nonlinearizable; cf. with Theorem 5.29 below and [**PM01**].

Remark 5.24. The condition on the rate of convergence of small denominators can be reformulated in terms of the growth rate of coefficients of decomposition of the irrational number $l \in \mathbb{R} \setminus \mathbb{Q}$ into the continuous fraction. This is a more standard way of formulating the Brjuno condition in the recent literature.

If a resonance occurs in the Siegel case, then the situation turns out to be even more complicated: a resonant conformal germ $f \in \operatorname{Diff}(\mathbb{C}, 0)$ with multiplicator $\mu \in \exp 2\pi i \mathbb{Q}$ is almost never analytically equivalent to its polynomial Poincaré–Dulac formal normal form described in Theorem 4.26. This result and its numerous developments are explained in detail in §21.

A two-dimensional analytic orbital classification of Siegel resonant vector fields (saddle-nodes and resonant saddles from Table I.1) is at least as difficult as the analytic classification of resonant germs from $\operatorname{Diff}(\mathbb{C}, 0)$. Indeed, in §7 we will show that the corresponding foliations have leaves with nontrivial (infinite cyclic) fundamental group, whose holonomy is generated by Siegel resonant germs from $\operatorname{Diff}(\mathbb{C}, 0)$. The details can be found in Chapter IV; see §22.

Somewhat unexpectedly, the cuspidal points behave better than their less degenerate brethren. In [**SZ02**] H. Żołądek and E. Stróżyna proved that one can always reduce a holomorphic planar vector field near a cuspidal singular point to a *holomorphic* normal form (4.17) (i.e., with converging series $a(x), b(x) \in \mathcal{O}(\mathbb{C}, 0)$) by a biholomorphic transformation. The direct and difficult proof from [**SZ02**] was recently replaced by beautiful geometric arguments by F. Loray [**Lor06**]. This proof, based on *nonlocal uniformization technique*, is split into a series of problems in §23 (Problems 23.6–23.13).

5F. Path method. In this section we outline another very powerful *analytic* method of reducing holomorphic vector fields and self-maps to their normal forms. This method is called *path method* (méthode de chemin, homotopy method) since it consists of connecting the initial object (field, self-map) with its normal form by a path (usually a line segment) and then looking for a flow of a nonautonomous vector field that would conjugate with each other all objects in this parametric family.

We *illustrate* the path method by proving two relatively simple results, analytic reducibility of one-dimensional holomorphic vector fields (cf. with Theorem 4.24) and one-dimensional hyperbolic self-maps to their normal form. Both results, however, can be proved by shorter arguments; see Problems 5.5 and 5.6 below.

Theorem 5.25. *Any analytic vector field* $F(x) = x^{k+1}(1+\cdots)\frac{\partial}{\partial x} \in \mathcal{D}(\mathbb{C},0)$ *is analytically conjugate to its polynomial formal normal form* $F_0(x) = (x^{k+1} + ax^{2k+1})\frac{\partial}{\partial x}$.

Proof. Without loss of generality we may assume from the beginning, that the jet of F of any specified order is already reduced to the normal form. Thus we can assume that the field $F = F_1$ is given as $F_0(x) + R(x)\frac{\partial}{\partial x}$, where R is as flat at the origin, as necessary. It will be sufficient to require that the function $R(x)$ has zero of multiplicity $2k+2$ at the origin, $R(x) = x^{2k+2}S(x)$, $S \in \mathcal{O}(\mathbb{C},0)$. We want to show that for all values of an auxiliary complex parameter z from some domain $U \subseteq \mathbb{C}$ containing the segment $[0,1]$, the vector fields $F_z(x) = F_0(x) + zR(x)\frac{\partial}{\partial x}$ are holomorphically equivalent to each other. This, in particular, would imply that F_0 and F_1 are holomorphically equivalent, which would immediately imply the assertion of the theorem.

Consider the planar domain $(\mathbb{C},0) \times U$ and the vector field on it,

$$\mathbf{F} = \mathbf{F}_0 + zR(x) \cdot \tfrac{\partial}{\partial x} + 0 \cdot \tfrac{\partial}{\partial z}, \qquad \mathbf{F}_0 = (x^{k+1} + ax^{2k+1})\tfrac{\partial}{\partial x}, \tag{5.19}$$

which is the suspension of the above parametric family of vector fields on the line.

Consider another planar vector field $\mathbf{H} \in \operatorname{Diff}\big((\mathbb{C}^1,0) \times U\big)$, $U \subseteq \mathbb{C}$, which has the form

$$\mathbf{H} = h(x,z)\tfrac{\partial}{\partial x} + 1 \cdot \tfrac{\partial}{\partial z}, \qquad h(0,z) \equiv 0. \tag{5.20}$$

Lemma 5.26 (Path method paradigm)**.** *If there exists a holomorphic vector field* $\mathbf{H} \in \operatorname{Diff}\big((\mathbb{C}^1,0) \times U\big)$ *of the form* (5.20) *which* commutes *with* $\mathbf{F}$*,*

$$[\mathbf{F}, \mathbf{H}] = 0, \tag{5.21}$$

then all germs of vector fields $F_z \in \mathcal{D}(\mathbb{C}^1,0)$ *are holomorphically equivalent to each other for all values of* $z \in U$*.*

Proof. If the vector fields $\mathbf{F}$ and $\mathbf{H}$ commute, then the flow of the vector field $\mathbf{H}$ commutes with the flow of $\mathbf{F}$ and hence the flow maps of $\mathbf{H}$ are symmetries of the field $\mathbf{F}$.

Because of the special structure of $\mathbf{H}$, its flow sends the lines $\{z = \text{const}\}$ into each other, each time fixing the origin $\{x = 0\}$. Thus the flow $\exp \mathbf{H}$ maps $\{z = 0\}$ into $\{z = 1\}$, is defined in some neighborhood of the origin and conjugating $\mathbf{F}|_{z=0} = F_0$ with $\mathbf{F}|_{z=1} = F_1$. □

Now we can complete the proof of Theorem 5.25, showing that in the assumptions of the theorem, such a vector field $\mathbf{H}$ indeed exists. The *homological equation* (5.21) is equivalent to a partial differential equation on the function H,

$$f \cdot \frac{\partial h}{\partial x} - h \cdot \frac{\partial f}{\partial x} = -R, \qquad f(x,z) = x^{k+1} + ax^{2k+1} + zR(x). \tag{5.22}$$

Yet in fact this equation can be considered as a linear first order ordinary (with respect to the x-variable) nonhomogeneous differential equation analytically depending on the parameter $z \in U$. The solution of the corresponding *homogeneous* equation is immediate,

$h_0(x,z) = f(x,z)$, using the ansatz $h(x,z) = s(x,z)h_0(x,z)$ we obtain the equation (recall that $R = x^{2k+2}S$),

$$f^2 \cdot \frac{\partial s}{\partial z} = -R(x), \quad \text{i.e.} \quad \frac{\partial s}{\partial x} = -\frac{S(x)}{\bigl(1 + ax^k + x^{k+1}S(x)\bigr)^2}. \tag{5.23}$$

Integration of the right hand side with the initial condition $s(0,z) = 1$ yields a solution $s = s(x,z)$ holomorphic at $x = 0$ for all $z \in U$. The vector field $\mathbf{H} = s(x,z)\mathbf{F} + 1 \cdot \frac{\partial}{\partial z}$ satisfies all conditions imposed by Lemma 5.26 and allows us to construct a holomorphic conjugacy between F_0 and F_1. □

Obviously, the polynomial normal form (4.21) can be replaced by the rational normal form (4.23).

Remark 5.27. Besides holomorphic differential vector fields, one may consider *meromorphic differential* 1*-forms* on the complex line (or, more precisely, their germs at the origin): the set of all such forms is naturally denoted by $\Lambda^1(\mathbb{C},0) \otimes \mathfrak{M}(\mathbb{C},0)$.

The group $\operatorname{Diff}(\mathbb{C},0)$ acts on such forms, so one can establish normal forms. Yet instead of developing parallel theory, one can use duality: a 1-form $\omega \in \Lambda^1(\mathbb{C},0)$ and a vector field $F \in \mathcal{D}(\mathbb{C},0)$ are called *dual*, if $\omega(F) \equiv 1$. Holomorphic transform of a dual pair is again a dual pair.

Meromorphic (i.e., with a pole at the origin) 1-forms have two obvious invariants that cannot be changed by holomorphic transformations: the *order of the pole* and the *residue* at this point.

The form dual to the rational vector field (4.23) is $\frac{dx}{x^{k+1}} - a\frac{dx}{x}$, and the formal invariant $a \in \mathbb{C}$ is the residue of this form (modulo the sign). This observation explains the role of the formal invariant.

As yet another application of the path method, we give an independent proof of the Schröder–Kœnigs Theorem 5.18.

Consider the analytic self-map $f \in \operatorname{Diff}(\mathbb{C},0)$, $f(x) = \mu x + r(x)$, with the multiplicator $\mu \in \mathbb{C}^*$, $|\mu| < 1$ and analytic nonlinearity $r(x) = O(x^2)$.

As before, we embed f into an analytic one-parameter deformation $f_z(x) = \mu x + zr(x)$ with a complex parameter $z \in U \subseteq \mathbb{C}$, $[0,1] \subseteq U$, and suspend it to the planar self-map $\mathbf{f} \in \operatorname{Diff}(\mathbb{C}^2,0)$,

$$\mathbf{f}\colon (x,z) \mapsto (\mu x + zr(x), z), \qquad (x,z) \in (\mathbb{C}^1,0) \times U. \tag{5.24}$$

The following lemma is a reformulation of the main paradigm of the path method (Lemma 5.26) for the current context.

Lemma 5.28. *If a vector field* $\mathbf{H}$ *as in* (5.20) *is preserved by the self-map* $\mathbf{f}$*, i.e.,*

$$\mathbf{f}_* \cdot \mathbf{H} = \mathbf{H} \circ \mathbf{f}, \qquad \mathbf{f}_* = \frac{\partial \mathbf{f}(x,z)}{\partial(x,z)}, \tag{5.25}$$

then all self-maps f_z *for all* $z \in U$*, are analytically equivalent, in particular,* $f_1 = f$ *is analytically equivalent to the linear map* f_0*. The conjugacy is achieved by the flow of the field* $\mathbf{H}$ *restricted on the lines* $\{z = \text{const}\}$*.* □

The proof of Lemma 5.28 almost literally reproduces that of Lemma 5.26 and is skipped. In order to prove Theorem 5.18, we need only to show that the homological equation (5.25) is solvable.

Alternative proof of Theorem 5.18. The identity (5.25) reduces to a single scalar linear nonhomogeneous functional equation

$$\frac{\partial f_z(x)}{\partial x} \cdot h(x,z) - h(f_z(x),z) = r(x). \tag{5.26}$$

This equation can be solved in two steps, solving first the corresponding homogeneous equation $(\frac{\partial f}{\partial x})u - u \circ f = 0$, and then looking for a solution of (5.26) in the form $h = su$, similar to the way the equation (5.22) was solved.

The homogeneous equation can be rewritten as a fixed point statement,

$$h = \left(\frac{\partial f}{\partial x}\right)^{-1} \cdot (h \circ f), \qquad f = f_z \in \operatorname{Diff}(\mathbb{C}, 0). \tag{5.27}$$

It has a trivial (zero) solution, yet we can restrict the operator occurring in the right hand side, on the subspace of functions tangent to identity, $h(x) = x + O(x^2)$.

Without loss of generality we may assume (passing to a sufficiently small neighborhood of the origin which is rescaled to the unit disk) that all maps f_z satisfy the inequalities

$$\begin{gathered} \left|\frac{\partial f}{\partial z}\right| \geqslant \mu_-, \qquad |f(x)| < \mu_+|x|, \qquad \forall x \in D_1 = \{|x| \leqslant 1\}, \\ 0 < \mu_- < |\mu| < \mu_+ < 1. \end{gathered} \tag{5.28}$$

Here $\mu_\pm$ are two positive constants which can be assumed to be arbitrarily close to $|\mu| < 1$.

First we show that the operator $\Phi\colon h \mapsto (\frac{\partial f}{\partial x})^{-1} \cdot (h \circ f)$ restricted on the subspace

$$\mathcal{M} = \{u \in \mathcal{A}(D_1) \colon u(0) = 0,\ \tfrac{du}{dx} = 1\}$$

of holomorphic functions tangent to the identity at the origin, is contracting in the sense of the usual supremum-norm $\|u\| = \max_{x \in D_1} |u(x)|$. Clearly, $\Phi(\mathcal{M}) \subseteq \mathcal{M}$.

Indeed, since Φ is linear, it is sufficient to show that $\|\Phi q\| < \lambda \|q\|$ for any $q \in \mathcal{A}(D_1)$ having a second order zero at the origin and some λ strictly between 0 and 1. Note that for any such function $q(x)$, we have the inequality $|q(x)| \leqslant \|q\| \cdot |x|^2$: it is sufficient to apply the maximum modulus principle to the holomorphic ratio $q(x)/x^2$. Then from (5.28) it immediately follows that

$$\|\Phi q\| \leqslant \max_{|x| \leqslant 1} \frac{1}{\mu_-} \|q\| \cdot |f(x)|^2 \leqslant \frac{\mu_+^2}{\mu_-} \cdot \max_{|x| \leqslant 1} \|q\|\, |x|^2 \leqslant \frac{\mu_+^2}{\mu_-} \cdot \|q\|.$$

Since the ratio μ_+^2/μ_- can be made arbitrarily close to $|\mu| < 1$, the operator Φ restricted on $\mathcal{M}$ is contracting and hence has a holomorphic fixed point u analytically depending on z and any additional parameters (if present).

Now a solution of the nonhomogeneous equation can be found using the ansatz $h = su$. Substituting this ansatz into the equation (5.26), we obtain the *Abel-type equation*

$$s - s \circ f = -R(x), \qquad R = R_z(x) = \frac{r(x)}{(\frac{\partial f}{\partial z}) \cdot u(x,z)}, \qquad f = f_z(x). \tag{5.29}$$

The function $R_z(x)$ is holomorphic and vanishes at the origin $x = 0$ for all values of x, since f_z has a simple zero and $r(x)$ has a double zero at the origin.

The formal solution of the equation (5.29) is given by the series

$$s = -\sum_{k=0}^{\infty} R \circ f^{\circ k}, \qquad s = s(\cdot, z),\ f = f_z,\ R = R_z, \tag{5.30}$$

which is well defined because f is contracting. Moreover, since R vanishes at the origin, we have $|R_z(x)| < C|x|$ for some $C < \infty$ and all $x \in D_1$. Combining this with the uniform bounds $|f^{\circ k}(x)| \leqslant \mu_+^k |x|$ implied by (5.28), we conclude that the series (5.30) converges uniformly on D_1 and hence its sum is a holomorphic function vanishing at $x = 0$. The holomorphic vector field $\mathbf{H} = s(x,z)u(x,z)\frac{\partial}{\partial x} + 1 \cdot \frac{\partial}{\partial z}$ solves the equation (5.25).

The alternative proof of Theorem 5.18 is complete. □

* * *

5G. Divergence dichotomy. As follows from the Poincaré, Siegel and Brjuno theorems, for most linear parts the linearizing series converges, and in the remaining cases the linearizing series *may diverge*. On the other hand, no matter how “bad” the linearization and its eigenvalues are, there are always nonlinear systems that can be linearized (e.g., linear systems in nonlinear coordinates). It turns out that in some precise sense for a given linear part, the convergence/divergence pattern is common for *most* nonlinearities.

Consider a *parametric* nonlinear system

$$\dot{x} = Ax + z\,f(x), \qquad x \in \mathbb{C}^n, \quad z \in \mathbb{C}, \tag{5.31}$$

holomorphic in some neighborhood of the origin with the *nonresonant* linearization matrix A and the nonlinear part linearly depending on the auxiliary complex parameter $z \in \mathbb{C}$. For such systems for each value of the parameter $z \in \mathbb{C}$ there is a *unique* (by Remark 4.6) formal series $H_z(x) = x + h_z(x) \in \operatorname{Diff}[[x,z]]$ linearizing (5.31). This series may converge for some values of z while diverging for the rest. It turns out that there is a strict alternative: either the linearizing series *converges for all values of z without exception*, or on the contrary *the series H_z diverges for all z outside a rather small exceptional set $K \Subset \mathbb{C}$.*

The exceptional sets are small in the sense that their (electrostatic) *capacity* is zero. The notion of capacity is formally introduced below in §5**H**, where some of its basic properties are collected. We mention here only that zero capacity implies zero Lebesgue measure for any compact set.

Theorem 5.29 (Divergence dichotomy, Yu. Ilyashenko [**Ily79a**], R. Perez Marco [**PM01**])**.** *For any nonresonant linear family* (5.31) *one has the following alternative*:

(1) *Either the linearizing series $H_z \in \operatorname{Diff}[[\mathbb{C}^n, 0]]$ converges for all values of $z \in \mathbb{C}$ in a symmetric polydisk $\{|x| < r\}$ of a positive radius $r = r(z) > 0$ decreasing as $O(|z|^{-1})$ as $z \to \infty$, or*

(2) *The linearizing series H_z diverges for all values of z except for a set $K_f \Subset \mathbb{C}$ of capacity zero.*

The proof is based on the following property of polynomials, which can be considered as a quantitative uniqueness theorem for polynomials. If K is a set of positive capacity and $p \in \mathbb{C}[z]$ a polynomial vanishing on K, then by definition p vanishes identically. One can expect that if p is small on K, then it is also uniformly small on *any* other compact subset, in particular, on all compact subsets of $\mathbb{C}$.

Theorem 5.30 (Bernstein inequality). *If $K \Subset \mathbb{C}$ is a set of positive capacity, then for any polynomial $p \in \mathbb{C}[z]$ of degree $r \geqslant 0$,*

$$|p(z)| \leqslant \|p\|_K \, \exp(rG_K(z)), \tag{5.32}$$

where $\|p\|_K = \max_{z\in K} |p(z)|$ is the supremum-norm of p on K, and $G_K(z)$ is the nonnegative Green function of the complement $\mathbb{C} \smallsetminus K$ with the source at infinity; see (5.36).

We postpone the proof of this theorem until §5**H** and proceed with deriving Theorem 5.29 from the Bernstein inequality.

Lemma 5.31. *Formal Taylor coefficients of the formal series linearizing the field* (5.31) *are polynomial in z.*

More precisely, every monomial x^α, $|\alpha| \geqslant 2$, enters into the vector series h_z with the coefficient which is a polynomial of degree $\leqslant |\alpha| - 1$ in z.

Proof. The equation determining $h = h_z$ is of the form

$$\left(\frac{\partial h_z}{\partial x}\right)(Ax + z\, f(x)) = Ah_z(x). \tag{5.33}$$

Collecting the terms of degree m in x, we obtain for the corresponding mth homogeneous (vector) components $h_z^{(m)}$, $f^{(l)}$, the recurrent identities

$$\left(\frac{\partial h_z^{(m)}}{\partial x}\right) Ax - Ah_z^{(m)} = -z \sum_{k+l=m,\, l\geqslant 2} \left(\frac{\partial h_z^{(k+1)}}{\partial x}\right) f^{(l)}.$$

From these identities it obviously follows by induction that each $h_z^{(m)}$ is a polynomial of degree $m-1$ in z for all $m \geqslant 1$ (recall that f does not depend on z). □

Proof of Theorem 5.29. Assume that the formal series $H_z(x) = x + h_z(x)$ linearizing the field $F_z(x) = Ax + z\, f(x)$ converges for values of z belonging to some set $K^* \subset \mathbb{C}$ of positive capacity.

Consider the subsets $K_{c\rho} \Subset \mathbb{C}$, $\rho > 0$, $c < +\infty$, defined by the condition

$$z \in K_{c\rho} \iff |h_z^{(m)}(0)| \leqslant c\rho^{-m} \quad \forall m \in \mathbb{N}.$$

By this definition, $K^* = \bigcup_{c,\rho} K_{c\rho}$, since a Taylor series converges if and only if satisfies some Cauchy-type estimate. Each of the sets $K_{c\rho}$ obviously is a compact subset of $\mathbb{C}$, being an intersection of semialgebraic compact sets.

The compacts $K_{c\rho}$ are naturally nested: $K_{c'\rho'} \subseteq K_{c\rho}$ if $\rho' > \rho$ and $c' < c$. Passing to a countable sub-collection, one concludes that the set K of positive capacity is a countable union of compacts $K_{c\rho}$. By Proposition 5.35 (see below), one of these compacts must also be of positive capacity. Denote this compact by $K = K_{c\rho}$; by its definition,

$$|h_z^{(m)}| \leqslant c\rho^{-m}, \qquad \forall z \in K, \ \forall m \in \mathbb{N}.$$

Since the capacity of K is positive, Theorem 5.30 applies. By this theorem and Lemma 5.31, the polynomial coefficients of the series h_z for *any* $z \in \mathbb{C}$ satisfy the inequalities

$$|h_z^{(m)}| \leqslant c\rho^{-m} \exp[(m-1)G_K(z)] \leqslant c(\rho/\exp G_K(z))^{-m}, \quad \forall z \in \mathbb{C}, \ \forall m \in \mathbb{N}.$$

This means that the series h_z converges for any $z \in \mathbb{C}$ in the symmetric polydisk $\{|x| < \rho/\exp G_K(z)\}$. Together with the asymptotic growth rate $G_K(z) \sim \ln|z| + O(1)$ as $z \to \infty$ (see (5.36)) this proves the lower bound on the convergence radius of H_z. □

The dichotomy established in Theorem 5.29 may be instrumental in constructing "nonconstructive" examples of diverging linearization series. Consider again the nonresonant case where the homological equation $\operatorname{ad}_A g = f$ is always formally solvable.

Theorem 5.32 ([**Ily79a**]). *Assume that the formal solution $g \in \mathcal{D}[[\mathbb{C}^n, 0]]$ of the homological equation $\operatorname{ad}_A g = f$ is divergent.*

Then the series linearizing the vector field $F_z(x) = Ax + z\,f(x)$, diverges for most values of the parameter z, eventually except for a zero capacity set.

Proof. Assume the contrary, that the linearizing series H_z converges for a positive capacity set. By Theorem 5.29, it converges then for all values of z, in particular, h_z is holomorphic in some small polydisk $\{|x| < \rho', \ |z| < \rho''\}$.

Differentiating (5.33) in z, we see that the derivative $g(x) = \frac{\partial h_z(x)}{\partial z}\big|_{z=0}$ is a converging solution of the equation $(\frac{\partial g}{\partial x})Ax - Ag = f$, contrary to the assumption of the theorem. □

Remark 5.33. The divergence assumption appearing in Theorem 5.32 can be easily achieved. Assume that A is a diagonal matrix with the spectrum $\{\lambda_j\}_1^n$ such that the differences $|\lambda_j - \langle\lambda, \alpha\rangle\,|$ decrease faster than *any* geometric progression $\rho^{|\alpha|}$ for any nonzero ρ. Assume also that the Taylor coefficients of f are bounded *from below* by *some* geometric progression. Then the series $\operatorname{ad}_A^{-1} f$ diverges.

It remains to observe that a set of positive measure is necessarily of positive capacity (Proposition 5.35), hence divergence guaranteed in the assumptions of Theorem 5.32, occurs for almost all z in the measure-theoretic sense, as stated in [**Ily79a**].

5H. Capacity and Bernstein inequality. The brief exposition below is based on [**PM01**] and the encyclopedic treatise [**Tsu59**].

Recall that the function $\ln|z-a|^{-1} = -\ln|z-a|$ is the electrostatic potential on the z-plane $\mathbb{C} \cong \mathbb{R}^2$, created by a unit charge at the point $a \in \mathbb{C}$ and harmonic outside a. If μ is a nonnegative measure (charge distribution) on the compact $K \Subset \mathbb{C}$, then its potential is the function represented by the integral $u_\mu(z) = \int_K \ln|z-a|^{-1}\,d\mu(a)$ and the energy of this measure is

$$E_\mu(K) = \iint_{K\times K} \ln|z-w|^{-1}\,d\mu(z)\,d\mu(w).$$

This energy can be either infinite for all measures, or $E_\mu(K) < +\infty$ for some nonnegative measures. In the latter case one can show that among all nonnegative measures normalized by the condition $\mu(K) = 1$, the (finite) minimal energy $E^*(K) = \inf_{\mu(K)=1} E_\mu(K)$ is achieved by a unique *equilibrium distribution* μ_K. The corresponding potential $u_K(z)$ is called the *conductor potential* of K.

Definition 5.34. The (harmonic, electrostatic) *capacity* of the compact K is either zero (when $E_\mu = +\infty$ for any charge distribution on K) or $\exp(-E^*(K)) > 0$ otherwise;

$$\varkappa(K) = \begin{cases} 0, & \text{if } \forall\mu\ E_\mu(K) = +\infty, \\ \sup\limits_{\mu(K)=1,\,\mu\geqslant 0} \exp(-E_\mu(K)), & \text{otherwise.} \end{cases} \tag{5.34}$$

Proposition 5.35. *Capacity of compact sets possesses the following properties*:

(1) *Countable union of zero capacity sets also has capacity zero.*

(2) $\varkappa(K) \geqslant \sqrt{\operatorname{mes}(K)/\pi e}$, *where* $\operatorname{mes}(K)$ *is the Lebesgue measure of* K, *in particular, if* K *is a set of positive measure, then* $\varkappa(K) > 0$.

(3) *If* K *is a Jordan curve of positive length, then* $\varkappa(K) > 0$.

Proof. All these assertions appear in [**Tsu59**] as Theorems III.8, III.10 and III.11 respectively. □

Proposition 5.36. *For compact sets of positive capacity, the conductor potential is harmonic outside* K, *and*

$$\begin{aligned} u_K \leqslant \varkappa^{-1}(K), \qquad u_K|_K = \varkappa^{-1}(K) \quad \text{a.e.}, \\ u_K(z) = -\ln|z| + O(|z|^{-1}) \quad \text{as } z \to \infty. \end{aligned} \tag{5.35}$$

Proof. [**Tsu59**, Theorem III.12] □

As a corollary, we conclude that for sets of the positive capacity there exists the Green function

$$G_K(z) = \varkappa^{-1}(K) - u_K(z) = \ln|z| + \varkappa^{-1}(K) + o(1) \quad \text{as } z \to \infty, \tag{5.36}$$

nonnegative on $\mathbb{C} \smallsetminus K$, vanishing on K and asymptotic to the fundamental solution of the Laplace equation with the source at infinity.

Proof of Theorem 5.30 (Bernstein inequality). Since the assertion is invariant by multiplication by scalars, it is sufficient to prove for monic polynomials only.

Let $p(z) = z^r + \cdots$ be a monic polynomial of degree r. Consider the function

$$g(z) = \ln|p(z)| - \ln\|p\|_K - rG_K(z), \qquad z \in \mathbb{C} \smallsetminus K.$$

We claim that this function is nonpositive, $g \leqslant 0$ outside K. Indeed, g is negative near infinity since $g(z) = -\ln\|p\|_K - r\varkappa^{-1}(K) + o(1)$ as $z \to \infty$ by (5.36). On K we have the obvious inequality $\ln|p(z)| \leqslant \ln\|p\|_K$, and the Green function G_k has zero limit on K by (5.35). By construction, the function g is harmonic in $\mathbb{C} \smallsetminus K$ outside the isolated zeros of p where it tends to $-\infty$. By the maximum principle, the function g is nonpositive everywhere, $\ln|p(z)| \leqslant \ln\|p\|_K + rG_K(z)$ for all $z \in \mathbb{C} \smallsetminus K$. After passing to exponents this nonpositivity proves the theorem. □

Example 5.37. Assume that $K = [-1, 1]$ is the unit segment. Its complement is conformally mapped into the exterior of the unit disk $D = \{|w| < 1\}$ by the function $z = \frac{1}{2}(w + w^{-1})$, $w = z + \sqrt{z^2 - 1}$. The Green function G_D of the exterior is $\ln|w|$. Thus we obtain the explicit expression for G_K,

$$G_K = \ln\left|z + \sqrt{z^2 - 1}\right|,$$

which implies the classical form of the Bernstein inequality,

$$|p(z)| \leqslant \left|z + \sqrt{z^2 - 1}\right|^{\deg p} \max_{-1 \leqslant z \leqslant +1} |p(z)|. \tag{5.37}$$

Exercises and Problems for §5.

Problem 5.1. Prove that if h is a solution for the homogeneous homological equation (5.27) with a hyperbolic map f, then $H = h(x)\frac{\partial}{\partial x} \in \mathcal{D}(\mathbb{C}, 0)$ is a vector field that only by a constant factor differs from the generator of the self-map f: $f = \exp cH$, for some $c \in \mathbb{C}$.

Problem 5.2. Supply a detailed proof of the Poincaré theorem for self-maps (Theorem 5.17).

Exercise 5.3. Let $l \in \mathbb{R}$ be an irrational number whose rational approximations have only sub-exponential accuracy,

$$|l - \tfrac{p}{q}| > Ce^{-q^{1-\varepsilon}} \qquad \text{for some } C, \varepsilon > 0, \tag{5.38}$$

and $\mu = \exp 2\pi i l$. Prove that for any holomorphic right hand side f the homological equation

$$h \circ \mu - \mu h = f, \qquad f \in \mathcal{O}(\mathbb{C}, 0), \tag{5.39}$$

has an analytic (convergent) solution $h \in \mathcal{O}(\mathbb{C}, 0)$.

Exercise 5.4. Let $l \in \mathbb{R}$ be an irrational number which admits infinitely many exponentially accurate rational approximations p/q such that $|l - \frac{p}{q}| < e^{-q}$. Prove that for some right hand sides f the homological equation (5.39) has only divergent solutions (cf. with Remark 5.33).

Problem 5.5. Let $\mathbf{F} = F(x)x\frac{\partial}{\partial x} \in \mathcal{D}(\mathbb{C}^1, 0)$ be the germ of a holomorphic vector field at a singular point of multiplicity $k+1 \geqslant 2$ at the origin, $F(x) = x^{k+1}(1+o(1))$, and $\mathbf{F}' = \mathbf{F} + \mathbf{o}(x^{2k+1}) \in \mathcal{D}(\mathbb{C}^1, 0)$ is another such germ with the same $2k+1$-jet.

(i) Prove that these two germs are analytically equivalent if and only if two meromorphic 1-forms ω and ω', dual to $\mathbf{F}$ and $\mathbf{F}'$ respectively, are holomorphically equivalent (cf. with Remark 5.27).

(ii) Show that in the assumptions of the problem, the orders of the poles and the Laurent parts of the 1-forms ω and ω' coincide so that the difference $\omega - \omega'$ is holomorphic.

(iii) Passing to the primitives and denoting by $a_k, \dots, a_1, a_0$ the common Laurent coefficients of the forms ω, ω', prove that the equation

$$\frac{a_k}{y^k} + \cdots + \frac{a_1}{y} + a_0 \ln y + O(y) = \frac{a_k}{x^k} + \cdots + \frac{a_1}{x} + a_0 \ln x + O(z)$$

with holomorphic terms $O(y)$ and $O(x)$, admits a holomorphic solution $y = y(x)$ tangent to identity (substitute $y = ux$ and apply the implicit function theorem to the function $u(x)$ with $u(0) = 1$).

Problem 5.6 (Yet another proof of Schröder–Kœnigs theorem; cf. with [**CG93**]). Let $f \in \operatorname{Diff}(\mathbb{C}, 0)$ be a contracting hyperbolic holomorphic self-map, $f(z) = \lambda z + \cdots$, $|\lambda| < 1$, and $g(z) = \lambda z$ its linearization (the normal form).

Prove that the sequence of iterations $h_n = g^{-\circ n} \circ f^{\circ n}$ is defined and converges in some small disk around the origin. The limit $h = \lim h_n$ conjugates f and g.

Problem 5.7. Prove Theorem 5.5 along the same lines (M. Villarini).

6. Finitely generated groups of conformal germs

Thus far we have studied classification and certain dynamic properties of *single* germs of vector fields and biholomorphisms. However, in §2**C** we introduced an important invariant of foliation, the holonomy group of a leaf $L \in \mathcal{F}$ with nontrivial fundamental group $\pi_1(L, a)$, $a \in L$. By construction, the holonomy is a representation of $\pi_1(L, a)$ by conformal germs $\operatorname{Diff}(\tau, a)$, where τ is a cross-section to L at a, and the holonomy group G is identified with the image of that representation. Usually if the fundamental group of a leaf of a holomorphic foliation is finitely generated, then so is the group G. We will consider only the case of holomorphic foliations on complex 2-dimensional surfaces, thus dealing only with finitely generated subgroups of the group $\operatorname{Diff}(\mathbb{C}, 0)$ of conformal germs.

In this section we study classification problems for *finitely generated groups* of conformal germs and their dynamic properties, focusing on the properties which will be later used in §11 and §28. In much more detail the theory is treated in the recent monograph [**Lor99**].

Problem 6.3. Compute all holonomy maps of an integrable foliation $\{du = 0\}$, $u \in \mathcal{O}(\mathbb{C}^2, 0)$, if $u = \prod u_j^{p_j}$ is the primary decomposition of the holomorphic germ u with irreducible factors u_j and natural exponents $p_j \in \mathbb{N}$.

Problem 6.4. Prove that a formally integrable holomorphic self-map (or a finitely generated group G of holomorphic germs of self-maps from $\operatorname{Diff}(\mathbb{C}, 0)$) is also analytically integrable; cf. with Theorem 6.8.

Suggestion. Use the formal chart in which $\widehat{u}(z) = z^m$.

Problem 6.5. Prove that an (orbital) symmetry of a holomorphic vector field on $(\mathbb{C}, 0)$ is necessary holomorphic itself.

Problem 6.6. Construct a finitely generated subgroup $G \subset \operatorname{Diff}(\mathbb{C}, 0)$, whose orbits are dense in each of the two half-planes $\{\pm \operatorname{Im} z > 0\}$ separately, yet both half-planes are invariant by G.

Generalize this example and find a group whose orbits are dense in each of $2p$ invariant sectors in $(\mathbb{C}, 0)$ for any $p > 1$ (cf. with Theorem 6.53).

Problem 6.7 (formal rigidity of generic groups)**.** Assume that two finitely generated subgroups $G, G' \subseteq \operatorname{Diff}(\mathbb{C}, 0)$ are *formally* equivalent and one of these groups contains a hyperbolic germ. Prove that in such case G and G' are holomorphically equivalent, moreover, any formal conjugacy between them is necessarily holomorphic (convergent).

7. Holomorphic invariant manifolds

In this short section we show that under rather weak conditions one can eliminate enough nonresonant terms to ensure existence of *holomorphic invariant (sub)manifolds*. Recall that a holomorphic submanifold $W \subset (\mathbb{C}^n, 0)$ is invariant for a holomorphic vector field F, if the vector $F(x)$ is tangent to W at any point $x \in W$. Traditionally the prefix 'sub' is omitted, though it plays an important role: in §14 we will discuss invariant analytic subvarieties that are *not* submanifolds because of their singularity.

7A. Invariant manifolds of hyperbolic singularities. Suppose that the spectrum $S \subset \mathbb{C}$ of linearization matrix A of a holomorphic vector field consists of two parts $S^{\pm} \subset \mathbb{C}$ separated by a real line (i.e., each part belongs to an open half-plane bounded by the line). In this case no eigenvalue from one part can be equal to a linear combination of eigenvalues from the other part with nonnegative coefficients,

$$\begin{gathered}\lambda_j^- - \sum \alpha_i \lambda_i^+ \neq 0, \qquad \lambda_i^+ - \sum \alpha_j \lambda_j^- \neq 0, \\ \lambda_i^+ \in S^+, \quad \lambda_j^- \in S^-, \quad \alpha_i, \alpha_j \in \mathbb{Z}_+,\end{gathered} \tag{7.1}$$

(we say that there are no *cross-resonances* between the two parts). Without loss of generality A can be assumed to be in the block diagonal form. By the Poincaré–Dulac theorem, there exists a formal transformation eliminating all nonresonant terms corresponding to the nonzero cross-combinations (7.1). The corresponding formal normal form has two invariant manifolds coinciding with the corresponding coordinate subspaces.

Moreover, all denominators (7.1) are obviously bounded from below. Therefore one can expect that the corresponding transformation converges and the invariant manifolds will exist in the analytic category. This is indeed the case, though the accurate proof is organized along different lines.

Theorem 7.1 (Hadamard–Perron theorem for holomorphic flows)**.** *Assume that the linearization operator of a holomorphic vector field $Ax + F(x)$ has a transversal pair of invariant subspaces $L^{\pm}$ such that the spectra of A restricted on these subspaces are separated from each other.*

Then the vector field has two holomorphic invariant manifolds $W^{\pm}$ tangent to the subspaces $L^{\pm}$.

However, the proof of this result is indirect. We start by formulating and proving a counterpart of Theorem 7.1 for biholomorphisms.

Definition 7.2. A holomorphic self-map $H \in \operatorname{Diff}(\mathbb{C}^n, 0)$, $x \mapsto Mx + h(x)$, $h(0) = \frac{\partial h}{\partial x}(0) = 0$, is said to be *hyperbolic* if no eigenvalue of the linearization matrix $M \in \operatorname{GL}(n, \mathbb{C})$ has modulus 1.

For a matrix M without eigenvalues on the unit circle, we denote $L^{\pm} \subseteq \mathbb{C}^n$ two invariant subspaces such that the restriction $M|_{L^-}$ is contracting (in a suitable Hermitian metric) and $M|_{L^+}$ expanding (i.e., $M^{-1}|_{L^+}$ is contracting).

To define invariant manifolds for biholomorphisms we need to be careful and replace sets by their germs at the fixed points. Otherwise it would be necessary to give different definitions for expanding and contracting submanifolds.

Definition 7.3. A holomorphic submanifold W passing through a fixed point of a biholomorphism $H\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is *invariant*, if *the germ* of $H(W)$ at the origin coincides with *the germ* of W.

Theorem 7.4 (Hadamard–Perron theorem for biholomorphisms)**.** *A hyperbolic holomorphism in a sufficiently small neighborhood of the fixed point at the origin has two holomorphic invariant submanifolds W^+ and W^-.*

These manifolds pass through the origin, transversal to each other and are tangent to the corresponding invariant subspaces $L^{\pm}$ of the linearized map $x \mapsto Mx$.

The dimensions of the invariant manifolds are necessarily equal to the dimension of the corresponding subspaces. The manifold W^+ is called *unstable manifold*, whereas W^- is referred to as the *stable manifold*, because the restriction of H on these manifolds is unstable and stable respectively.

Proof. The linearization matrix M of a holomorphic biholomorphism $H\colon (\mathbb{C}^n,0)\to(\mathbb{C}^n,0)$ can be put into the block diagonal form. Choosing appropriate system of local holomorphic coordinates $(x,y)\in(\mathbb{C}^k,0)\times(\mathbb{C}^l,0)$, $k+l=n$, one can always assume that the map H has the form

$$H\colon \begin{pmatrix} x\\ y\end{pmatrix}\longmapsto\begin{pmatrix} Bx+g(x,y)\\ Cy+h(x,y)\end{pmatrix},\qquad (x,y)\in(\mathbb{C}^k,0)\times(\mathbb{C}^l,0).\tag{7.2}$$

Here the square matrices B,C and the nonlinear terms g,h of order $\geqslant 2$ satisfy the conditions

$$\begin{aligned} |B|\leqslant\mu,\quad |C^{-1}|\leqslant\mu,\qquad &\mu<1,\\ |f(x,y)|+|g(x,y)|<|x|^2+|y|^2,\qquad &\text{for } |x|<1,\ |y|<1. \end{aligned}\tag{7.3}$$

with some *hyperbolicity parameter* $\mu<1$.

It is sufficient to prove the existence of the *stable* manifold only; the unstable manifold for H is the stable manifold of the inverse map H^{-1} which is also hyperbolic.

The stable manifold W^+ tangent to $L^+=\{(x,0)\}$ is necessarily the graph of a holomorphic vector function $\varphi\colon\{|x|\leqslant\varepsilon\}\to\{|y|\leqslant\varepsilon\}$ defined in a sufficiently small polydisk, $\varphi(0)=0$, $\frac{\partial\varphi}{\partial x}(0)=0$. For this manifold to be invariant, the function φ must satisfy the functional equation

$$\varphi\bigl(Bx+g(x,\varphi(x))\bigr)=C\varphi(x)+h(x,\varphi(x)).\tag{7.4}$$

This equation can be transformed to the fixed point form as follows:

$$\varphi=\mathcal{H}\varphi,\qquad (H\varphi)(x)=C^{-1}\varphi\bigl(Bx+g(x,\varphi(x))\bigr)-h(x,\varphi(x)).\tag{7.5}$$

All assertions of Theorem 7.4 follow from the contracting map principle and the following Lemma 7.5. □

The "linearization" (removal of all nonlinear terms of order 2 and higher) of the operator $\mathcal{H}$ at the "point" $\varphi=0$ results in the operator

$$\varphi(x)\mapsto C^{-1}\varphi(Bx),\qquad |B|,\ |C^{-1}|\leqslant\mu<1,$$

which is obviously contracting. Lemma 7.5 shows that nonlinear terms do not affect this property.

Denote by $\mathcal{A}_\varepsilon$ the Banach space of functions holomorphic in the open disk of radius $\varepsilon>0$ and continuous on the closure.

Lemma 7.5. *Under the assumptions* (7.3), *the nonlinear operator* $\mathcal{H}$ *has the following properties*:

(1) *$\mathcal{H}$ is well defined for φ in the ball $\mathcal{B}_\varepsilon = \{\varphi\colon \sup_{|x|<\varepsilon} |\varphi(x)| < \varepsilon\}$ inside the space $\mathcal{A}_\varepsilon$, and takes this ball into itself,*

(2) *the subset $\mathcal{B}^1_\varepsilon$ of functions in $\mathcal{B}_\varepsilon$ with the Lipschitz constant $\leqslant 1$, is preserved by $\mathcal{H}$,*

(3) *the operator $\mathcal{H}$ is contracting on $\mathcal{B}^1_\varepsilon$,*

provided that the value $\varepsilon > 0$ is sufficiently small.

Proof. To prove the first assertion, note that $|Bx + g(x, \varphi(x))| < \mu|x| + |x|^2 + |\varphi|^2 < \mu\varepsilon + 2\varepsilon^2 < \varepsilon$ for $|x| < \varepsilon$, if ε is sufficiently small. Thus the composition occurring in the definition of $\mathcal{H}$ makes perfect sense and $\mathcal{H}\varphi$ is well defined. For the same reason, $|\varphi|$ never exceeds $\mu\varepsilon + 2\varepsilon^2 < \varepsilon$ which means that $\mathcal{B}_\varepsilon$ is taken by $\mathcal{H}$ into itself.

The Jacobian matrix $J(x) = \frac{\partial\varphi}{\partial x}$ is transformed into $J' = C^{-1}J(\cdots)(B + \frac{\partial g}{\partial x} + \frac{\partial g}{\partial y}J) + (\frac{\partial h}{\partial x} + \frac{\partial h}{\partial y}J)$. Since the terms g, h are of order $\geqslant 2$, their derivatives vanish at the origin and therefore the Jacobian is no greater (in the sense of the matrix norm) than $(\mu^2 + O(\varepsilon))|J|$. As $\mu < 1$, this proves the assertion about the Lipschitz constant.

To prove the last assertion that $\mathcal{H}$ is contractive, notice that the operator $\varphi(x) \mapsto h(x, \varphi(x))$ is strongly contracting:

$$|h(x, \varphi_1(x)) - h(x, \varphi_2(x))| \leqslant \left|\tfrac{\partial h}{\partial y}\right| |\varphi_1(x) - \varphi_2(x)| \leqslant O(\varepsilon)\|\varphi_1 - \varphi_2\|_\varepsilon. \quad (7.6)$$

Consider the operator $\varphi \mapsto \mathcal{G}\varphi = \varphi(Bx + g(x, \varphi))$ and the difference of the values it takes on two functions $\varphi_1, \varphi_2 \in \mathcal{B}^1_\varepsilon$: by the triangle inequality,

$$\begin{aligned} |\mathcal{G}\varphi_1(x) - \mathcal{G}\varphi_2(x)| &= |\varphi_1(Bx + g_1(x)) - \varphi_2(Bx + g_2(x))| \\ &\leqslant |\varphi_1(Bx + g_2(x)) - \varphi_2(Bx + g_2(x))| \\ &\quad + |\varphi_1(Bx + g_1(x)) - \varphi_1(Bx + g_2(x))|, \end{aligned}$$

where we denoted $g_i(x) = g(x, \varphi_i(x))$ for brevity. The first term does not exceed $\|\varphi_1 - \varphi_2\|_\varepsilon$. Since the vector function $\varphi_1 \in \mathcal{B}^1_\varepsilon$ has Lipschitz constant 1, the second term does not exceed $|g_1(x) - g_2(x)| = |g(x, \varphi_1(x)) - g(x, \varphi_2(x))|$. Similarly to (7.6), this part is no greater than $O(\varepsilon)\|\varphi_1 - \varphi_2\|_\varepsilon$. Finally, we conclude that $\mathcal{G}$ is Lipschitz on $\mathcal{B}^1_\varepsilon$: $\|\mathcal{G}\varphi_1 - \mathcal{G}\varphi_2\|_\varepsilon \leqslant (1 + O(\varepsilon))\|\varphi_1 - \varphi_2\|$.

Adding all terms together for $\mathcal{H} = C^{-1}\mathcal{G} - h(x, \cdot)$, we conclude that if $\varphi_{1,2} \in \mathcal{B}^1_\varepsilon$, then

$$\|\mathcal{H}\varphi_1 - \mathcal{H}\varphi_2\|_\varepsilon \leqslant (\mu + O(\varepsilon))\, \|\varphi_1 - \varphi_2\|_\varepsilon.$$

Since $\mu < 1$, the operator $\mathcal{H}$ is contracting on the closed subset $\mathcal{B}^1_\varepsilon$ of the complete metric space $\mathcal{B}_\varepsilon \subset \mathcal{A}_\varepsilon$. □

Remark 7.6. Characteristically for the proofs based on the contracting map principle, the germs of invariant manifolds are automatically unique.

Now we can derive Theorem 7.1 from Theorem 7.4.

Proof. Passing if necessary to an orbitally equivalent field, one may assume that the linearization $A = \operatorname{diag}\{A_+, A_-\}$ is block diagonal with the spectra of the blocks are *separated by the imaginary axis.*

Consider the flow maps $\varPhi^t = \exp tF\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ for $t = 1/k$, $k = 1, 2, \dots$. Each of them is a biholomorphism with the linear part $x \mapsto \exp tAx$ whose eigenvalues are the corresponding exponentials $\{\exp t\lambda_i \colon \lambda_i \in S\}$ separated by the unit circle $\{|\lambda| = 1\}$. In the assumptions of the theorem, each flow map $\varPhi^t$ is hyperbolic for the specified values of $t \in 1/\mathbb{N}$. By Theorem 7.4, the map $\varPhi^t$ has a pair of invariant manifolds $W_t^\pm$, tangent to the corresponding invariant subspaces $L^\pm$ common for all $t \in \mathbb{R}$.

Apriori, the invariant subspaces $W_t^\pm$ do not have to coincide. However, $\left(\varPhi^{1/k}\right)^k = \varPhi^1$, therefore manifolds invariant for $\varPhi^{1/k}$, are invariant also for $\varPhi^1$. Since the invariant manifolds for the latter map are unique, we conclude that all the maps $\varPhi^{1/k}$ leave the pair $W^\pm = W_1^\pm$ invariant.

In other words, an analytic trajectory $x(t)$ of the vector field which begins on, say, W^-, $x(0) \in W^-$, remains on W^- for $t = 1/k$. Since isolated zeros of analytic functions cannot have accumulation points, $x(t)$ is on W^- for all (sufficiently small) values of $t \in (\mathbb{C}, 0)$. Then W^- is invariant for the vector field $Ax + F(x)$. The proof for W^+ is similar. □

Remark 7.7. Intersection of invariant manifolds is again an invariant manifold. This observation allows us to construct small-dimensional invariant manifolds for holomorphic vector fields. For instance, if the linearization matrix $\varLambda$ has a simple eigenvalue $\lambda_1 \ne 0$ such that $\lambda_1/\lambda_j \notin \mathbb{R}_+$ for all other eigenvalues λ_j, $j = 2, \dots, n$, then the vector field has a *one-dimensional holomorphic invariant manifold* (curve) tangent to the corresponding eigenvector.

The Hadamard–Perron theorem for holomorphic flows, as formulated above, is the nearest analog of the Hadamard–Perron theorem for smooth flows in $\mathbb{R}^n$. There are known stronger results in this direction; see [**Bib79**].

7B. Hyperbolic invariant curves for saddle-nodes. Consider a holomorphic vector field on the plane $(\mathbb{C}^2, 0)$ with the saddle-node at the origin. Recall that by Definition 4.28, this means that exactly one of the eigenvalues is zero, while the other eigenvalue must be nonzero. The null space (line) of the linearization operator is called the *central* direction. The direction of eigenvector with the nonzero eigenvalue is referred to as *hyperbolic.*

The nonzero eigenvalue cannot be separated from the null one, thus the Hadamard–Perron theorem cannot be applied. However, the invariant manifold (smooth holomorphic curve) tangent to the eigenvector with nonzero eigenvalue, exists and is unique in this case as well. As before, we start with the case of biholomorphisms with one contracting eigenvalue $|\mu| < 1$ and the other eigenvalue equal to 1. For obvious reasons, such maps are called *saddle-node* biholomorphisms.

Any saddle-node biholomorphism $H\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ can be brought into the form

$$H\colon \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} \mu x + g(x,y) \\ y + y^2 + h(x,y) \end{pmatrix}, \qquad \mu \in (0,1) \subset \mathbb{R}, \tag{7.7}$$

with g, h holomorphic nonlinear terms of order $\geqslant 3$, by a suitable holomorphic choice of coordinates x, y. Indeed, all other quadratic terms are nonresonant and can be removed (Exercise 4.8).

Theorem 7.8. *The biholomorphism* (7.7) *has a unique holomorphic invariant manifold* (*curve*) *tangent to the eigenvector* $(1, 0) \in \mathbb{C}^2$.

Proof. The manifold $W = \operatorname{graph} \varphi$ is invariant for the saddle-node self-map H of the form (7.7) if the function φ satisfies the functional equation

$$\varphi(\mu x + g(x, \varphi(x))) = \varphi(x) + \varphi^2(x) + h(x, \varphi(x)). \tag{7.8}$$

This equation can be represented under the fixed point form $\mathcal{H}\varphi = \varphi$ using the operator $\mathcal{H}$ defined as follows:

$$(\mathcal{H}\varphi)(x) = \varphi\bigl(\mu x + g(x, \varphi(x))\bigr) - \varphi^2(x) - h(x, \varphi(x)). \tag{7.9}$$

This operator is no longer contracting: its linearization at $\varphi = 0$ is the operator $\varphi(x) \mapsto \varphi(\mu x)$ which keeps all constants fixed. To restore the contractivity, we have to restrict this operator on the subspace of functions vanishing at the origin, with the norm $\|\varphi\|' = \sup_{x \neq 0} \frac{|\varphi(x)|}{|x|}$. Technically it is more convenient to substitute $\varphi(x) = x\psi(x)$ into the functional equation (7.8) and bring it back to the fixed point form. As a result, we obtain the equation

$$(\mu x + g(x, x\psi(x)) \cdot \psi\bigl(\mu x + g(x, x\psi(x))\bigr) = x\psi(x) + x^2\psi^2(x) + h(x, x\psi(x)),$$

which yields the nonlinear operator $\mathcal{H}'$,

$$(\mathcal{H}'\psi)(x) = \bigl(\mu + g'(x, \psi(x))\bigr) \cdot \psi\bigl(\mu x + g(x, x\psi(x))\bigr) - x\psi^2(x) - h'(x, \psi). \tag{7.10}$$

Here the holomorphic functions $g'(x,y) = g(x,xy)/x$, $h'(x,y) = h(x,xy)/x$ are of order $\geqslant 2$ at the origin.

The proof of Lemma 7.5 carries out almost literally for the operator $\mathcal{H}'$ as in (7.10), proving that it is contractible on the space of functions $\psi\colon \{|x| < \varepsilon\} \to \{|y| < \varepsilon\}$ with respect to the usual supremum-norm on sufficiently small discs. $\square$

Completely similar to derivation of Theorem 7.1 from Theorem 7.4 in the hyperbolic case, Theorem 7.8 implies the following result concerning holomorphic saddle-nodes.

Theorem 7.9. *A holomorphic vector field on the plane* $(\mathbb{C}^2, 0)$ *having a saddle-node singularity* (*one eigenvalue zero, another nonzero*) *at the origin, admits a unique holomorphic nonsingular invariant curve passing through the singular point and tangent to the hyperbolic direction.* □

This curve is called the *hyperbolic invariant manifold.*

It is important to conclude this section by the explicit example showing that the other invariant manifold, the *central manifold* tangent to the central direction, may not exist in the analytic category. Note, however, that the formal invariant manifold always exists and is unique: this follows from the formal orbital classification of saddle-nodes (Proposition 4.29).

Example 7.10 (L. Euler)**.** The vector field

$$x^2 \frac{\partial}{\partial x} + (y - x) \frac{\partial}{\partial y} \tag{7.11}$$

has vertical hyperbolic direction $\frac{\partial}{\partial y}$ and the central direction $\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$. The central manifold, if it exists, must be represented as the graph of the function $y = \varphi(x)$, $\varphi(x) = x + \sum_{k \geqslant 2} c_k x^k$. However, this series diverges, as was noticed already by L. Euler. Indeed, the function φ must be the solution to the differential equation

$$\frac{d\varphi}{dx} = \frac{\varphi(x) - x}{x^2}$$

which implies the recurrent formulas for the coefficients,

$$k\, c_k = c_{k+1}, \qquad k = 1, 2, \dots, \qquad c_1 = 1.$$

The factorial series with $c_k = (k-1)!$ has zero radius of convergence, hence no analytic central manifold exists.

However, sufficiently large “pieces” of the central manifold for the saddle-node can be shown to exist; see §22**I**.

Exercises and Problems for §7.

Exercise 7.1. Prove that a nonresonant hyperbolic self-holomorphism is analytically linearizable on its holomorphic invariant manifolds W^+ and W^-.

Problem 7.2. Prove that if a hyperbolic self-map analytically depends on additional parameters (and remains hyperbolic for all values of these parameters), then the invariant manifolds $W^{\pm}$ also depend analytically on the parameters.

Problem 7.3. Formulate and prove a parallel statement for a saddle-node.

Exercise 7.4. Describe possible number and relative position of analytic separatrices of elementary planar singularities of holomorphic vector fields.

24. Nonaccumulation theorem for hyperbolic polycycles

> [*What can be said about*] the maximal number of and position of Poincaré's boundary cycles (*cycles limites*) for a differential equation of the first order and degree of the form $\frac{dy}{dx} = \frac{Y}{X}$, where X and Y are rational integral functions of nth degree in x and y?
>
> *D. Hilbert*, 1901, reprinted from [**Hil00**]

This second part of Hilbert's sixteenth problem appears to be one of the most elusive in his famous list [**Hil00**], second only to the Riemann ζ-function Conjecture. In the introductory subsection §24**A** based on [**Ily02**], we briefly describe the current status of this problem.

The body of the section is devoted to investigation of limit cycles of analytic vector fields[6]. The central result of this section, Theorem 24.24 on finiteness of limit cycles of analytic vector fields having only nondegenerate singular points, was proved by Yu. Ilyashenko in [**Ily84**].

24A. Legends and truth on the limit cycles. As most problems from the Hilbert's list, the sixteenth problem is formulated very broadly and can be made precise in a variety of ways.

24$\mathbf{A}_1$. *Various flavors of Hilbert's sixteenth.* By different placement of quantifiers the Hilbert's question can be transformed into three problems in increasing order of strength as follows.

Problem I. *Is it true that a planar polynomial vector field may have only finitely many limit cycles?*

Problem II. *Can the number of limit cycles be bounded by a constant depending only on the degree n of the vector field?*

Assuming the affirmative answer to Problem II, denote by $H(n)$ the *Hilbert number*, the conjectural bound for the number of limit cycles that a polynomial vector field of degree n may exhibit. Linear vector fields have no limit cycles, hence $H(1) = 0$. Finiteness of $H(2)$ is already an open problem.

Problem III. *Give an upper bound for $H(n)$.*

Only Problem I is solved now. The affirmative answer was proved independently in [**Ily91**] and [**Eca92**].

To separate analytic and algebraic aspects of the Hilbert problem, we will consider the following two questions concerning *real analytic* rather than polynomial vector fields.

---

[6]Recall that the limit cycle of a vector field is an isolated compact leaf of the real foliation defined by the vector on the real plane, real 2-sphere or the projective plane $\mathbb{R}P^2$; cf. with Definition 9.11

Problem IV. *Is it true that a real analytic vector field on the* 2*-sphere* $\mathbb{S}^2$ *has only a finite number of limit cycles?*

The "purely analytic" counterpart of Problem II has the following form.

Problem V. *Given a parametric family of real analytic vector fields on the* 2*-sphere, analytically depending on finitely many parameters varying over a compact subset in the parameter space, is it true that the number of limit cycles in this family is uniformly bounded?*

An affirmative answer in Problem V implies solutions of the Problems I, II and IV, since polynomial vector fields can be extended as real analytic foliations of the 2-sphere, and constitute a finite-parametric family parameterized by the coefficients of the vector fields varying over the projective space (see §25**A** for detailed explanations). In fact, it is the solution of Problem IV that is achieved in [**Eca92**] and [**Ily91**]. In other words, the known *individual finiteness* of limit cycles for polynomial vector fields has analytic rather than algebraic nature.

Clearly, all these questions reformulated literally for C^∞-smooth rather than real analytic vector fields, have negative answers; see §9**F**. Yet somewhat surprisingly there are meaningful questions which are reasonable "smooth analogs" of the above analytic problems. The following formulation is an implicit conjecture that the exotic smooth vector fields with infinitely many limit cycles constitute a subset of *infinite codimension* in the total space of C^∞-smooth vector fields on the sphere.

Problem VI (Hilbert–Arnold problem)**.** *Given a* generic *n-parametric family of C^∞-smooth vector fields on the* 2*-sphere, smoothly depending on parameters varying over a compact subset in the parameter space, is it true that the number of limit cycles in this family is uniformly bounded?*

A restricted version of this problem (under the additional assumption on the types of singular points that are allowed to occur in the family) is solved in [**IY95**]. We wish to stress that this formulation is unrelated (neither implies nor is implied by) to any of the algebraic/analytic Problems I to V.

24**A**$_2$. *Historical sketch.* As is typical for most of the problems from Hilbert's list, the sixteenth problem lies on the crossroads of many different directions and served as a motivation for many developments. Yet its own history is rather dramatic: several times it was believed to be proved only to later discover gaps.

Before Hilbert, Henri Poincaré considered polynomial vector fields in the plane, in the framework of his geometric theory of differential equations. He introduced the notion of limit cycle and proved that a planar polynomial vector field *without saddle connexions* has only a finite number of limit cycles.

In 1923, Dulac [**Dul23**] claimed a solution of Problem I in full generality. In the mid-fifties of the twentieth century, Petrovskii and Landis published a solution to Problem III [**PL55, PL57**]. They claimed that $H(n)$ is bounded by a certain polynomial of degree 3 in n, and $H(2) = 3$. In the early sixties a severe error in the arguments by Petrovskii and Landis was revealed by S. Novikov and Yu. Ilyashenko. Later quadratic vector fields with 4 limit cycles were explicitly constructed in [**CW79, Shi80b**].

In 1981, a ruinous gap was found in Dulac's solution of Problem I (cf. [**Ily85**]): Dulac was operating with asymptotic series as if they were convergent. Thus after eighty years of intense efforts our knowledge on Hilbert's sixteenth problem was still almost the same as at the time when the problem was formulated.

24**A**$_3$. *Some recent progress on the Hilbert's sixteenth problem.* The principal achievement is the general theorem solving Problems I and IV.

Theorem 24.1 (Individual finiteness theorem, [**Ily91, Eca92**])**.** *A polynomial vector field in the plane has only a finite number of limit cycles. The same is true for analytic vector fields on the 2-sphere.*

After some preliminary work described in §24**B**–§24**D**, the Finiteness Theorem 24.1 follows from the Nonaccumulation Theorem 24.23 formulated below. It is the Nonaccumulation theorem that is the most difficult result, whose proof occupies hundreds of pages. We will not discuss it, though the analytic normal forms for parabolic singularities and saddle resonant vector fields obtained in §21–§22 play the key role in this analysis. Ecalle's theory of resurgent functions is presented in [**Eca85, Eca92**].

The infinitesimal Hilbert's sixteenth problem deals with limit cycles that appear by perturbation of Hamiltonian vector fields that do not have limit cycles at all. Its main tool is investigation of Abelian integrals considered as analytic multivalued functions of complex parameters. These questions are discussed in detail in §26 below.

Bifurcation theory is intimately related to Hilbert's sixteenth. Indeed, the function "number of limit cycles of the equation" has points of discontinuity corresponding to equations whose perturbations generate limit cycles via bifurcations. Limit cycles may bifurcate from separatrix polygons, also known as *polycycles* (defined in §24**C**). The *cyclicity* of a polycycle in a family of equations is the maximal number of limit cycles that may bifurcate from the polycycle in this family, very much like cyclicity of singular point introduced in §12**A**, p. 201. Using the notion of cyclicity, one can formulate the Hilbert-type problems in the language of bifurcations theory.

Problem VII. *Is it true that a polycycle occurring in a finite parameter family of planar analytic vector fields has only finite cyclicity?*

Problem VIII. *Is it true that a polycycle occurring in a generic k-parameter family of smooth planar vector fields may generate only a finite number of limit cycles, with an upper bound depending on k only?* (*This latter quantity is denoted by $B(k)$.*)

The affirmative answer in Problem VII would imply a solution of Problem II and existence of the Hilbert number $H(n)$ for any finite n (without giving the slightest idea of how this number can be computed). The affirmative answer in Problem VIII would lead to an instant solution of the Hilbert-Arnold Problem VI. These implications are proved by using simple compactness arguments due to R. Roussarie [**Rou98**]. Both Problems VII and VIII remain unsolved, yet the latter seems to be easier than the former, in light of the recent achievements.

More precisely, denote by $E(k)$ the maximal cyclicity of a polycycle that can occur in a *generic* k-parameter family of smooth vector fields, under the additional assumption that all singular points on this polycycle are *elementary*.

Theorem 24.2 (Ilyashenko and Yakovenko [**IY95**])**.** *For any k, the number $E(k)$ is finite and bounded from above by an elementary function of k.*

As a corollary, one can immediately conclude that the Hilbert–Arnold problem has the affirmative answer if restricted on the smooth vector fields having only elementary singularities on the 2-sphere.

The proof of Theorem 24.2 is constructive and yields an algorithmic expression for the upper bound. Further elaborating this construction, V. Kaloshin in [**Kal03**] obtained a simple explicit upper bound,

$$E(k) \leqslant 2^{25k^2}. \tag{24.1}$$

The Kaloshin bound is apparently very much excessive, yet it is one of the first *Hilbert-type* numbers (bounds pertinent to the number of limit cycles) obtained during the hundred years of quest.

In the rest of this section we illustrate the power of the analytic normal forms theory and prove the Individual Finiteness Theorem 24.1 under the additional assumption that all singular points of the vector field and nondegenerate saddles. To present the complete proof, we have to go back to the early times of the geometric theory of differential equations.

24B. Poincaré–Bendixson theory revisited. One of the highlights of the geometric theory of real planar vector fields is the Poincaré–Bendixson theorem. It describes the limit behavior of phase trajectories of vector fields without singular points in domains on the 2-sphere using purely topological arguments. In the next three subsections we apply similar methods to describe limit sets of aperiodic trajectories for spherical vector fields with singularities and the accumulation sets for their periodic trajectories.

Here and below we consider smooth real vector fields on the sphere and their trajectories parameterized by real values of the time. Since the sphere is compact, any such trajectory can be extended for all values of the time $t \in \mathbb{R}$. Let $v \in \mathcal{D}(\mathbb{S}^2)$ be a vector field and $\varphi\colon \mathbb{R} \to \mathbb{S}^2$ its trajectory.

Definition 24.3. An ω-limit set of a trajectory φ is the set of all points $y \in \mathbb{R}^2$ which are limits of sequences of points $\varphi(t_n)$ corresponding to sequences of time $t_n \to \infty$. An α-limit set of a trajectory $\varphi(t)$ is the ω-limit set of the trajectory $\varphi(-t)$, i.e., after the time reversal.

We will denote these limits by $\omega(\varphi)$ and $\alpha(\varphi)$ respectively.

Remark 24.4. The definition of an ω- (resp., α-) limit set can be modified for noncomplete vector fields or for fields defined in noninvariant domains. It is sufficient to require that φ be defined for all sufficiently large *positive* (resp., *negative*) values of time.

One can give an alternative description for $\omega(\varphi)$. For any $T > 0$ denote by φ_T the restriction of the phase curve on the semi-interval $[T, +\infty)$. This is a forward invariant set whose closure $\overline{\varphi_T} \subset \mathbb{S}^2$ is also forward invariant (forward invariance is invariance by the real flow maps $\Phi_t = \exp tv$ of the field $v \in \mathcal{D}(\mathbb{S}^2)$ for nonnegative times $t \in \mathbb{R}_+$). These sets form a family of nested connected compacts on the sphere, whose intersection, as one can easily see, coincides with $\omega(\varphi)$:

$$\varnothing \neq \omega(\varphi) = \bigcap_{T>0} \overline{\varphi_T} \Subset \mathbb{S}^2. \tag{24.2}$$

From the description (24.2) one can easily derive the following properties of limit sets on the sphere.

Proposition 24.5. *The ω-limit set of a trajectory of the spherical vector field is a closed connected set invariant by both positive and negative flow of the field.* □

Remark 24.6. The same definitions can be given for a vector field on the plane $\mathbb{R}^2$, but in this case the sets φ_T can be unbounded, $\overline{\varphi_T}$ noncompact and, as a result, the ω-limit set can be empty or nonconnected.

Example 24.7. The phase portraits sketched on Fig. IV.5 show that ω-limits of trajectories on the sphere can be singular points, cycles (periodic orbits) or more complicated objects which consist of several singular points

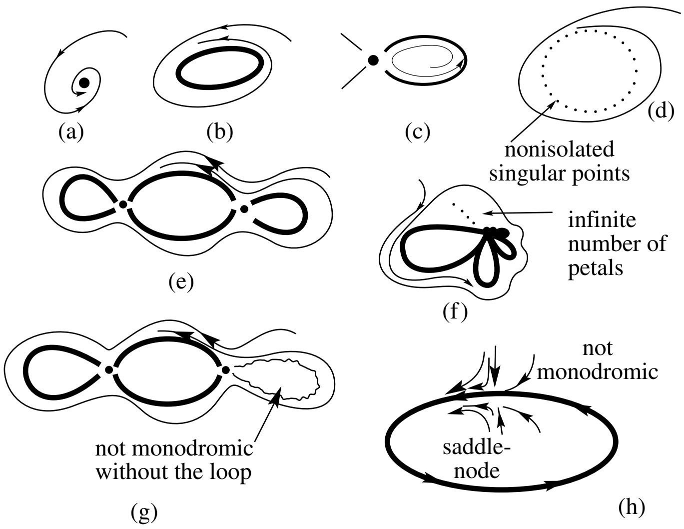

Figure IV.5. Zoo of limit periodic sets. (a) Isolated singular point. (b) Periodic orbit. (c) Separatrix loop. (d) Curve of nonisolated singular points. (e) Monodromic polycycle. (f) Singular point with infinitely many homoclinic trajectories. (g) Part of a polycycle is a polycycle but not monodromic. (h) Oriented but not monodromic saddle-node loop

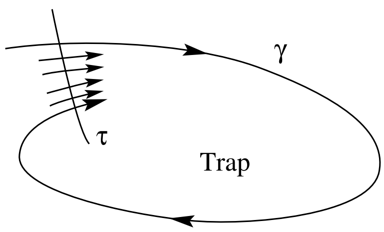

Figure IV.6. Bendixson trap

together with several orbits which are bi-asymptotic to these singular points as $t \to \pm\infty$.

In order to describe ω-limit sets, we introduce a simple but powerful construction designed by Bendixson.

Definition 24.8. A *Bendixson trap* for a vector field v on the sphere is a closed oriented piecewise-smooth curve which consists of two smooth parts:

(1) a piece of *nonperiodic* phase trajectory γ oriented by the field and thus defining the orientation of the trap, and
(2) a smooth arc τ transversal to the field at all its points.

By the Jordan theorem, any Bendixson trap divides the sphere into two connected domains, one of them invariant by the flow of v in the forward time (it will be referred to as *interior* to justify the term "trap"), the other ("*exterior*") invariant in the reverse time. Note that the orientation of the trap can be opposite to the orientation of the boundary of the interior part.

Lemma 24.9. *No point on the transversal arc of a Bendixson trap can belong to an ω-limit set of any trajectory.*

In particular, the invariant arc of the trap cannot be an ω-limit set.

Proof of the lemma. Any orbit starting on the transversal arc enters the interior domain either immediately, or at worst after traversing the invariant arc of the trap, and never leaves it since that moment. In particular, it can never return to a sufficiently small neighborhood of the arc τ. □

As an immediate consequence, we can prove that a trajectory accumulates to its ω-limit set from one side only.

Proposition 24.10. *If $\gamma = \omega(f)$ contains a nonsingular point a and $\tau\colon (\mathbb{R}^1, 0) \to \mathbb{S}^2$ is a cross-section to γ at a, then all intersections of φ with τ occur only on one side of the cross-section.*

Proof. If φ intersects τ at two points p and q on two different sides of τ, then the closed line formed by the arc $\varphi|_p^q$ of φ from p to q and the arc $\tau|_q^p$ of τ from q to p is a trap. The point $a \in \tau|_q^p$ is hence a point of a limit set which lies on the transversal arc of a trap, in contradiction with Lemma 24.9. □

The following result constitutes the most familiar part of the Poincaré–Bendixson theory.

Theorem 24.11 (H. Poincaré, 1886, I. Bendixson, 1901)**.** *An ω-limit set which does not contain singular points of the field, is necessarily a periodic orbit.*

Proof. Let $\gamma = \omega(\varphi)$ be the limit set and $a \in \gamma$ a nonsingular point on it. Consider a cross-section τ to γ at a as in Proposition 24.10. The trajectory φ crosses τ infinitely many times. Consider the positive orbit $\psi \subseteq \gamma$ starting at a. It must intersect τ some time in the future. Indeed, otherwise the closure $\overline{\psi(t)|_{[1,+\infty)}}$ would be a compact subset of the sphere disjoint from τ, and since the orbit φ must remain in a neighborhood of this compact, it would be unable to cross τ infinitely many times.

Hence ψ crosses τ again. If this intersection occurs at a point b different from a, then the closed curve formed by $\psi|_a^b$ and $\tau|_b^a$ would be a trap in contradiction with Lemma 24.9.

The only remaining possibility is that ψ crosses τ at the same point $a \in \tau \cap \psi$. Then ψ and hence γ is a periodic orbit of v. □

In the presence of singular points the limit sets can be more complicated, as mentioned in Example 24.7. Still these limit sets admit a rather simple description.

A trajectory φ of a vector field is called bi-asymptotic to two points a, b if $\{a\} = \alpha(\varphi)$, $\{b\} = \omega(\varphi)$. Clearly, in such a case both a and b must be singular points; the case $a = b$ is *not* excluded.

Theorem 24.12. *Any limit set of a vector field on the sphere consists of singular points and entire trajectories of the field, bi-asymptotic to some of these singular points.*

To prove this theorem, we reformulate it in the language of *iterated* limit sets. Being invariant, an ω-limit set of any orbit φ consists of entire trajectories of the field. This allows us to iterate the construction of limit sets.

Definition 24.13. The iterated limit set $\omega^2(\varphi)$ is the union of ω-limit sets of *all* trajectories forming $\omega(\varphi)$.

If γ is a singular or periodic orbit, then $\omega(\gamma) = \omega^2(\gamma) = \gamma$. The set $\omega^2(\varphi)$ is also closed and invariant by the flow, but may well be nonconnected.

In the same way higher iterated ω-limit sets can be defined inductively as unions of limit sets of all trajectories forming a previous iteration. By construction, they constitute a sequence of nested compacts. Yet it turns out that on the plane this generalization does not lead to anything new. The core statement of the Poincaré–Bendixson theory asserts that the iterated ω-limit sets on the sphere in fact stabilize from the second step. The following statement has no analogs for vector fields on higher-dimensional manifolds.

Lemma 24.14. *For any vector field with isolated singular points on the sphere, the ω^2-limit of any trajectory is either a periodic orbit, or a collection of singular points.*

Proof. Suppose that $\Gamma = \omega^2(\varphi)$ contains a nonsingular point a of the field, and let τ be a cross-section to Γ at a. This means that some invariant trajectory γ from $\omega(\varphi)$ must cross τ infinitely many times. But the contour formed by an arc of γ between two *subsequent* crossings and a segment of the cross-section will be a Bendixson trap unless γ is periodic. This would contradict Lemma 24.9. □

Proof of Theorem 24.12. By Lemma 24.14, both α- and ω-limit sets of any nonconstant trajectory $\gamma \subseteq \omega(\varphi)$ are singular points. □

We conclude this section by an example showing that on surfaces other than the sphere (with its simple topological properties), Theorem 24.12 fails completely.

Example 24.15. The constant vector field $dy/dx = \alpha$, $\alpha \in \mathbb{R}$ on the 2-torus $\mathbb{T}^2 = (\mathbb{R} \bmod \mathbb{Z})^2$ has all trajectories periodic for $\alpha \in \mathbb{Q}$. However, if $\alpha \notin \mathbb{Q}$ is irrational, the ω-limit of any orbit coincides with the entire torus $\mathbb{T}^2$.

24C. Polycycles, monodromy, correspondence maps. Without further assumptions on the vector field it is difficult to describe more precisely possible limit sets of trajectories on the sphere.

From this moment on we will assume that all vector fields satisfy the following two finiteness assumptions:

(1) the field has only isolated singular points on the sphere, and
(2) each singular point has only finitely many hyperbolic sectors (cf. Definition 9.2).

These assumptions are automatically satisfied for *real analytic* vector fields.

By Theorem 24.12, in these assumptions the ω-limit set Γ of any trajectory φ is a planar (more accurately, spherical) finite graph consisting of finitely many vertices (singular points) connected by edges (trajectories bi-asymptotic to these vertices).

This graph is co-oriented: by Proposition 24.10, every edge $\gamma \subset \Gamma$ has a "positive" side, from which the trajectory φ accumulates to Γ, and the "negative" side. Therefore among the connected components of $\mathbb{S}^2 \smallsetminus \Gamma$ (faces of the spherical graph) there is a distinguished component Ω containing φ; see Fig. IV.7. Each connected component C of the boundary $\partial\Omega \subseteq \Gamma$ is an "almost circle", i.e., the image of the circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ by a continuous map $\iota\colon \mathbb{S}^1 \to C$ bijective except finitely many points that are mapped into singular points of v. Since the curve φ cannot (again by Jordan theorem) approach any point from $\partial\Omega \smallsetminus \Gamma$, we conclude that $\partial\Omega = \Gamma$ and hence $\partial\Omega$ must be connected, $\partial\Omega = C$. In other words, $\Gamma = \omega(\varphi)$ which is not a singular point or a cycle, is a closed continuous curve bounding a spherical domain, whose self-intersections can occur only at singularities. Such an object is called a *polycycle*.

Definition 24.16. A *polycycle* of a vector field is a finite oriented spherical graph Γ such that:

(1) topologically Γ is a continuous image of the circle $\mathbb{S}^1$,
(2) vertices of Γ are at the singular points of the field,
(3) edges of Γ are infinite trajectories of the field.

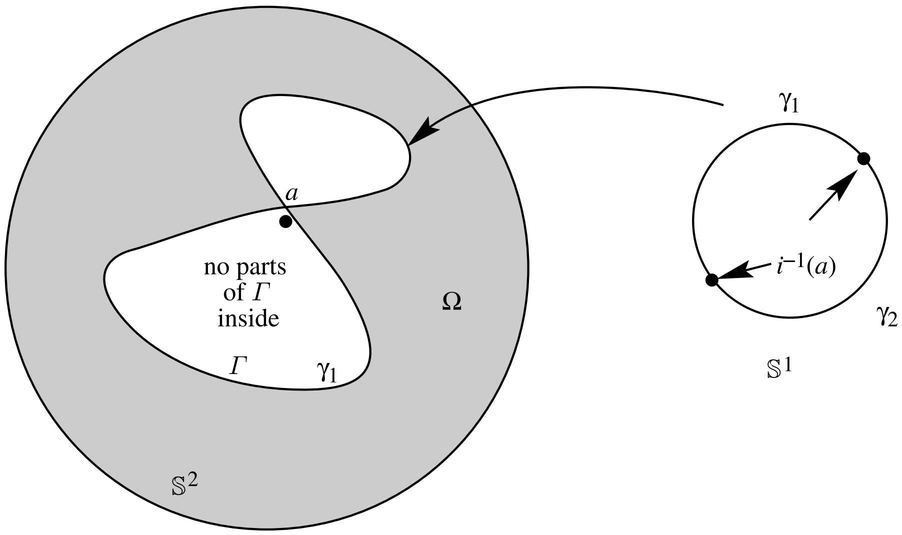

Figure IV.7. The continuous "almost one-to-one" image of the circle Γ bounding the connected domain Ω

Note that among the singular points (also cyclically enumerated) repetitions are allowed whereas the edges are all distinct.

Let $\tau_+ : (\mathbb{R}^1_+, 0) \smallsetminus \{0\} \to (\mathbb{S}^2, a)$ be a *semi-section*, the restriction of a cross-section τ at a nonsingular point $a \in \Gamma$, on the "positive" open semi-interval (i.e., such that $\varphi \cap \tau_+$ is nonvoid).

Proposition 24.17. *There is a well-defined first return map* (*also called* monodromy map) $\Delta_\Gamma : \tau_+ \to \tau_+$ *such that for any point* $p \in \tau_+$ *the orbit of* v *starting at* p*, intersects* τ_+ *for the first time again at* $\Delta_\Gamma(p)$*.*

Proof. Consider the infinite sequence of points $x_1, x_2, \dots,$ which are *subsequent* intersections of the trajectory φ with the semi-section τ_+; this sequence converges to the base point a of the semi-section.

Consider the trap T formed by the arc of φ from x_1 to x_2 and a piece of τ_+ between these points. The trajectory φ starting from the point x_2 entirely belongs to the annulus $T \smallsetminus \Omega$, where Ω is the spherical domain containing φ. Without loss of generality we may assume that this annulus contains no singular points of the field other than belonging to the polycycle (recall that singularities of v are isolated).

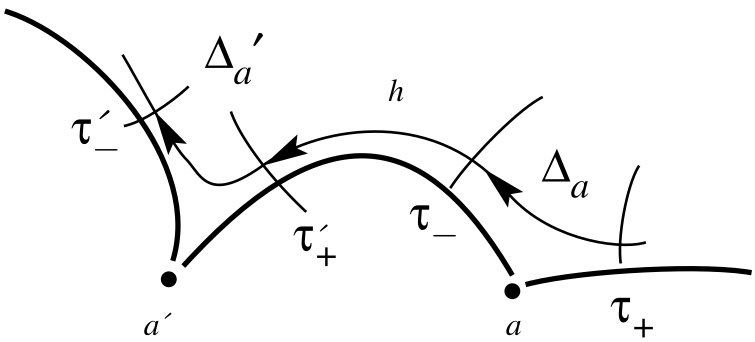

Figure IV.8. Correspondence maps

Consider the strip Π formed by two arcs $\varphi' = \varphi|_{x_1}^{x_2}$ and $\varphi'' = \varphi|_{x_2}^{x_3}$ of the trajectory φ and two segments $\tau' = \tau_+|_{x_1}^{x_2}$ and $\tau'' = \tau_+|_{x_2}^{x_3}$ on the cross-section. We claim that any other trajectory ψ starting on τ', crosses τ'' at some time in the future.

Indeed, ψ cannot cross the arcs φ', φ'' as they are phase curves of the field. If ψ does not cross τ'', then its ω-limit must be nonvoid. Since Π does not contain singular points, the ω-limit set must be a cycle by the Poincaré–Bendixson Theorem 24.11. But by the Poincaré–Hopf index theorem, each cycle must contain a singular point in its interior, leading again to the contradiction.

Therefore the first return map Δ_Γ is well defined on τ' and takes values on τ''. For the same reasons Δ_Γ is well defined on any segment $\tau_+|_{x_n}^{x_{n+1}} \subset \tau_+$. Since these segments together cover the entire semi-section τ_+, the proposition is proved. □

Remark 24.18 (terminological)**.** Note that the first return map Δ_Γ constructed in the proof of Proposition 24.17, possesses the following property: for all points $p \in \tau_+$ sufficiently close to a, the orbit connecting p with $\Delta_\Gamma(p)$ remains in an arbitrarily small neighborhood of the polycycle. This condition excludes some polycycles, e.g., those sketched on Fig. IV.5 (g), (h), from being limit sets of trajectories. In the future we will call a polycycle Γ *monodromic*, if it admits the first return map along orbits that remain in an arbitrarily small neighborhood of Γ.

Consider a singular point $a \in \Gamma$ on a monodromic polycycle Γ, and let $\gamma_+, \gamma_- \subseteq \Gamma$ be two trajectories such that $\omega(\gamma_+) = a = \alpha(\gamma_-)$ (the loop case where $\gamma_+ = \gamma_-$ is not excluded). Let $\tau_\pm$ be two semi-sections to the curves $\gamma_\pm$ at two points $a_\pm$ respectively, from the "positive" side of each of them. The same arguments as in the proof of Proposition 24.17 show that each trajectory starting on τ_+ sufficiently close to a_+, crosses also τ_- somewhere near a_-. This allows us to define the *correspondence map* $\Delta_a\colon \tau_+ \to \tau_-$ associated with the singular point $a \in \Gamma$. This map, in general, not analytically extendable to the point a_+, remains continuous after setting $\Delta_a(a_+) = a_-$. By this construction, Δ_a is defined modulo the freedom in choosing the cross-sections $\tau_\pm$, i.e., modulo a conjugacy by analytic germs $h_\pm \in \mathrm{Diff}(\mathbb{R}^1, 0)$ from left and right, $h_- \circ \Delta_a \circ h_+$.

We will summarize the results of this section as follows.

Theorem 24.19. *Assume that a smooth vector field on the sphere has only isolated singular points, each of them having at most finitely many hyperbolic sectors.*

Then an ω-limit set of any orbit of this field is either a singular point, or a cycle (periodic orbit) or a finite monodromic polycycle Γ.

In the latter case the first return map of this polycycle Δ_Γ is well defined on any semi-section τ_+ to Γ at a nonsingular point of the latter, and expands as a finite composition of the form

$$\Delta_\Gamma = h_n \circ \Delta_{a_n} \circ h_{n-1} \circ \Delta_{a_{n-1}} \circ \cdots \circ h_1 \circ \Delta_{a_1} \circ h_0. \tag{24.3}$$

Here Δ_{a_i} are correspondence maps associated with the singular points $a_i \in \Gamma$, and h_i are some real analytic maps. □

24D. Accumulation of limit cycles. Recall (see Definition 9.11) that a limit cycle is an *isolated* periodic trajectory of a vector field.

As the first step towards the solution of Problem I (finiteness problem for limit cycles) we will describe possible accumulation sets for limit cycles of smooth vector fields. Such fields may indeed have an infinite number of limit cycles, but these cycles must accumulate to a monodromic polycycle. To make this statement precise, we need the notion of the Hausdorff distance.

Definition 24.20. Let A, B be two subsets of a metric space M. The Hausdorff distance between them is the nonnegative number

$$\mathrm{dist}(A, B) = \max[\sup_{a\in A} \mathrm{dist}(a, B),\ \sup_{b\in B} \mathrm{dist}(b, A)], \tag{24.4}$$

where $\mathrm{dist}(x, Y) = \inf_{y\in Y} \mathrm{dist}(x, y)$ is the distance between a point x and any subset $Y \subset M$.

One can easily verify (see [**BBI01**, Chapter 7]) that the Hausdorff distance satisfies the triangle inequality and defines a *metric* on the space of *closed* subsets: if A, B are closed and $\mathrm{dist}(A, B) = 0$, then $A = B$.

A sequence of subsets $A_1, A_2, \dots, A_n, \dots \subseteq M$ converges in the sense of Hausdorff distance to a limit A, if every point of $a \in A$ is the limit of a sequence of points $a_1, a_2, \dots$ such that $a_i \in A_i$. An alternative description of the limit is similar to (24.2)

$$A = \bigcap_{n=1}^{\infty} \overline{\bigcup_{i=n}^{\infty} A_i};$$

see [**BBI01**, Exercise 7.3.4]. The following result is elementary but very useful.

Theorem 24.21 (W. Blaschke)**.** *If the metric space M is compact, then the space of compact subsets of M equipped with the Hausdorff distance is also compact.*

Proof. See [**BBI01**], Theorem 7.3.8. □

Theorem 24.22. *Assume that a smooth vector field on the sphere $\mathbb{S}^2$ has only isolated singular points, each of them having at most finitely many hyperbolic sectors.*

If this field has infinitely many limit cycles, then there exists an infinite sequence of these cycles $\{\gamma_i\}_{i=1}^{\infty} \subset \mathbb{S}^2$ converging in the sense of the Hausdorff distance to a singular point, a cycle (periodic orbit) or a monodromic polycycle Γ.

In the latter case if $\Delta_\Gamma\colon \tau_+ \to \tau_+$ is the monodromy map of the polycycle, then the intersection points $p_i = \gamma_i \cap \tau_+$ are isolated fixed points for Δ_Γ accumulating to the base point of the semi-section τ_+.

Proof. By Blaschke Theorem 24.21, an infinite number of limit cycles on the compact 2-sphere must contain an infinite sequence of cycles that accumulates in the sense of the Hausdorff distance to a compact subset $\Gamma \subseteq \mathbb{S}^2$. We show that if Γ contains a nonsingular point of v, then Γ is either a cycle or a monodromic polycycle.

To do this, one can modify slightly the arguments leading to the proof of Theorem 24.19. Yet we can reduce Theorem 24.22 to Theorem 24.19 directly, using a *plug* as on Fig. IV.9.

Let $a \in \Gamma$ be a nonsingular point. Consider *two* close semi-sections τ_+, τ'_+ at the points $a \neq a'$ to the trajectory γ passing through a, and denote by p_i, p'_i the corresponding intersection points between the cycles γ_i with these cross-sections.

Consider the narrow strip Π ("plug") bounded by $\gamma|_a^{a'}$ and the two semi-sections τ_+, τ'_+ (the outer bound can be chosen rather arbitrarily). Let w be a C^∞-smooth vector field which coincides with v everywhere outside of Π and on the boundary $\tau_+ \cup \tau'_+ \cup \gamma|_a^{a'}$ of the latter, such that its orbits which begin at p_i pass through Π and end at p'_{i+1}.

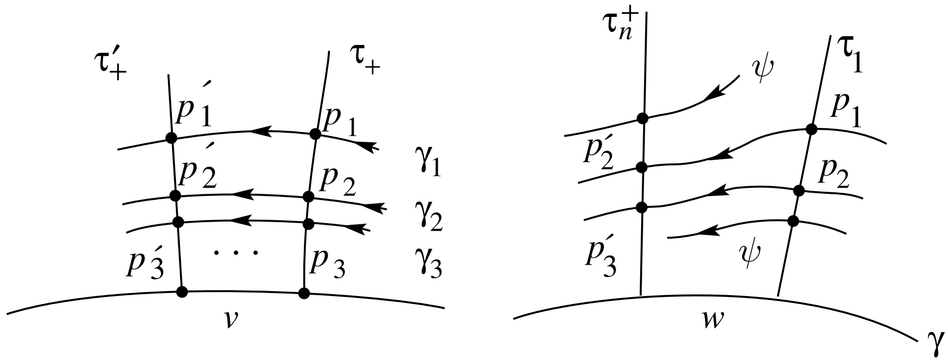

Figure IV.9. The plug: modification of a vector field in a small strip Π between two semi-sections

Then all cycles γ_i of the initial field v belong to the single trajectory ψ of the field w. Obviously, the ω-limit set of ψ coincides with the Hausdorff limit set Γ for the sequence of the limit cycles γ_i. By Theorem 24.19, the former is a cycle or a monodromic polycycle. □

Finiteness Theorem 24.1 for limit cycles of planar and spherical analytic vector fields follows now from a purely analytic local property of the monodromy map of polycycles of such fields.

Theorem 24.23 (General finiteness theorem, Yu. Ilyashenko [**Ily91**], J. Écalle [**Eca92**])**.** *The monodromy map of a polycycle of an analytic vector field in the plane cannot have an infinite number of isolated fixed points.*

We will prove here this theorem under a simplifying assumption that the polycycle is *hyperbolic*, i.e., it carries only nondegenerate saddles at the vertexes. This implies the following theorem which is the main result of this section.

Theorem 24.24 (Easy finiteness theorem)**.** *A real analytic vector field on the 2-sphere, having only nondegenerate singular points, may have only finitely many limit cycles.*

The proof is based on investigation of the individual correspondence maps for analytic hyperbolic saddles and their compositions with holomorphic germs.

24E. Almost regular germs and monodromy of hyperbolic polycycles. Developing the ideas of Dulac [**Dul23**], we introduce a class of germs with two competing properties. On one hand, this class is large enough as to include monodromy transformations of hyperbolic polycycles $\Delta_\Gamma\colon (\mathbb{R}^1_+, 0) \to (\mathbb{R}^1_+)$, $z \mapsto \Delta_\Gamma(z)$, which are in general not analytic at $z = 0$.

On the other hand, this class is so close to the class of analytic functions that germs of this class are uniquely determined by their asymptotic expansions.

In this section we will mostly work in the *logarithmic chart* $\zeta = -\ln z$: in this chart the interval $z \in (0, \varepsilon)$ becomes a neighborhood of infinity, $\zeta \in (\frac{1}{\varepsilon}, +\infty)$.

Definition 24.25. A *standard* (*quadratic*) *domain* Ω_C is the image of the right half-plane $\mathbb{C}_+ = \{\operatorname{Re}\zeta > 0\}$ by the map

$$\varphi_C \colon \zeta \mapsto \zeta + C\sqrt{1+\zeta}, \qquad C > 0. \tag{24.5}$$

The constant C is a parameter determining the "size" of the standard domain Ω_C.

Definition 24.26. An *exponential series*, or *Dulac series*, is the formal series

$$S = \alpha\zeta + \beta + \sum_{j=1}^{\infty} p_j(\zeta)\exp(-\nu_j\zeta), \qquad \alpha,\beta \in \mathbb{R}, \quad p_j \in \mathbb{R}[\zeta], \tag{24.6}$$

in which

$$\alpha \geqslant 0, \qquad 0 < \nu_1 < \nu_2 < \cdots < \nu_n < \cdots, \qquad \lim \nu_j = +\infty.$$

No assumptions on convergence of the series (24.6) is made.

A function f defined in some standard domain Ω_C is said *to admit an expansion in the Dulac series* (24.6) (to be *expandable*, for short), if for any order $\nu > 0$ there exists a partial sum S_ν of this series, such that

$$|f(\zeta) - S_\nu(\zeta)| = o\big(\exp(-\nu\zeta)\big) \qquad \text{as } |\zeta| \to \infty \text{ in } \Omega_C. \tag{24.7}$$

Definition 24.27. The germ of a real analytic map $f\colon (\mathbb{R}^1_+, 0) \to (\mathbb{R}^1_+, 0)$ is called *almost regular*, if in the logarithmic chart the germ $-\ln f\big(\exp(-\zeta)\big)$ has a representative that can be extended as a biholomorphic map between two standard domains and expanded in a Dulac exponential series there.

Remark 24.28. Apriori in the Definition 24.27 one can allow dependence of the Dulac series on the order ν to which it approximates the almost regular germ f. Yet the asymptotic series (24.6), if it exists, is unique, and this is proved exactly like the uniqueness of the Taylor asymptotic series. Assume that for any ν there exists a Dulac polynomial $S_\nu(\zeta)$ (a finite sum of the form (24.6) with positive exponents ν_j not exceeding ν) such that the difference $f - S_\nu$ is decreasing as $o\big(\exp(-\nu\zeta)\big)$. Then all polynomials S_ν are necessarily truncations of a single Dulac series S as in (24.6) which is an asymptotic series for the function f. Indeed, if $\nu' > \nu$, then S_ν is a truncation of S_ν. Otherwise their difference cannot be decreasing as $o\big(\exp(-\nu'\zeta)\big)$ as $\zeta \to \infty$ in Ω_C.

The condition of almost regularity is weaker than analyticity at the point $z = 0$. Indeed, any converging Taylor series $f(z) = a_1 z + a_2 z^2 + \cdots$ in the logarithmic chart becomes a uniformly *convergent* Dulac series

$$\begin{aligned}-\ln f(\zeta) &= \ln a_1 + \zeta + \ln\bigl(1 + \tfrac{a_2}{a_1}\exp(-\zeta) + \tfrac{a_3}{a_1}\exp(-2\zeta) + \cdots\bigr)\\ &= \zeta + \beta + (\text{Dulac series without affine part}).\end{aligned}$$

Yet the following property means that in some respects almost regular germs are similar to analytic germs which were called *regular* in the old-fashioned language of the nineteenth century (this explains the choice of the term "almost regular").

Theorem 24.29. *An almost regular germ is uniquely determined by its asymptotic Dulac series: two almost regular germs with the same series coincide identically in their common domain.*

In other words, not only the Dulac asymptotic series is uniquely defined by an almost regular germ as Remark 24.28 notes, but the germ itself is completely determined by its series.

It turns out that the class of almost regular germs is large enough for our purposes.

Theorem 24.30. *The germ of the monodromy map of a hyperbolic polycycle is almost regular.*

The Nonaccumulation Theorem 24.24 is an almost direct consequence of these two theorems, as the following argument shows.

Proof of Theorem 24.24. Suppose that limit cycles accumulate to a hyperbolic polycycle Γ. Then the monodromy map $\Delta = \Delta_\Gamma \colon (\mathbb{R}^1_+, 0) \to (\mathbb{R}^1_+)$ has an infinite number of isolated fixed points accumulating to $z = 0$, as explained in §24**D**.

By Theorem 24.30, in the logarithmic chart $\zeta = -\ln z$ the monodromy map $f(\zeta) = -\ln \Delta(\exp -\zeta)$ admits an exponential asymptotic series S of the form (24.6) and has infinitely many real fixed points accumulating to $\zeta = +\infty$. We claim, following Dulac [**Dul23**], that this series is in fact an identity, $S = \zeta$.

Indeed, consider the difference $S - \zeta$ which also admits the exponential series (24.6). If this difference is nonzero, then its leading term is either affine $(\alpha - 1)\zeta + \beta$, or exponential $p_1(\zeta)\exp(-\nu_1\zeta)$. In both cases the difference between the monodromy map $f(\zeta)$ itself and the identity ζ has the form $g(\zeta)(1 + o(1))$, where $g(\zeta)$ is a real analytic function on $\mathbb{R}_+$ *with only finitely many* (real) *zeros*, which contradicts the assumption that these zeros are accumulating to infinity. Hence the series S must be identical, $S = \zeta$.

Thus the asymptotic series S of the map f is identity. On the other hand, Δ is almost regular by Theorem 24.30. Theorem 24.29 implies that in this case the map f itself is identity, $f(\zeta) \equiv \zeta$, and hence $\Delta(z) \equiv z$. Thus Δ cannot have isolated fixed points at all. The contradiction proves the Nonaccumulation Theorem 24.24. □

Remark 24.31. In [**Dul23**] Dulac tacitly assumed that the monodromy map with the identical Dulac series, is itself identity, circumventing Theorem 24.29. However, this assertion is wrong in absence of hyperbolicity of the polycycle. In [**Ily84**] one can find an example of a (nonhyperbolic) polycycle whose monodromy differs from identity by a flat (decaying faster than any exponential of ζ) nonzero function.

The rest of this section is devoted to the proof of the two key facts: Theorem 24.29 is proved in §24**G**, while the proof of Theorem 24.30 is postponed until subsection §24**H**. In order to carry out the proofs, we need some elementary properties of almost regular maps.

24F. Elementary properties of almost regular maps. The class of almost regular germs is rather natural. As was already noted, it contains all germs regular at $z = 0$.

Example 24.32. The power map $z \mapsto cz^\lambda$ for $\lambda > 0$ is almost regular. Indeed, in the logarithmic chart this map becomes *affine*, $\zeta \mapsto \lambda\zeta + \beta$, $\beta = -\ln c$. The corresponding Dulac series is finite, and it remains only to verify that it maps any standard domain into another standard domain. One can easily verify that the image of the standard domain Ω_C belongs to the standard domain $\Omega_{C'}$ if $C' = \alpha^{1/2}C + C_0$ for C_0 sufficiently large.

Rather expectedly, the class of almost regular germs is closed by composition.

Lemma 24.33. *Composition of two almost-regular germs is again an almost regular germ.*

Proof. It is convenient to treat separately the *affine germs* of the form $\zeta \mapsto \alpha\zeta + \beta$, $\alpha \geqslant 0$, $\beta \in \mathbb{C}$, and the *parabolic almost regular germs* whose Dulac series starts with the identical term,

$$S = \zeta + \sum_{\nu>0} p_\nu(\zeta)\exp(-\nu\zeta). \qquad (24.8)$$

Let us check that if $f(\zeta)$ is a function holomorphic in a standard domain Ω_C and admits there an estimate $|f(\zeta) - \zeta| < \exp(-\varepsilon\zeta)$ for some $\varepsilon > 0$, then the image of Ω_C by f contains a standard domain $\Omega_{C'}$ for C' sufficiently large. Indeed, the exponential small "perturbation" cannot change the asymptotic behavior of the curve

$$\operatorname{Re}\zeta = C\,|\operatorname{Im}\zeta|^2 + O(1), \qquad \operatorname{Im}\zeta \to \pm\infty \qquad (24.9)$$

which is the boundary $\partial\Omega_C$. Preservation of the class of standard domains under action of affine maps is discussed in Example 24.32.

Thus composition of almost regular germs is defined (after analytic continuation) in some standard domain and takes it into another standard domain. It remains to verify the existence of an asymptotic Dulac expansion for a composition of two almost regular maps.

Note that if $R = \sum_{\nu>0} p_\nu(\zeta)\exp(-\nu\zeta)$ is a Dulac series *without the affine part* (with only positive exponents), then all its powers $R^2, R^3, \dots$ and any product $\exp(-\mu\zeta)R$, $\mu > 0$, are also of the same form. Therefore the formal exponent

$$\exp(-\mu R) = 1 + \sum_{k>0} (-\mu R)^k/k!$$

is also a well-defined Dulac series. The direct substitution now shows immediately that the composition of two parabolic series

$$(\zeta + R') \circ (\zeta + R) = (\zeta + R) + \sum_{\mu>0} p_\mu(\zeta + R)\exp(-\mu\zeta)\exp(-\mu R) = \zeta + R''$$

is a parabolic Dulac series.

It remains only to notice that composition of a parabolic Dulac series with an affine map $a\colon \zeta \mapsto \alpha\zeta + \beta$ (in any order) is obviously a Dulac series, and moreover, parabolic germs constitute a normal subgroup: if $f(\zeta) = \zeta + R$ is a parabolic germ, then $a^{-1} \circ f \circ a$ is again a parabolic germ. □

Remark 24.34. Since the maps holomorphic at infinity are automatically almost regular, the definition of the almost regular maps does not depend on the coordinate chart: by Lemma 24.33, the composition $g^{-1} \circ f \circ g$ is again a map defined in a standard domain and asymptotic to a Dulac series there.

24G. Phragmén–Lindelöf principle for almost regular germs. In this subsection we prove Theorem 24.29. It is a purely analytic fact closely related to the enhanced version of the maximum modulus principle known as the *Phragmen–Lindelöf principle.*

Recall that the maximum modulus principle asserts that a function $f = f(z)$ holomorphic in a (bounded) domain $z \in D$ and continuous on the boundary achieves the maximal value of its modulus $|f(z)|$ somewhere on the boundary ∂D. If the continuity assumption fails at a single point of the boundary, the function may well be unbounded.

Example 24.35. The function $f(z) = \exp(1/z)$ is holomorphic in the disk $|z - 1| < 1$ and continuous on its boundary except the single point $\{z = 0\}$. Yet this function is unbounded in D, despite the fact that its modulus is constant on the boundary $\partial D \smallsetminus \{1\}$. The latter fact becomes obvious in the conformal chart $\zeta = 1/z$ which transforms the function f into the exponent $\exp\zeta$ and the domain into the half-plane $\operatorname{Re}\zeta > 1/2$. The restriction of f on the boundary has constant modulus $m = \exp\frac{1}{2}$.

This example illustrates the phenomenon that lies at the core of the Phragmén–Lindelöf principle: the maximum modulus principle may fail if the boundary of the domain contains a point a near which f is unbounded, but only if the growth of f when approaching such a point is sufficiently fast; the "critical threshold" for the growth rate depends on the geometry of the boundary ∂D near a.

For our applications it is sufficient to consider only domains on the Riemann sphere, bounded by two circular arcs. In a suitable chart they become sectors with the vertex at the origin with an opening angle $2\pi/\alpha$, symmetric with respect to the real ray $\mathbb{R}^1_+ \subset \mathbb{C}$.

Theorem 24.36 (Phragmén–Lindelöf, 1908)**.** *Assume that a function $f(z)$ is holomorphic in the sector $S_\alpha = \{z : |\operatorname{Arg} z| < \frac{\pi}{2\alpha}\}$ for some $\alpha \geqslant 1$ and is continuous and bounded on the boundary of this sector,*

$$|f(z)| \leqslant M \qquad \text{for all } z \text{ such that } \operatorname{Arg} z = \pm\tfrac{\pi}{2\alpha}. \tag{24.10}$$

If the growth of f admits a uniform apriori bound

$$|f(z)| = O(\exp|z|^\beta), \qquad |z| \to \infty, \quad z \in S_\alpha, \tag{24.11}$$

for some $\beta < \alpha$, then in fact f is bounded in S_α by the same constant, $|f(z)| \leqslant M$ for all $z \in S_\alpha$.

Proof. Consider, following [**Tit39**, §5.6], the auxiliary function $g(z) = \exp(-\varepsilon z^\gamma) \cdot f(z)$ with an arbitrary small positive $\varepsilon > 0$ and some γ between α and β. We have

$$|g(z)| = \exp\bigl(-\varepsilon|z|^\gamma \cdot \cos(\gamma \operatorname{Arg} z)\bigr)\,|f(z)|.$$

Since $\gamma < \alpha$, we have $\cos(\frac{\pi\gamma}{2\alpha}) > 0$ and hence

$$|g(z)| \leqslant M \qquad \forall z \in \partial S_\alpha = \{\operatorname{Arg} z = \pm\tfrac{\pi}{2\alpha}\}.$$

On the circular arcs $\{|z| = r\} \cap S_\alpha$ by the growth assumption on f we have the estimates

$$|g(z)| \leqslant \exp\bigl(-\varepsilon r^\gamma \cos\tfrac{\gamma\pi}{2\alpha}\bigr) \cdot |f(z)| \leqslant C\exp\bigl(r^\beta - \varepsilon r^\gamma \cos\tfrac{\gamma\pi}{2\alpha}\bigr).$$

As $\gamma > \beta$ and $\varepsilon > 0$, the latter expression tends to zero as $r \to \infty$, hence the maximum modulus principle applied to the bounded sector $S_\alpha \cap \{|z| < r\}$ for all sufficiently large r yields the inequality $|g(z)| \leqslant M$ there. Since r can be arbitrarily large, $|g(z)| \leqslant M$ everywhere in S_α.

The last inequality, transformed to the form $|f(z)| \leqslant M \exp(\varepsilon |z|^\gamma)$, for any finite $z \in S_\alpha$ admits passing to the limit as $\varepsilon \to 0^+$, yielding the inequality $|f(z)| \leqslant M$ in S_α. □

To apply this result to the half-plane $\mathbb{C}_+$ corresponding to $\alpha = 1$, we would have to require that f grows subexponentially as $|z| \to \infty$. Yet this growth condition can be relaxed if f is controlled along the real axis.

Lemma 24.37. *Let f be a function holomorphic in the half-plane $\mathbb{C}_+$ and continuous and bounded on the imaginary axis $i\mathbb{R} = \partial\mathbb{C}_+$. Assume that f grows at most exponentially in $\mathbb{C}_+$, i.e., $|f(z)| \leqslant C \exp(\mu|z|)$ for some $\mu > 0$.*

Then under this apriori growth assumption:

(1) *if f is bounded on the real axis $\mathbb{R}_+ \subset \mathbb{C}_+$, then f is bounded everywhere in $\mathbb{C}_+$ and the maximum of its absolute value is achieved somewhere on the boundary*;

(2) *if f decreases faster than any exponent along the real axis $\{z > 0\}$, $|f(z)| \leqslant C_\rho \exp(-\rho z)$ for any large $\rho > 0$, then f is identically zero, $f \equiv 0$.*

Moreover, these assumptions hold if the half-plane $\mathbb{C}_+$ is replaced by the standard domain Ω_C.

Proof. By Theorem 24.36 applied with $\alpha = 2$, $\beta = 1$ to each of the quarter-planes $\mathbb{C}_+ \cap \{\pm \operatorname{Im} z > 0\}$, we conclude that f is bounded in each of them, proving thus the first assertion of the lemma.

To prove the second assertion, consider the family of functions $f_\varepsilon(z) = f(z)\exp(z/\varepsilon)$ for arbitrarily small $\varepsilon > 0$. Any such function still has exponential growth in $\mathbb{C}_+$. Since the exponent has modulus equal to 1 on $i\mathbb{R}$ for any $\varepsilon > 0$, the maximum absolute value M achieved by f_ε on the boundary, does not depend on ε. Finally, if f decreases faster than any exponent along $\mathbb{R}_+$, so does each f_ε. Applying the first assertion of the lemma to f_ε, we arrive at the inequality $|f_\varepsilon(z)| \leqslant M$ for all $z \in \mathbb{C}_+$ and all $\varepsilon > 0$. Rewriting this inequality in the form $|f(z)| \leqslant M\,|\exp(-z/\varepsilon)|$ and passing to the limit as $\varepsilon \to 0^+$, we conclude that $f(z)$ must vanish identically in $\mathbb{C}_+$.

Finally, if f satisfies the assumptions of the lemma in a standard domain Ω_C, then $f \circ \varphi_C$ obviously satisfies the same assumptions in $\mathbb{C}_+$, where $\varphi_C : \mathbb{C}_+ \to \Omega_C$ is the map (24.5) occurring in the definition of the standard domain. □

Proof of Theorem 24.29. Theorem 24.29 is an immediate corollary to Lemma 24.37.

If two almost regular germs g and h have the same asymptotic expansions (24.6), then their difference germ $g-h$ has zero asymptotic expansion. Let f be a representative of this difference. By Definition 24.27, it can be holomorphically extended to some standard domain Ω_C, and grows no faster than a linear function there. On the other hand, f decays at infinity faster than any exponential, since its asymptotic series is identically zero. By Lemma 24.37, $f \equiv 0$, hence, $g \equiv h$. □

24H. Correspondence map of a hyperbolic saddle. The proof of Theorem 24.30 rests upon the following result.

Theorem 24.38. *The correspondence map of a hyperbolic saddle is almost regular.*

To prove this theorem, we first note that the correspondence map of a hyperbolic saddle *in the formal normal form* (22.3) is almost regular; moreover, in this case the corresponding Dulac series is convergent.

If the normal form is linear, then the correspondence map is a pure power, $w = cz^\lambda$, which becomes affine $\zeta \mapsto \lambda\zeta + \ln c$, in the logarithmic charts. Thus only a nonlinear normal form should be studied.

Consider the saddle vector field in the formal normal form, defined by the ordinary differential equations

$$\begin{cases} \dot w = -\lambda w(1+q(u)), & u(z,w) = z^m w^n, \\ \dot z = z, & \lambda = \frac{m}{n}, \end{cases} \qquad q(u) = \frac{u^{p+1}}{1+\alpha u^p}. \tag{24.12}$$

Let τ_+ and τ_- be the cross-sections $\{w=1\}$ and $\{z=1\}$ to the vector field (24.12) with the charts z and w on them respectively. The correspondence map $\Delta\colon \tau_+ \to \tau_-$ is well defined for $z>0$ and takes positive values.

Proposition 24.39. *The correspondence map Δ between the cross-sections τ_+ and τ_- for the vector field* (24.12) in the formal normal form*, is almost regular.*

Moreover, the corresponding Dulac series in this case is convergent: there exists a real analytic function $G \in \mathcal{O}(\mathbb{R}^2, 0)$ such that

$$-\ln\Delta(\zeta) = \lambda\zeta + G\big(\exp(-m\zeta), \zeta\exp(-m\zeta)\big), \qquad \zeta = -\ln z. \tag{24.13}$$

Proof. The assertion follows from integrability of the vector field (24.12) which allows us to compare the value of the resonant monomial on the intersections $(z,1)$ and $(1,w)$ of an arbitrary integral trajectory of (24.12) with the cross-sections $\tau_\pm$. One has to prove that the solution of the initial value problem for the quotient differential equation

$$\frac{du}{dt} = -n\lambda\frac{u^{p+2}}{1+\alpha u^p}, \qquad u(0) = z^m,$$

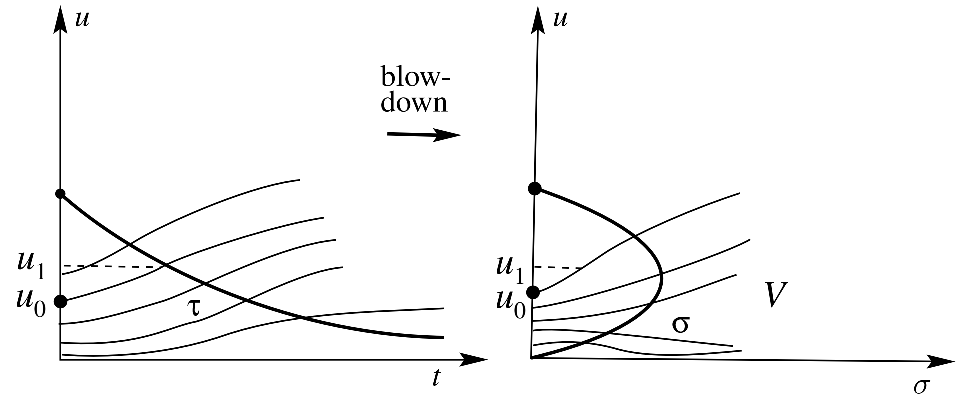

Figure IV.10. Integration of the quotient equation via blow-down

evaluated at the moment $t = -\ln z$, is (after extracting of the mth order root) an almost regular function $w(z)$ of z. The proof can be achieved by explicit integration and investigation of the resulting algebraic relation between z, $\ln z$ and w.

Yet one can avoid intermediate calculations applying the following geometric construction (a particular nonparametric case of [**IY91**, Lemma 11]). The quotient equation can be coupled with the trivial equation $\dot t = 1$, resulting in a vector field in the positive quadrant of the (t, u)-plane,

$$\begin{cases} \dot u = u[-n\lambda\, q(u)], \\ \dot t = 1, \end{cases} \qquad t, u \geqslant 0. \tag{24.14}$$

Suppose that a trajectory γ of the initial field (24.12) crosses τ_+ at the point $(1, z)$ corresponding some value $u_0 = z^m$. Then the travel time necessary to reach τ_- is equal to $-\ln z = -\frac{1}{m}\ln u_0$.

Consider on the (t, u)-plane the curve $\tau = \{t = -\frac{1}{m}\ln u\}$; the value u at the moment of intersection between γ and τ_- is the u-coordinate of the intersection of the respective trajectory of (24.14) with τ (Fig. IV.10).

The system (24.14) *admits a simple blow-down of the t-axis*: after passing to the coordinates u and $v = tu$, we obtain

$$\dot u = u\bigl(-n\lambda\, q(u)\bigr), \qquad \dot v = u + v\bigl(-n\lambda\, q(u)\bigr) = u\left(1 - n\lambda \frac{vu^p}{1+\alpha u^p}\right) \tag{24.15}$$

(we use the fact that $q(u)$ is divisible by u). After division by u we obtain a *nonsingular vector field V* in a neighborhood of the origin on the (u, v)-plane, tangent to the v-axis. The curve τ blows down to the curve σ defined by the equation $v = -\frac{1}{m}u\ln u$ which tends to the origin as $u \to 0^+$.

The vector field V, being transversal to the u-axis and tangent to the v-axis, admits a real analytic first integral $\Phi(u,v) = uF(u,v)$, $F(0,0) = 1$, uniquely defined by the Cauchy boundary data $\Phi(u,0) \equiv u$. From the above description of the correspondence map, we conclude that the u-value u_1 at the moment when the trajectory crosses the exit section τ_-, is equal to the value of Φ restricted on σ, i.e., $u_1 = u_0 F(u_0, -\frac{1}{m} u_0 \ln u_0)$.

Returning to the coordinates $z = u_0^{1/m}$ and $w = u_1^{1/n}$, we conclude that the correspondence map for the saddle in the formal normal form can be expressed as

$$\begin{aligned} w &= [z^m F(z^m, -z^m \ln z)]^{1/n} \qquad && G \in \mathcal{O}(\mathbb{R}^2, 0), \\ &= z^\lambda G(z^m, z^m \ln z), && G(0,0) = 1, \end{aligned} \tag{24.16}$$

where the function $G = F^{1/n}$ is real analytic in its two variables since $F(0,0) = 1$.

In the logarithmic chart $\zeta = -\ln z$ the correspondence map $-\ln w$ defined by the expressions (24.16) becomes a convergent Dulac series. □

For a saddle *not* in the formal normal form, we can no longer claim that the correspondence map is represented by a *convergent* Dulac series; for instance, this is impossible for formally linearizable but analytically nonlinearizable saddles (for more examples see [**Tri90**]). Nevertheless, we will show that this map extends analytically into sufficiently large domain in the logarithmic chart and admits an *asymptotic* Dulac series there.

Lemma 24.40. *The correspondence map of a saddle in the logarithmic chart extends to a standard domain Ω_C for a sufficiently large $C > 0$.*

Proof. The meromorphic nonlinear differential equation

$$\frac{dw}{dz} = -\lambda \frac{w}{z} \cdot (1 + \Psi(z,w)), \qquad z, w \in \mathbb{C}, \tag{24.17}$$

in the logarithmic chart takes the form

$$\frac{dw}{d\zeta} = -\lambda w (1 + \psi(w, \zeta)). \tag{24.18}$$

The function ψ holomorphic in the product $\mathbb{C}_+ \times \{|w| < 1\}$ can without loss of generality be assumed uniformly arbitrarily small there, in particular, it is sufficient if $|\psi| < \lambda/2$.

Consider the function $W(\zeta, \eta)$ of two complex variables, which is initially only locally defined near the diagonal $\{\zeta = \eta\}$ as the solution of the equation (24.18) with the initial condition $W(\eta, \eta) = 1$.

An oriented path $\gamma = \gamma(\eta)$ in the ζ-half-plane $\mathbb{C}_+$, connecting the point $\zeta = \eta$ with the point $\zeta = 0$ will be called *admissible*, if $W(\,\cdot\,, \eta)$ can be

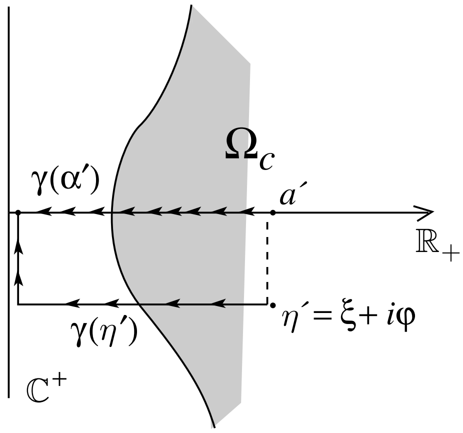

Figure IV.11. Analytic continuation of saddle correspondence map

analytically continued along this path from its initial value $W(\eta,\eta) = 1$, and this continuation satisfies the restriction $\left|W(\,\cdot\,,\eta)|_\gamma\right| \leqslant 1$ along this path.

If γ is an admissible path, then it defines the germ at $\zeta = \eta$ of *some* branch $\Delta(\eta) = W(0,\eta)$ of the complexified correspondence map; here the right hand side is obtained by the above continuation along γ.

If $\eta_+ \in \mathbb{R}_+$ is a point on the real axis, then the path $\gamma(\eta_+) = [\eta_+, 0]$ (the real segment) is admissible, since $W(\cdot,\eta_+)$ is increasing on the real axis. The function $\Delta(\eta_+)\colon \mathbb{R}_+ \to \mathbb{R}_+$ obtained by continuation along these paths defines the *real branch* of the correspondence map.

In order to obtain the analytic continuation of this real branch to a point $\eta \in \mathbb{C}_+$, one should find an admissible path $\gamma(\eta) = \gamma_0$ which can be continuously deformed *within a family of admissible paths* γ_s, $s \in [0,1]$, into a real segment $[0,\eta'] = \gamma_1$.

Let $\eta = \varrho + i\varphi$ be a point in the half-plane $\mathbb{C}_+$. We claim that the path $\gamma(\eta)$ which consists of the segment of length ϱ from η to the point $i\varphi \in i\mathbb{R} = \partial\mathbb{C}_+$, and the segment of length $|\varphi|$ on imaginary axis, continuing the path to the origin, is admissible provided that η belongs to some standard domain Ω_C.

Indeed, for points $\varrho + i\varphi$ inside the standard domain Ω_C, we have the asymptotic representation $|\varphi| = (\varrho/C)^{1/2} + O(1)$ as $\varrho \to +\infty$. Along the first segment of the corresponding path $\gamma = \gamma(\varrho + i\varphi)$ the modulus of W decreases exponentially from 1 to a small value not exceeding $\exp(-\lambda\varrho/2)$ if $|\psi| < \frac{1}{2}$, since $\operatorname{Re}(\lambda + \psi(z,w)) \geqslant \lambda/2$ along this path.

The Cauchy operator $F^{i\varphi}$ of analytic continuation (flow) along the vertical segment can be represented in the form

$$F^{i\varphi} = F^{i\theta} \circ \Delta_z^{nk}, \qquad 0 \leqslant \theta < 2\pi n, \ k \in \mathbf{N},$$

where $\Delta_z(w) = (\exp 2\pi i \frac{m}{n})w + O(w^2) \in \mathrm{Diff}(\mathbb{C}, 0)$ is the holonomy (monodromy) operator associated with the standard loop $z = \exp 2\pi i t$, $t \in [0, 1]$, on the z-axis. The linear part of Δ_z is a rational rotation, so that the nth iterate $\Delta_z^n(w) = w + O(w^2) \in \mathrm{Diff}_1(\mathbb{C}, 0)$ is parabolic (tangent to the identity). Because of the inequality between $|\varphi|$ and ϱ implied by the condition $\varrho + i\varphi \in \Omega_C$, we have an upper bound $k = O\big((\varrho/C)^{1/2}\big)$.

Let L be the maximal Lipschitz constant of the flow map $F^{i\theta}$ over $0 \leqslant \theta \leqslant 2\pi n$ on the disk $\{|w| \leqslant 1\}$. Clearly, $L < +\infty$.

The growth of iterates of $\Delta_z^{nk}(w)$ of the parabolic germ Δ_z^n as $k \to \infty$ can be estimated comparing the growth of the cascade of iterates $\sigma_a : r \mapsto r + ar^2$ with the growth of solutions of the auxiliary differential equation $\dot{r} = br^2$ on the time interval $t \in [0, k]$ (here a, b are real parameters). Indeed, since Δ_z^n is parabolic, $|\Delta^n(z) - z| < a|z|^2$, thus the absolute value of the iterates $\Delta_z^{kn}(z)$ does not exceed $\sigma_a^k(r)$, $r = |z|$. The flow $\sigma_b' = \exp br^2 \frac{\partial}{\partial r}$ of the auxiliary equation can also be immediately computed: $\sigma_b'(r) = r + br^2/(1 - br)$. Thus for each $a > 0$ one can find $b > 0$ such that the flow majorizes the cascade, $\sigma_a^k < {\sigma_b'}^k$ for all k on a sufficiently small interval $r < \varepsilon$.

The flow of this equation with the initial condition $r(0) = |W(i\varphi, \eta)| \leqslant \exp(-\lambda\varrho/2)$ at the moment $k = O\big((\varrho/C)^{1/2}\big)$ does not exceed the reciprocal $\big|\exp(\lambda\varrho/2) - O(\varrho/C)^{1/2}\big|^{-1}$ (the auxiliary flow is constant in the chart $1/w$). Thus we conclude that along the path $\gamma(\eta)$ the function $W(\,\cdot\,, \eta)$ is bounded in the absolute value by $L\big|\exp(\lambda\varrho/2) - O(\varrho/C)^{1/2}\big|^{-1}$ which is less than 1 if $\varrho > C$ (as is the case if $\eta \in \Omega_C$) and C is sufficiently large.

The path $\gamma(\eta)$, $\eta = \varrho + i\varphi$, can be deformed to a segment of the real axis as follows: its endpoint $\eta_s = \varrho + i\varphi(1 - s)$ moves parallel to the imaginary axis towards $\varrho \in \mathbb{R}$, and $\gamma(\eta_s)$ as before consists of a horizontal segment of the same length ρ and contracting vertical segments of length $(1 - s)|\varphi|$. All estimates remain the same during this deformation, hence the paths $\gamma(\eta_s)$ are admissible for all $s \in [0, 1]$. □

Proof of Theorem 24.38. After the existence of analytic continuation of the saddle correspondence map into a standard domain is proved, the Proximity lemma 22.6 together with Proposition 24.39 allow us to prove that this map admits an asymptotic expansion in the Dulac series. This will complete the proof of Theorem 24.38.

Consider an arbitrary saddle vector field F. By Proposition 24.39, without loss of generality we may assume that the coordinates are chosen that F differs from its formal normal form F_0 by N-flat terms as in (22.4).

We compare the correspondence maps Δ and Δ_0 for the two saddle fields, F and F_0 respectively. Both maps are defined in some standard domain Ω_C, and the correspondence map Δ_0 for F_0 is represented as a convergent Dulac series there.

By the Proximity Lemma 22.6, the correspondence map Δ for F differs from Δ_0 in Ω_C by the term that decays sufficiently fast to infinity,

$$\Delta(\zeta) - \Delta_0(\zeta) = O(\exp(-N\zeta/2)) \qquad \text{as } |\zeta| \to \infty, \quad \zeta \in \Omega_C.$$

This means that the Dulac series Δ_0 approximates Δ with an accuracy corresponding to $\nu = N/2$ in (24.7). Since N can be arbitrary, this (together with Remark 24.28) proves that the correspondence map for any saddle vector field is almost regular.

The assertion of Theorem 24.30 follows immediately from Lemma 24.33, as a composition of almost regular germs is almost regular. □

Exercises and Problems for §24.

Exercise 24.1. What step of the proof of the Poincaré–Bendixson fails when attempting to literally reproduce it for the 2-torus?

Problem 24.2. Let $H(x+iy) = y^2 - x^2 + y^4$ be a polynomial on the plane $\mathbb{R}^2 \cong \mathbb{C}^1$. Plot the phase portrait of the vector field $\dot z = ie^{iH(z)}\,(\frac{\partial H}{\partial x} + i\frac{\partial H}{\partial y})$ (in the complex notation).

Exercise 24.3. Modify the previous example to construct explicitly a polynomial vector field with an ω-limit set which carries any number of singular points on it.

Problem 24.4. Prove that sums and products of almost regular germs are almost regular.

Problem 24.5. Give an example of a saddle, for which the correspondence map has a divergent Dulac series.

Problem 24.6. Prove, using the Phragmen–Lindelöf theorem, that if one of the sectorial components of a *normalizing* map-cochain (Definition 21.9) is identical, then all other components are also identical (cf. with Problem 21.3).