NOTES ON O-MINIMALITY

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Contents

1. Ordered Structures

1.1. Preliminaries.

- By *an ordered structure* we mean a first order structure $\mathcal{M} = \langle M, <, \ldots \rangle$ where $<$ is *a dense* linear ordering on M.
- We fix an ordered structure $\mathcal{M}$.
- By definable we mean definable with parameters from M.
- By an interval we mean an interval in M with endpoints in $M \cup \{\pm\infty\}$.
- For a function f we will denote by $\Gamma(f)$ the graph of f.
- For definable $X \subseteq M^n$ and $Y \subseteq M^k$, as usual, we say that a function $f\colon X \to Y$ is definable if the graph of f is a definable subset of $M^n \times M^k$.
- For a set $X \subseteq M^k$ we will denote by X^c the complement of X, i.e. $M^k \setminus X$.
- We use $\langle a, b \rangle$ to denote an ordered pair.
- **Topology:** we use the order topology on M and the product topology on M^k.

1.2. **Definability in Ordered Structures.**

Proposition 1.1. *If X is a definable subset of M^n then the topological closure and interior of X are definable.*

Proof. **Exercise 1.1**. $\square$

Proposition 1.2. *Let $A \subseteq M^n$ be a definable set and $f\colon A \to M$ a definable function.*

(1) The set $\{a \in A\colon f \text{ is continuous at a } \}$ is definable.
(2) The function $x \mapsto \lim_{t\to x} f(t)$ is definable (i.e. its domain is a definable set and the function is definable).

Proof. **Exercise 1.2**. $\square$

Proposition 1.3 (Uniform definability)**.** *Let $\{X_a\colon a \in M^k\}$ be a uniformly definable family of subsets of M^n (i.e. there is definable $X \subseteq M^k \times M^n$ such that for every $a \in M^k$ we have $X_a = \{x \in M^n\colon \langle a, x\rangle \in X\}$). Then*

(1) The family $\{cl(X_a)\colon a \in M_k\}$ is also uniformly definable.
(2) The sets of all $a \in M_k$ such that M_a is a discrete set, an open set, a closed set, a bounded set, nowhere dense set are definable.

Proof. **Exercise 1.3**. $\square$

Ecercise 1.4. Let $\mathcal{M}$ be an $\aleph_1$-saturated ordered structure.

(1) Show that a sequence $(a_i)_{i\in\mathbb{N}}$ in $\mathcal{M}$ is convergent if and only if it is eventually constant.

(2) Show that M is not topologically connected.

(3) Show that every compact subset of M is finite.

1.3. **Definable Connectedness.**

Definition 1.4. A subset $A \subseteq M^n$ is *definably connected* if there are no definable open $U_1, U_2 \subseteq M^n$ such that $A \cap U_1 \cap U_2 = \varnothing$ and both $A \cap U_1$ and $A \cap U_2$ are nonempty.

Ecercise 1.5.

(1) Show that the image of a definably connected set under a definable continuous map is definably connected.

(2) Let $X_1, X_2 \subseteq M^n$ be definable connected sets with $cl(X_1) \cap X_2 \neq \varnothing$. Show that $X_1 \cup X_2$ is definably connected.

1.4. **O-minimal Structures.**

Definition 1.5. An ordered structure $\mathcal{M}$ is called *o-minimal* if every definable subset $A \subseteq M$ is a finite union of points and intervals.

Proposition 1.6. *If $\mathcal{M} = \langle M, < \rangle$ is a densely ordered set then $\mathcal{M}$ is o-minimal.*

Proof. By quantifier elimination every definable subset $A \subseteq M$ is a Boolean combination of sets $x < a, x = a, a < x$.

Ecercise 1.6. Show that an ordered structure $\mathcal{M}$ is o-minimal if and only if every definable subset $A \subseteq M$ is a Boolean combination of points and intervals.

□

Ecercise 1.7. Let $\mathcal{V} = \langle V, <, +, (\lambda_k)_{k \in K} \rangle$ be an ordered vector space over an ordered field K. Show that $\mathcal{V}$ is o-minimal. (You may use quantifier elimination for $\mathcal{V}$.)

Ecercise 1.8. Show that the ordered field of real numbers $\bar{\mathbb{R}} = \langle \mathbb{R}, <, +, -, \cdot, 0, 1 \rangle$ is o-minimal. (You may use quantifier elimination for $\bar{\mathbb{R}}$.)

Ecercise 1.9. Let $\mathcal{M}$ be an o-minimal structure.

(1) Show that every interval $I \subseteq M$ is definably connected.
(2) Show that M^n is definably connected.
(3) (Intermediate Value Theorem) Let $f, g\colon I \to M$ be definable continuous functions on an open interval I such that for any $x \in I$ we have $f(x) \neq g(x)$. Show that either $f(x) > g(x)$ on I, or $f(x) < g(x)$ on I.
(4) Show that every infinite definable subset of M contains an interval.
(5) Show that if $A \subseteq M$ is a definable subset then the frontier of A $(fr(A) = cl(A) \setminus int(A))$ is finite.
(6) Show that a definable bounded from above $A \subseteq M$ has a least upper bound.
(7) Let $\{X_a\colon a \in M^k\}$ be a uniformly definable family. Show that the set $\{a \in M^k\colon X_a$ is finite $\}$ is definable.
(8) Let $\bar{G} = \langle G, <, \cdot\rangle$ be an ordered group. Assume $\bar{G}$ is o-minimal. Show that $\bar{G}$ has no definable nontrivial proper subgroups and it is abelian.
(9) Let $\bar{R} = \langle R, <, \cdot, +, -, \cdot, 0, 1\rangle$ be an ordered field. Assume $\bar{R}$ is o-minimal. Show that $\bar{R}$ is real closed.

Claim 1.7. *Let $\mathcal{M}$ be an o-minimal structure and $X \subseteq M$ be a definable set. For $a \in M$ exactly one of the following holds*

(1) There is $\varepsilon > a$ such that $(a, \varepsilon) \subseteq X$;
(2) There is $\varepsilon > a$ such that $(a, \varepsilon) \subseteq X^c$.

Proof. **Exercise 1.10** □

2. UNIFORM FINITENESS

➢ We fix an o-minimal structure $\mathcal{M}$.

Theorem 2.1 (Main Theorem)**.** *If* $\mathcal{N} \equiv \mathcal{M}$ *then* $\mathcal{N}$ *is o-minimal.*

Theorem 2.2 (Uniform Finiteness)**.** *Let* $\{X_a : a \in M^k\}$ *be a uniformly definable family of subsets of* M*. Then there is* $k \in \mathbb{N}$ *such that for all* $a \in M^k$ *we have*

$$|X_a| > k \iff X_a \text{ is infinite.}$$

Ecercise 2.1. Show that Theorem 2.1 implies Theorem 2.2 and vice versa.

Ecercise 2.2. Show that there are elementary equivalent structures $\mathcal{A}$ and $\mathcal{B}$ such that:

(a) Every definable subset of A is either finite or co-finite;
(b) The is an infinite and co-infinite definable subset of B.

2.1. **Monotonicity Theorem.**

Our first goal is to show that every definable function $f: M \to M$ is piece-wise continuous and monotone.

We need some technical claims first.

Claim 2.3. *Let $I \subseteq M$ be an open interval and $f : I \to M$ a definable function such that $x < f(x)$ for every $x \in I$. Then there is an open interval $J \subseteq I$ and $c > J$ such that $f(x) > c$ for all $x \in J$.*

Proof. We assume $I = (a, b)$. Let

$$B = \{d \in I : f(x) \leq f(d) \text{ for all } x \in (a, d)\}.$$

Case 1: There is $\varepsilon > a$ such that $(a, \varepsilon) \subseteq B$.
Then f is increasing on (a, ε) and we can take $J = (\alpha, \beta) \subseteq (a, \varepsilon)$ with $a < \alpha < \beta < f(\alpha)$ and $c = f(\alpha)$.
Case 2: not Case 1. Then, by Claim 1.7, there is $\varepsilon > a$ such that $(a, \varepsilon) \subseteq B^c$. Decreasing ε slightly if needed we may assume $\varepsilon \in B^c$.
We take $c = f(\varepsilon)$ and claim that the set $\{x \in (a, \varepsilon) : f(x) > c\}$ is infinite, hence contains an interval.

Indeed, since $\varepsilon \in B^c$, there is $t_0 \in (a, \varepsilon)$ with $f(t_0) > c$. Since $t_0 \in B^c$, there is $t_1 \in (a, t_0)$ with $f(t_1) > f(t_0)$. Continuing we get an infinite sequence $\ldots < t_2 < t_1 < t_0 < \varepsilon$ with $c > f(t_0) > f(t_1) > \ldots$. □

Theorem 2.4. *Let $I \subseteq M$ be an open interval, and $X \subseteq I \times I$ a definable set. Then there is an open interval $J \subseteq I$ such that either $\{\langle x, y\rangle \in J \times J : x < y\} \subseteq X$ or $\{\langle x, y\rangle \in J \times J : x < y\} \subseteq X^c$.*

Proof. For $a \in I$ we will denote by X_a the set $\{x \in M : \langle a, x\rangle \in X\}$.

Let $Y = \{a \in I : (a, \varepsilon) \subseteq X_a$ for some $\varepsilon > a\}$. If Y is finite then, by Claim 1.7, the set $\{a \in I : (a, \varepsilon) \subseteq X_a^c$ for some $\varepsilon > a\}$ is infinite. Replacing X with X^c if needed, we may assume that Y is infinite, hence contains an open interval I'.

For every $a \in I'$ let $f(a) = \sup\{b \in I' : (a, b) \in X_a\}$. It is easy to see that f is a definable function. Since $I' \subseteq Y$, we have $f(a) > a$ for every $a \in I'$. Applying the previous claim we can find an interval $J \subseteq I'$ and $c > J$ such that $f(a) > c$ for all $a \in J$. It is not hard to see that $\{\langle x, y\rangle \in J \times J : x < y\} \subseteq X$. □

Corollary 2.5. *Let $I \subseteq M$ be an open interval, and assume $I \times I \subseteq X_1 \cup \ldots \cup X_r$ for some definable $X_i, i = 1, \ldots, r$. Then there is an open interval $J \subseteq I$ and $k \in \{1, \ldots r\}$ such that $\langle a, b\rangle \in X_k$ for every $a < b \in J$.*

Theorem 2.6 (Monotonicity Theorem)**.** *Let $f : I \to M$ be a definable function on some open interval $I = (a, b)$. Then there are $a = a_0 < a_1 < \ldots < a_n = b$ such that on each (a_i, a_{i+1}) the function f is either constant or strictly monotone and continuous.*

Proof. We first show *monotonicity.* Consider the following definable subset of M:

$$\begin{aligned} A_= &= \{x \in I\colon f \text{ is locally constant in a neighborhood of } x\} \\ A_< &= \{x \in I\colon f \text{ is locally increasing in a neighborhood of } x\} \\ A_> &= \{x \in I\colon f \text{ is locally decreasing in a neighborhood of } x\} \end{aligned}$$

We claim that the set $I \setminus (A_= \cup A_< \cup A_>)$ is finite. If not, then it is infinite and contains an interval J. Consider the sets

$$X_\square = \{\langle x, y\rangle \in J^2\colon f(x) \square f(y)\}$$

where $\square \in \{<, =, >\}$. These sets cover $J \times J$, hence, by Corollary 2.5, there is an open interval $J' \subseteq J$ and $\square \in \{=, <, >\}$ such that for all $x < y \in J'$ we have $f(x)\square f(y)$. It is easy to get a contradiction now.

We leave an **Exercise** to show that a definable function $f\colon I \to M$ locally increasing (decreasing,constant) at every $a \in I$ is increasing (decreasing, constant) on I.

Continuity: Using monotonicity, we may assume that f is either constant or strictly monotone on I. The set $I_0 = \{a \in I\colon f \text{ is continuous at } a\}$ is definable, and we need to show that it is co-finite in I. If not then there would be an interval $J \subseteq I$ such that f is nowhere continuous on J. Using monotonicity we may assume that f is strictly monotone on J. The image of J under f is infinite hence it contains an interval J'. We leave it as an **Exercise** to show that $f^{-1}(J')$ is an interval and f is continuous on it. $\square$

Corollary 2.7. *Let* $f\colon (a,b) \to M$ *be definable. Then for every* $c \in (a,b)$ *both one-sided limits* $\lim_{x\to c^+} f(x)$ *and* $\lim_{x\to c^-} f(x)$ *exist in* $M \cup \{\pm\infty\}$. *Also the limits* $\lim_{x\to a^+} f(x)$ *and* $\lim_{x\to b^-} f(x)$ *exist*

Corollary 2.8. *Let* $f\colon [a,b] \to M$ *be a definable continuous function. Then* f *takes a maximum and minimum values on* $[a,b]$.

2.2. **Uniform Finiteness and Cell Decomposition in M^2.**

➢ In this section for a definable set $A \subseteq M^2$ and $x \in M$ we will denote by A_x the fiber $\{y \in M\colon \langle x, y\rangle \in A\}$.

Our goal is to prove the following uniform finiteness lemma.

Lemma 2.9 (Finiteness Lemma)**.** *Let* $A \subseteq M^2$ *be a definable subset such that for every* $x \in M$ *the set* A_x *is finite. Then there is* $K \in \mathbb{N}$ *such that* $|A_x| < K$ *for all* $x \in M$.

Instead of subsets A as in Lemma 2.9 it is more convenient to consider small sets.

Definition 2.10. A definable set $A \subseteq M^2$ is *small* if the set $\{x \in M \colon A_x \text{ is infinite}\}$ is finite.

Lemma 2.11. *Let $A \subseteq M^2$ be a definable small set. Then there are points $-\infty = a_0 < a_1 < a_2 < \ldots < a_k < a_{k+1} = +\infty$ and natural numbers $k_i \in \mathbb{N}$ such that for every $x \in (a_1, a_{i+1})$ we have $|A_x| = k_i$.*

We will prove Lemma 2.11 by a series of claims.

Let $A \subseteq M^2$ be a small definable set. We say that a point $\langle a, b\rangle \in A$ is *normal in* A if there is an open box $U = I \times J$ in M^2 containing $\langle a, b\rangle$ such that $U \cap A = \Gamma(f)$ for some definable continuous function $f \colon I \to M$. We will denote by $G(A)$ the set of all points normal in A.

Claim 2.12. *Let $A \subseteq M^2$ be a definable small set. If $\pi_1(A)$ is infinite then there is an open interval I and a definable continuous function $f \colon I \to M$ such that $\Gamma(f) \subseteq A$.*

Claim 2.13. *Let $A \subseteq M^2$ be a definable small set, $I \subseteq M$ an open interval and $f \colon I \to M$ a definable continuous function such that $\Gamma(f) \subseteq A$. Then there is $x_0 \in I$ such that $\langle x_0, f(x_0)\rangle$ is normal in A.*

Proof. **Exercise 2.3**. □

Corollary 2.14. *If $A \subseteq M^2$ is a definable small set then $\pi_1(A \setminus G(A))$ is finite.*

Claim 2.15. *If $A \subseteq M^2$ is a definable small set then $\mathrm{cl}(A)$ is also small.*

Claim 2.16. *If $A \subseteq M^2$ is a definable small set then $\pi_1(cl(A) \setminus A)$ is finite.*

Proof. **Exercise 2.4**. □

We say that a definable set $A \subseteq M^2$ is locally bounded at $a \in M$ if there is an open interval I containing a and a bounded open interval J such that $(I \times M) \cap A \subseteq I \times J$.

Claim 2.17. *If $A \subseteq M^2$ is a definable small set then A is locally bounded at all but finitely many $a \in M$.*

Proof. **Exercise 2.5**. □

We now ready to finish the proof of Lemma 2.11. Let $A \subseteq M^2$ be a definable small set. Using above claims we can find $-\infty = a_0 < a_1 < a_2 < \ldots < a_k < a_{k+1} = +\infty$ such that for every $I_i = (a_i, a_{i+1})$ we have:

- The set A is locally bounded at every $x \in I_i$.
- Every point in $(I_i \times M) \cap A$ is normal in A.
- $(I_i \times M) \cap A$ is closed in $I_i \times M$.

Ecercise 2.6. Let $i \in \{0, \ldots, k\}$. Show that for all $x, y \in I_i$ we have $|A_x| = |A_y|$.

It finishes the proof of Lemma 2.11.

In fact we have obtained a description of small sets.

Lemma 2.18. *Let* $A \subseteq M^2$ *be a definable small set. Then there are points* $-\infty = a_0 < a_1 < a_2 < \ldots < a_k < a_{k+1} = +\infty$ *such that the intersection of* A *with each vertical strip* $(a_i, a_{i+1}) \times M$ *has the form* $\Gamma(f_{i,1}) \cup \Gamma(f_{i,2}) \cup \ldots \cup \Gamma(f_{i,k_i})$ *for some definable continuous functions* $f_{i,j} \colon (a_i, a_{i+1}) \to M$ *with* $f_{i,1}(x) < f_{i,2}(x) < \ldots f_{i,k_i}(x)$ *for* $x \in (a_i, a_{i+1})$.

2.3. **Uniform Finiteness and Cell Decomposition.**

Definition 2.19. For every $n \in \mathbb{N}$, we define k-*cells* in M^n by induction on n as follows:

(I) A 0-cell in M is a point; an 1-cell in M is an open interval.

(II) Assume $C \subseteq M^n$ is a definable k-cell.

(a) If $f\colon C \to M$ is a definable continuous function then $\Gamma(f)$ is a k-cell in M^{n+1}.

(b) If $f, g\colon C \to M$ are definable continuous functions with $f(x) < g(x)$ for all $x \in C$ (f, g may be constant functions $-\infty, +\infty$) then the set $\{\langle x, y\rangle \in M^n \times M\colon x \in C, f(x) < y < g(x)\}$ is a $(k+1)$-cell in M^{n+1}.

Ecercise 2.7. (1) Show that if $C \subseteq M^n$ is an n-cell then C is open.

(2) If $C \subseteq M^n$ is a k-cell and $k < n$ then C has an empty interior.

(3) If $X \subseteq M^n$ is a union of finitely many $(n-1)$ cells then $M^n \setminus X$ is dense in M^n and has a nonempty interior.

(4) Every cell is locally closed, i.e. it is open in its closure.

(5) Every cell is homeomorphic under an appropriate projection to an open cell.

(6) Every cell is definably connected.

(7) We say that two cells C_1, C_2 are adjacent if either $C_1 \cap cl(C_2) \neq \varnothing$ or $C_2 \cap cl(C_1) \neq \varnothing$.

Let X be a finite union of the cells $C_1, \ldots, C_k \subseteq M^n$. Show that X is definably connected if and only there is an ordering of the cells such that any two consecutive cells in this ordering are adjacent.

(8) Give an example of a cell $C \subseteq \mathbb{R}^2$ (in the language of real closed fields) such that $C^{-1} = \{\langle y, x\rangle \in \mathbb{R}^2 \colon \langle x, y\rangle \in C\}$ is not a cell.

Definition 2.20. We define *a cell decomposition of* M^n by induction on n.

(I) A cell decomposition of M is a partition of M into finitely many points and open intervals (i.e. a partition of M into finitely many cells).

(II) A cell decomposition of M^{n+1} is a partition of M^{n+1} into finitely many cells $C_i, i = 1, \ldots, m$, such that the set of projections $\{\pi(C_i) \colon i = 1, \ldots, m\}$ is a cell decomposition of M^n.

If $A \subseteq M^n$ is a definable set and $\mathcal{D}$ is a cell-decomposition of M^n then we say that $\mathcal{D}$ is compatible with A if every cell $C \in \mathcal{D}$ is either part of A or is disjoint from A.

The following is the fundamental cell decomposition theorem.

Theorem 2.21 (Cell Decomposition Theorem)**.** *(I) If $A_1, ..., A_k$ are definable subsets of M^n then there is a cell decomposition of M^n compatible with each A_i.*
(II) For each definable function $f\colon A \to M$, $A \subseteq M^n$, there is a cell decomposition of M^n compatible with A such that f is continuous on every cell.

The proof of this theorem is done by induction on n and is quite lengthy. Note that we have proved it for $n = 1$, and the part (I) for $n = 2$ can be derived from Lemma 2.18.

The following claim provides sufficiently many definable continuous functions.

Claim 2.22. *Let $U \subseteq M^n$ be an open set, $I \subseteq M$ an interval, and $f\colon U \times M \to M$ a functions such that for each $\langle u, r\rangle \in U \times M$*

(a) $f(u, \cdot)$ is continuous and monotone on I;
(b) $f(\cdot, r)$ is continuous on U.

Then f is continuous.

Proof. **Exercise 2.8**. □

2.4. **Some Consequences of Cell Decomposition Theorem.** For a definable set $X \subseteq M^n$ a *definably connected component* of X is a maximal definable definably connected subset of X.

Corollary 2.23. *Let $X \subseteq M^n$ be a nonempty definable set. Then X has only finitely many definably connected components. They are open and closed in X and form a partition of X.*

Proof. **Exercise 2.9**. □

In the following statements for a definable subset $X \subseteq M^{k+n}$ and $a \in M^k$ we will denote by X_a the set $\{b \in M^n \colon \langle a, b\rangle \in X\}$.

Proposition 2.24. *Let $\mathcal{D}$ be a cell decomposition of M^{k+n} and $a \in M^k$. Then the collection $\mathcal{D}_a = \{C_a \colon C \in \mathcal{D}\}$ is a cell decomposition of M^n.*

Corollary 2.25. *Let $\{X_a : a \in M^k\}$ be uniformly definable family of subsets of M^n. Then there is $K \in \mathbb{N}$ such that each X_a has at most K definably connected components.*

Proof. **Exercise 2.10**. □

Corollary 2.26. *If $\{X_a \colon a \in M^k\}$ is a uniformly definable family of subsets M^n then there is $K \in \mathbb{N}$ such that $|X_a| > K \iff X_a$ is infinite.*

Proof. **Exercise 2.11**. □

Corollary 2.27. *If $\mathcal{N} \equiv \mathcal{M}$ then $\mathcal{N}$ is o-minimal.*

3. Dimension

➢ We fix an o-minimal structure $\mathcal{M}$.

We are going to define two notions of dimensions for sets definable in $\mathcal{M}$ and show that they coincide.

Example 3.1. Let $X \subseteq \mathbb{C}^n$ be an algebraic variety defined over a countable subfield k. Then $\dim_{\mathbb{C}}(X) = max\{tr.deg(k(a)/k) \colon a \in X\}$.

3.1. Algebraic Dimension.

Recall that if $A \subseteq M$ and $b \in M$ then we say that b *is algebraic over* A if there is a formula $\varphi(x)$ over A such that $\mathcal{M} \models \varphi(b)$ and $\mathcal{M} \models \exists^{<k} x\, \varphi(x)$ for some $k \in \mathbb{N}$. We say that b *is definable over* A if we can choose $\varphi(x)$ as above with $\mathcal{M} \models \exists! x\, \varphi(x)$.

For a set $A \subseteq M$ *the algebraic closure of* A is the set

$$acl(A) = \{b \in M \colon b \text{ is algebraic over } A\},$$

and *the definable closure of* A is the set

$$dcl(A) = \{b \in M \colon b \text{ is definable over } A\}.$$

Ecercise 3.1. Show that in the field $\mathbb{C}$ we have $\sqrt{2} \in acl(\mathbb{Q})$, but $\sqrt{2} \notin dcl(\mathbb{Q})$.

Ecercise 3.2. Show that $b \in acl(A) \iff b \in dcl(A)$, and $b \in dcl(A)$ if and only if there is a partial function $f(\bar{x})$ definable over $\varnothing$ and $\bar{a} \in A$ such that $b = f(\bar{a})$.

In order to develop acl-dimension we need Exchange Lemma.

Lemma 3.2 (Exchange Lemma). *If $A \subseteq M$ and $b, c \in M$ with $b \in acl(Ac) \setminus acl(A)$ then $c \in acl(Ab)$.*

For a set $A \subseteq M$ and $I \subseteq M$ we say that *I is independent over A* if for all $x \in I$ we have $x \notin acl\big(A \cup (I \setminus \{x\})\big)$.

Definition 3.3. For a set $A \subseteq M$ and a tuple $\bar{a} \in M^n$ *the acl-dimension of $\bar{a}$ over A*, $a\text{-}\dim(\bar{a}/A)$, is the least cardinality of a subtuple $\bar{a}'$ of $\bar{a}$ such that $\bar{a} \subseteq acl(A\bar{a}')$.

Ecercise 3.3.
(1) $a\text{-}\dim(\bar{a}/A)$ is the cardinality of any maximally independent over A subtuple $\bar{a}'$ of $\bar{a}$.
(2) If $A \subseteq B$ then $a\text{-}\dim(\bar{a}/A) \geq a\text{-}\dim(\bar{a}/B)$.
(3) (Additivity) $a\text{-}\dim(\bar{a}\bar{b}/A) = a\text{-}\dim(\bar{a}/A\bar{b}) + a\text{-}\dim(\bar{b}/A)$.

In order to define correctly acl-dimension of a definable set we need to work in a saturated enough structure. So we also fix a κ-saturated elementary extension $\widetilde{\mathcal{M}}$ of $\mathcal{M}$, where $\kappa > |M|$. For a definable set $X \subseteq M^n$ we will denote by $\widetilde{X}$ the subset of $\widetilde{M}^n$ defined in $\widetilde{\mathcal{M}}$ be the same formula that defines X in $\mathcal{M}$.

Definition 3.4. (1) Let $X \subseteq \widetilde{M}^n$ be a set defined over $A \subseteq \widetilde{M}$ with $|A| < \kappa$. We define *the acl-dimension* of X to be $a\text{-}\dim(X) = max\{a\text{-}\dim(b/A) \colon b \in X\}$.
(2) For a definable set $X \subseteq M^n$ defined over a set $A \subseteq M$ we define $a\text{-}\dim(X) = a\text{-}\dim(\widetilde{X})$.

Ecercise 3.4. (1) Show that acl-dimension of a set does not depend on the choice of A, i.e. if X is also defined over some $A' \subseteq \widetilde{M}$ with $|A'| < \kappa$ then $a\text{-}\dim(X) = max\{a\text{-}\dim(b/A') \colon b \in X\}$.
(2) Show that acl-dimension of a definable set $X \subseteq M^n$ does not depend on the choice of $\widetilde{\mathcal{M}}$.

Ecercise 3.5.

(1) Let $X, Y \subseteq M^n$ be definable sets.
 (a) Show that $a\text{-}\dim(X \cup Y) = max(a\text{-}\dim(X), a\text{-}\dim(Y))$.
 (b) Show that $X \subseteq Y$ implies $a\text{-}\dim(X) \leq a\text{-}\dim(Y)$.
 (c) Show that $a\text{-}\dim(X \times Y) = a\text{-}\dim(X) + a\text{-}\dim(Y)$.
(2) Let $f \colon X \to M^k$ be a definable map. Show that $a\text{-}\dim(f(X)) \leq a\text{-}\dim(X)$, with equality if f is injective.
(3) Let $C \subseteq M^n$ be a k-cell. Show that $a\text{-}\dim(C) = k$.

3.2. **Geometric Dimension.**

Definition 3.5. For a definable set $X \subseteq M^n$ we define *the dimension of* X to be the largest d such that X contains a d-cell.

Theorem 3.6. *For a definable* $X \subseteq M^n$ *and* $d \in \mathbb{N}$ *the following conditions are equivalent.*

(1) $\dim(X) = d$*.*
(2) $a\text{-}\dim(X) = d$*.*
(3) d *is the largest integer such that* $\pi(X)$ *has a non-empty interior for some coordinate projection* $\pi\colon M^n \to M^d$*.*
(4) d *is the largest integer such that* $f(X)$ *has a non-empty interior for some definable* $f\colon X \to M^d$*.*

Proof. **Exercise 3.6**. □

Corollary 3.7. *Let* $A \subseteq M^n$ *be a definable set of dimension and* $f\colon A \to M^k$ *be a definable map. Then there is a definable set* $U \subseteq A$ *such that* f *is continuous on* U *and* $\dim(A \setminus U) < \dim(A)$*.*

Corollary 3.8. *Let* X, Y *be definable sets. Then* $\dim(X \times Y) = \dim(X) + \dim(Y)$*.*

Claim 3.9. *If* $X \subseteq M^n$ *is a definable set then* $\dim(cl(X) \setminus X) < \dim(X)$*.*

3.2.1. *Definability of dimension.*

Claim 3.10. *Let* $\{A_a\colon a \in M^k\}$ *be a uniformly definable family of subsets of* M^n*. Then for every* $\mathrm{d} \in \mathbb{N}$ *the set* $\{a \in M^k\colon \dim(A_a) = d\}$ *is definable.*

Proof. **Exercise 3.7**. □

4. Definable Choice

➢ In this we fix an o-minimal expansion of an ordered group $\mathcal{M} = \langle M, <, +, 0, \ldots \rangle$.

Theorem 4.1 (Definable Choice). *Let* $\{X_a \colon : a \in M^k\}$ *be a uniformly definable family of subsets of* M^n. *Then there is a definable function* $f \colon M^k \to M^n$ *such that* $f(a) \in X_a$ *for every non-empty* X_a, *and* $X_a = X_b$ *implies* $f(a) = f(b)$.

Corollary 4.2. *Let* $E \subseteq M^{2n}$ *be a definable equivalence relation on* M^n. *Then there is a definable function* $f \colon M^n \to M^k$ *such that* $aEb \iff f(a) = f(b)$.

Corollary 4.3 (Curve Selection). *Let* $X \subseteq M^n$ *be a definable set and* $a \in cl(X)$. *Then there is a definable map* $\sigma \colon (0, \varepsilon) \to X$ *such that* $\lim_{t \to 0^-} \sigma(t) = a$.

Corollary 4.4. *Let* $A \subseteq M$ *be a nonempty set different from* $\{0\}$. *Then* $dcl(A)$ *is the universe of an elementary substructure of* $\mathcal{M}$.

Proof. Follows from Tarski-Vaught Test and Definable Choice. □

Ecercise 4.1. Let $B \subseteq M^n$ be a definable bounded closed set and $f \colon \ \to M$ a definable continuous function. Show that f takes maximum and minimum values on B.

5. SMOOTHNESS

➢ In this section we work in o-minimal expansion of a real closed field $\mathcal{R} = \langle R, <, +, \cdot, 0, 1, \ldots \rangle$.

Definition 5.1. Let $I \subseteq R$ be an open interval. A definable function $f\colon I \to R$ is *differentiable at $a \in I$ with the derivative d* if

$$\lim_{t \to 0} \frac{f(a+t) - f(a)}{t} = d.$$

As usual we write $f'(a) = d$.

It is easy to see that if $f\colon I \to R$ is a definable function then the set $\{x \in I\colon f \text{ is differentiable at x}\}$ is definable and the function $x \mapsto f'(x)$ is definable on this set.

Theorem 5.2. *Let I be an open interval and $f\colon I \to R$ be a definable function. Then f is differentiable at all but finitely many points.*

Proof. For $x \in I$ let

$$f'(x^+) = \lim_{t \to 0^+} \frac{f(x+t) - f(x)}{t} \quad \text{and} \quad f'(x^-) = \lim_{t \to 0^-} \frac{f(x+t) - f(x)}{t}.$$

By o-minimality both these limits exist in $R \cup \{\pm\infty\}$, and f is differentiable at x if and only if $f'(x^+) = f'(x^-) \in R$.

Step 1. The set $\{x \in I \colon f'(x^+) \neq f'(x^-)\}$ is finite.
Assume not. Then there is an open interval $J \subseteq I$ such that $f'(x^+) \neq f'(x^-)$ at any $x \in J$. Decreasing J if needed we may assume that both $f'(x^+)$ and $f'(x^-)$ are continuous on J. Then either $f'(x^+) > f'(x^-)$ on J or $f'(x^+) < f'(x^-)$. We assume $f'(x^+) > f'(x^-)$. Then there is $c \in R$ and an open interval $J' \subseteq J$ such that $f'(x^+) > c > f'(x^-)$ on J'. Let $J'' \subseteq J'$ be an open interval such that the function $F(x) = f(x) - cx$ is continuous and strictly monotone on J''. It is easy to see that $F'(x^+) > 0$ and F is increasing on J'', and also $F'(x^-) < 0$ and F is decreasing on J''. A contradiction.
Step 2. The set $\{x \colon f'(x^+) \in \{\pm\infty\}\}$ is finite.
Assume that $f'(x^+) = +\infty$ at infinitely many x. Then we can find $a, b \in I$ such that f is continuous on $[a, b]$ and $f'(x^+) = f'(x^-) = +\infty$ on (a, b).

Let $h(x) = \lambda x + c$ be an affine function such that $h(a) = f(a)$ and $h(b) = f(b)$. Consider the function $F(x) = f(x) - h(x)$. It is easy to see that $F'(x^+) = F'(x^-) = +\infty$. Since F is continuous on $[a, b]$ and $F(a) = F(b) = 0$, F attends a maximum or minimum value at some $c \in (a, b)$. If F has maximum at c then $F'(c^+) \leq 0$, a contradiction. If F has minimum at c then $F'(c^-) \leq 0$, a contradiction. □

Corollary 5.3. *Let $a < b \in R$ and $f\colon (a,b) \to R$ be a definable function. Then for every $r \in \mathbb{N}$ there are $a = a_0 < a_1 < \ldots < a_k = b$ such that f is C^r on each (a_i, a_{i+1}).*

Ecercise 5.1.

(1)(Mean Value Theorem) Assume $a < b \in R$, $f\colon [a,b]$ is a definable function continuous on $[a,b]$ and differentiable on (a,b). Then there is $c \in (a,b)$ such that $f(b) - f(a) = f'(c)(b-a)$.
(2) Assume $f\colon (a,b) \to R$ is a definable function differentiable on $(a.b)$. If $f'(x) = 0$ on (a,b) then f is constant on (a,b).

Definition 5.4. Let $U \subseteq R^n$ be a definable open set and $f = (f_1, \ldots, f_k)\colon U \to R^k$ a definable map. For $r \geq 1$ we say that $f = (f_1, \ldots, f_k)\colon U \to R^k$ is a C^r*-map* if all the partial derivatives $\frac{\partial f_i}{\partial x_j}$ are C^{r-1}-functions on U.

Ecercise 5.2. Let $U \subseteq R^n$ be a definable open set and $f\colon U \to R^k$ be a definable continuous map. Then for every $r \geq 1$ there is a definable open $V_r \subseteq U$ such f is C^r on V_r and $\dim(U \setminus V_r) < n$.

Definition 5.5. Let $U \subseteq R^n$ be a definable open set and $f = (f_1, \ldots, f_)k\colon U \to R^k$ be a definable C^1-map. For $a \in U$ the $k \times n$ matrix of partial derivatives $\left(\frac{\partial f_j}{\partial x_j}(a)\right)$ is called *the Jacobian matrix of f at a* and is denoted by $J_f(a)$.
The linear map $x \mapsto J_f(a)x$ is called *the differential of f at a* and is denoted $d_a(f)$.

Theorem 5.6 (Inverse Function Theorem). *Let $U \subseteq R^n$ be a definable open set, $f\colon U \to R^n$ a definable C^r map, and $a \in U$. If $d_a(f)$ is invertible then there are definable open neighborhoods $U' \subseteq U$ of a and V of $f(a)$ such that f maps U' homeomorphically onto V and f^{-1} is also C^r.*

Theorem 5.7 (Implicit Function Theorem). *Let $U \subseteq R^{k+n}$ be a definable open set and $F = (F_1, \dots, F_n)\colon U \to R^n$ a definable C^r-map. Let $\langle x_0, y_0 \rangle$ be in U such that $F(x_0, y_0) = 0$ and the $n \times n$ matrix*

$$\left(\frac{\partial F_i}{\partial y_j}(x_0, y_0) \right)_{\substack{1 \le i \le n \\ 1 \le j \le n}}$$

is invertible. Then there are open definable neighborhoods V of x_0 in R^k and W of y_0 in R^n, and there is a definable C^r map $\varphi\colon V \to W$ such that $V \times W \subseteq U$ and for all $\langle x, y \rangle \in V \times W$ we have

$$F(x, y) = 0 \iff y = \varphi(x).$$

Proof. Apply Inverse Function Theorem to the map $\langle x, y \rangle \mapsto \langle x, F(x, y) \rangle$. □

5.1. **Smooth Cell Decomposition.**

Definition 5.8. Let $A \subseteq R^n$ be a definable set and $f\colon A \to R^m$ a definable map. We say that f *is* C^r *on* A if there is an open $U \subseteq R^n$ and a definable C^r-map $F\colon U \to R^m$ extending f.

Definition 5.9. A cell $C \subseteq R^n$ is a C^r*-cell* if all functions used in forming C are C^r.

Theorem 5.10 (Smooth Cell Decomposition)**.** *Let* $r \geq 1$.
(1) For any definable $A_1, \dots, A_k \subseteq R^n$ *there is a* C^r*-cell decomposition of* R^n *compatible with each* A_i.
(2) For any definable function $f\colon A \to R$, $A \subseteq R^n$ *there is a* C^r *cell decomposition of* R^n *compatible with* A *such that* $f \restriction C$ *is* C^r *on each cell* $C \subseteq A$

The proof of Smooth Cell Decomposition is based on the following claim.

Claim 5.11. *Let* $C \subseteq R^n$ *be a* k*-cell,* $f\colon C \to R$ *a definable function, and* $r \in \mathbb{N}$. *Then there is a definable subset* $C' \subseteq C$ *such then* $\dim(C \setminus C') < k$ *and* $f \restriction C'$ *is* C^r.

5.2. **Definable Triangulation.**

We say that $a_0, \dots, a_d \in R^n$ are *affine independent* if the vectors $a_1 - a_0, \dots, a_d - a_0$ are linearly independent.

For $a_0, \dots, a_d \in R^n$ let $(a_0, \dots, a_d) = \{\sum t_i a_i \colon t_i > 0, \sum t_i = 1\} \subseteq R^n$.

Ecercise 5.3. Show that $a_0, \dots, a_d \in R^n$ are affine independent if and only if $\dim((a_0, \dots, a_d)) = d$.

If $a_0, \dots, a_d \in R^n$ are affine independent then $(a_0, \dots, a_d)$ is called *a d-simplex in R^n spanned by* $a_0, \dots, a_d$.

The closure of $(a_0, \dots, a_d)$ is denoted by $[a_0, \dots, a_d]$. It is easy to see that

$$[a_0, \dots, a_d] = \left\{\sum t_i a_i \colon t_i \geq 0, \sum t_i = 1\right\} \subseteq R^n$$

We call $a_0, \dots, a_d$ the *vertices* of $(a_0, \dots, a_d)$ (and $[a_0, \dots, a_d]$).

A face of a simplex $(a_0, \dots, a_d)$ is a simplex spanned by a non-empty subset of $\{a_0, \dots, a_d\}$.

For simplexes σ and τ we write $\tau < \sigma$ is τ is a proper dace of σ.

Definition 5.12. *A complex* in R^n is a finite collection K of simplexes in R^n such that for $\sigma_1, \sigma_2 \in K$ either $cl(\sigma_1) \cap cl(\sigma_2) = \varnothing$ or $cl(\sigma_1) \cap cl(\sigma_2) = cl(\tau)$ for some common face τ of σ_1 and σ_2. (τ is not required to be in K !).

For a simplex K in R^n, *the polyhedron spanned by* K is
$|K| =$ union of all simplexes in K, and *the set of vertices of* K is
$Vert(K) =$ the set of all vertices of the simplexes in K.

Theorem 5.13 (Triangulation Theorem)**.** *Let $S_1, \dots, S_k \subseteq R^n$ be definable sets. Then there is a complex K is R^n and a homeomorphism $\Phi \colon R^n \to |K|$ such that $\Phi(S_i)$ is a union of simplexes in K.*

For $N \in \mathbb{N}$ let K_N be the complex consisting of the simplex $(e_1, \dots, e_N)$ and all its faces, where $e_1, \dots, e_N$ is the standard basis of R^N.

Claim 5.14. *For every definable set $A \subseteq R^n$ there is $N \in \mathbb{N}$ and a subcomplex K of K_N such that A is definably homeomorphic to $|K|$.*

Proof. Let L be a complex in R^n such that A is definably homeomorphic to $|L|$. Let $V = \{v_1, \dots, v_N\}$ be the set of vertices of L.

Let $F \colon V \to R^N$ be the map $v_i \mapsto e_i$, and
$K = \{(F(v_{i_1}), \dots, F(v_{i_s})) \colon (v_{i_1}), \dots, v_{i_s}) \in L\}$.

Ecercise 5.4. K is a subcomplex of K_N and F extends to a homeomorphism from $|L|$ onto $|K|$.

□

Corollary 5.15. *Up-to a definable homeomorphism there are at most countably many definable sets.*

Theorem 5.16. *Let $\{S_a\colon a \in R^k\}$ be a uniformly definable family of subsets of R^n. Then there is $N \in \mathbb{N}$ and a partial definable map $f\colon R^k \times R^n \to R^N$ such that for each $a \in R^k$ the map $f_a\colon x \to f(a,x)$ is a homeomorphism from S_a onto a union of faces of K_N.*

Proof. The type

$$\Sigma(x) = \{x \in R^k\} \bigcup_{N \in \mathbb{N}} \{\neg\exists\, z \big(\varphi(u,v,z) \text{ defines a graph of a homeomorphism from } S_x \text{ onto a union of faces of } K_N\big)\colon \varphi(u,v,z) \text{ is an } \mathcal{L}\text{-formula.}\}$$

is inconsistent. Hence we can partition R^k into finitely many definable sets A_i such that for each A_i there is $N_i \in \mathbb{N}$ and a formula $\varphi_i(u, v_i, z_i)$ such that for $a \in A_i$, $\varphi_i(u, v_i, b_a)$ defines a homeomorphism from S_a into a union of faces of K_{N_i}, for some b_a.

It is not hard to see that we can put all A_i together and assume one φ works for all R^k. Now we use definable choice. □

5.3. **Definable Trivialization.**

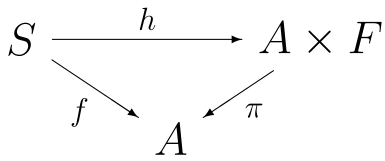
Definition 5.17. Let $f\colon S \to A$ be a definable map. We say that f *is trivial* if there is a definable set F and a definable homeomorphism $h\colon S \to A \times F$ such that the following diagram is commutative

$$\begin{array}{ccc} S & \xrightarrow{\quad h \quad} & A \times F \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & A & \end{array}$$

We say that f *is trivial* over a definable set $B \subseteq A$ if for $S_B = f^{-1}(B)$ the map $f \restriction S_B\colon S_B \to B$ is trivial.

Theorem 5.18 (Definable Trivialization)**.** *For a definable continuous map $f\colon S \to A$ there is a definable partition of $A = A_1 \cup \ldots \cup A_l$ such that f is trivial over each A_i.*

Proof. Using Theorem 5.16, after partitioning A if needed, we can assume that there is a definable set F and a definable bijection $h\colon S \to A \times F$ such that the following diagram is commutative



and for each $a \in A$, h maps $f^{-1}(a)$ homeomorphically onto $\{a\} \times F$. $\square$

Claim 5.19. *Let $S \subseteq A \times R^n$ be a definable set such that for each $x \in A$ the fiber $S_x = \{y \in R^n \colon \langle x, y\rangle \in S\}$ is closed in R^n. Then there is a partition of A into finitely many set A_i such that $S \cap (A_i \times R^n)$ is closed in $A_i \times R^n$.*

Proof. We do it by induction on $\dim(A)$. If $\dim(A) = 0$ then A is a finite set and S is closed.

Assume $\dim(A) > 0$. Let $A' = \pi(cl(S) \setminus S)$, where $\pi\colon S \to A$ is a projection.

For $A_0 = A \setminus A'$ we have that $S \cap (A_0 \times R^n)$ is closed in $A_0 \times R^n$. Thus, by the induction hypothesis, it is sufficient to show that $\dim(A') < \dim(A)$.

Assume not, i.e. $\dim(A') = \dim(A)$. Using definable choice we can find a definable function $\alpha\colon A' \to R^n$ such that $\alpha(a) \in R^n \setminus S_a$ and $\langle a, \alpha(a)\rangle \in cl(S)$ for all $a \in A'$.

Since each S_a is closed in R^n, using definable choice, we can also find a definable function $\gamma\colon A' \to R$ such that for all $a \in A'$ we have $B_{\gamma(a)}(\alpha(a)) \cap S_a = \varnothing$, where $B_{\gamma(a)}(\alpha(a))$ is an open ball in R^n of radius $\gamma(a)$ centered at a.

Let $A'' \subseteq A'$ be a definable set with $\dim(A' \setminus A'') < \dim(A')$ such that both α and γ are continuous on A''. Since $\dim(A'') = \dim(A)$, A'' contains a definable open in A set U. The set $\{\langle x, y\rangle \colon x \in U, y \in B_{\gamma(x)}(\alpha(x))\}$ is open in $A \times R^n$, disjoint from S and also contains points $\langle x, \alpha(x)\rangle$ in the closure of S. A contradiction. $\square$

6. Some o-minimal structure over the reals

The following structure are o-minimal:

(1) $\bar{\mathbb{R}} = \langle \mathbb{R}, <, +, -, \cdot, 0, 1\rangle$ -the field of real numbers;
(2) $\mathbb{R}_{an}$ - the field of real numbers expanded by all restricted analytic functions;
(3) $\mathbb{R}_{an,exp}$ - the expansion of $\mathbb{R}_{an}$ by the function $x \mapsto e^x$.

6.1. **The structure $\mathbb{R}_{an}$ and subanalytic sets.** Let A be a real analytic manifold of dimension n and $X \subseteq A$. Then the X is *subanalytic in A* if for every point $a \in A$ there is an open neighborhood U of a in A and an analytic bijection $f\colon U \to V$, where V is an open subset of $\mathbb{R}^n$ such that $f(X \cap U)$ is definable in $\mathbb{R}_{an}$.

Example 6.1.

(a) The set $\{\langle x, \sin(x)\rangle \colon x \in \mathbb{R}\}$ is a subanalytic subset of $\mathbb{R}^2$, but it is not definable in $\mathbb{R}_{an}$.
(b) The set $\{\langle x, \sin(1/x)\rangle \colon x \in \mathbb{R}^*\}$ is not subanalytic in $\mathbb{R}^2$, but it is subanalytic in $\mathbb{R}^* \times \mathbb{R}$.

Claim 6.2. *The set $X \subseteq \mathbb{R}^n$ is definable in $\mathbb{R}_{an}$ if and only if the set $\Pi(X)$ is subanalytic in $\mathbb{R}^n$, where $\Pi(x)\colon \mathbb{R}^n \to \mathbb{R}^n$ is the map*

$$\langle x_1, \ldots, x_n\rangle \mapsto \left\langle \frac{x_1}{\sqrt{1+x_1^2}}, \ldots, \frac{x_n}{\sqrt{1+x_n^2}} \right\rangle.$$

Proof. **Exercise 6.1**. □

6.2. **Growth Dichotomy.** There is a fundamental difference between structures $\mathbb{R}_{an}$ and $\mathbb{R}_{exp}$.

Definition 6.3. Let $\mathcal{R} = \langle R, <, +, \cdot, \ldots \rangle$ be an o-minimal expansion of a real closed field. We say that the structure $\mathcal{R}$ is *polynomially bounded* if for every definable function function $f\colon [c, +\infty) \to R$ there is $N \in \mathbb{N}$ such that $|f(x)| < x^N$ for all sufficiently large positive x.

Example 6.4. The structure $\mathbb{R}_{exp}$ **IS NOT** polynomially bounded.

Fact 6.5. *The structures* $\mathbb{R}_{an}$ **IS** *polynomially bounded.*

Theorem 6.6 (Growth Dichotomy)**.** *Let* $\mathcal{R} = \langle \mathbb{R}, <, +, \ldots \rangle$ *be an o-minimal expansion of the field of reals. If* $\mathcal{R}$ *is not polynomially bounded then the function* $x \mapsto e^x$ *is definable in* $\mathcal{R}$*.*

The proof uses computations in the Hardy field $\mathcal{H}_{\mathcal{R}}$ of germs at $+\infty$ of $\mathcal{R}$-definable functions.

6.3. **Field of Germs at 0^+.**

➢ We fix an o-minimal extension $\mathcal{R} = \langle R, <, +, -, \cdot, \dots, \rangle$ of a real closed field.

Let $\mathcal{R}'$ be a saturated enough elementary extension of $\mathcal{R}$, and $\tau \in R'$ a positive $\mathcal{R}$-infinitesimal element, i.e. $0 < \tau < r$ for all $r > 0 \in R$. Let $R_\tau = dcl(R \cup \{\tau\})$. Then, by Corollary 4.4, R_τ is the universe of an elementary substructure $\mathcal{R}_\tau$ of $\mathcal{R}'$, and it is an elementary extension of $\mathcal{R}$.

Notice, that for every element $a \in R_\tau$ there is an $\mathcal{R}$-definable function $\alpha\colon R_\tau \to R_\tau$ such that $a = \alpha(\tau)$, and for any formula $\varphi(x)$ we have
$\mathcal{R}_\tau \models \varphi(a)$ if and only if $\mathcal{R} \models \varphi(\alpha(t))$ for all small enough $t > 0$.

Ecercise 6.2. Let $X \subseteq R_\tau^n$ be an $\mathcal{R}_\tau$-definable set. Then its $\mathcal{R}$-trace $X \cap R^n$ is an $\mathcal{R}$-definable subset of R^n.

7. Tame extensions

7.1. **Tame extensions.**

Definition 7.1. A proper elementary extension $\widetilde{\mathcal{R}} \succ \mathcal{R}$ is called *tame* if for every $\gamma \in \widetilde{R}$ the set $\{x \in R\colon x < \gamma\}$ is $\mathcal{R}$-definable.

Example 7.2. 1. The extension $\mathcal{R}_\tau$ of $\mathcal{R}$ is tame.
2. The field of real numbers $\bar{\mathbb{R}}$ is not a tame extension of the field of algebraic real numbers.

Ecercise 7.1. If $\mathcal{R}$ is an o-minimal expansion of the field $\bar{\mathbb{R}}$ then every proper elementary extension $\widetilde{\mathcal{R}}$ of $\mathcal{R}$ is tame.

Theorem 7.3 (Definability of Types)**.** *Assume $\widetilde{\mathcal{R}} \succ \mathcal{R}$ is a tame extension. If $X \subseteq (\widetilde{R})^n$ is an $\widetilde{\mathcal{R}}$-definable set then the set $X \cap R^n$ is $\mathcal{R}$-definable subset of R^n.*

7.1.1. *Standard Part Map.* Let $\widetilde{\mathcal{R}} \succ \mathcal{R}$ be a tame extension, and $\gamma \in \widetilde{R}$. The set $A = \{r \in R\colon r < \gamma\}$ is definable in $\mathcal{R}$. Let $r = \sup_{\mathcal{R}}(A)$. We call r *the standard part of* γ and denote by $st(\gamma)$. Thus $st\colon \widetilde{R} \to R \cup \{\pm\infty\}$.

Ecercise 7.2. If $\mathrm{st}(\gamma) \in R$ then $st(\gamma)$ is unique element $r \in R$ such that $|\gamma - r| < \delta$ for all $0 < \delta \in R$.

Ecercise 7.3. Let $\alpha\colon R \to R$ be an $\mathcal{R}$-definable function. Consider the elementary extension $\mathcal{R}_\tau$ as above. Let $a = \alpha(\tau)$. Show that $st(a) = \lim_{t \to 0^+} \alpha(t)$,

Ecercise 7.4. Let $\widetilde{\mathcal{R}} \succ \mathcal{R}$ be a tame extension, and $X \subseteq (\widetilde{R})^n$ an $\widetilde{\mathcal{R}}$-definable R-bounded set. Then the set $st(X) = \{st(x)\colon x \in X\}$ is $\mathcal{R}$-definable subset of R^n.

8. HAUSDORFF LIMITS

For an element $x \in \mathbb{R}^n$ and a subset $Y \subseteq \mathbb{R}^n$ we put
$d(x,Y) = \inf\{d(x,y) \colon y \in Y\}$.

We will denote by $\mathcal{K}(\mathbb{R}^n)$ the collection of all compact subsets of $\mathbb{R}^n$.
The Hausdorff distance on $\mathcal{K}(\mathbb{R}^n)$ is defined as

$$d_H(X,Y) = sup\{d(x,Y), d(y,X) \colon x \in X, y \in Y\}.$$

Ecercise 8.1. Show that d_H is a metric on $\mathcal{K}(\mathbb{R}^n)$.

For a family $\mathcal{C} \subseteq \mathcal{K}(\mathbb{R}^n)$ we will denote by $cl_H(\mathcal{C})$ the topological closure of $\mathcal{C}$ in $\mathcal{K}(\mathbb{R}^n)$ with respect to the topology induced by d_H.

Theorem 8.1. *Let $\mathcal{R} = \langle \mathbb{R}, <, +, -, \cdot, \dots, \rangle$ be an o-minimal expansion of the field of real numbers. Let $\mathcal{C} = \{X_a \colon a \in \mathbb{R}^m\}$ be a uniformly definable family of compact subsets of $\mathbb{R}^n$ and $Y \in \mathcal{K}(\mathbb{R}^n)$. If $Y \in cl_H(\mathcal{C})$ then Y is definable.*

Proof. Let $X \subseteq \mathbb{R}^{m+n}$ be a definable set such that for $a \in R^m$ we have
$X_a = \{x \in \mathbb{R}^n \colon \langle a, x \rangle \in X\}$.

Let $\widetilde{\mathcal{R}}$ be an $\aleph_1$-saturated elementary extension of $\mathcal{R}$. Notice that by Exercise 7.1 $\widetilde{\mathcal{R}}$ is a tame extension of $\mathcal{R}$. We will denote by $\widetilde{X}$ the subset of $\widetilde{R}^{m+n}$ defined by the same formula as X in $\mathbb{R}^{m+n}$.

By Exercise 7.4, the Theorem will follow from the following claim.

Claim 8.2. *For a set $Y \in \mathcal{K}(\mathbb{R}^n)$ we have $Y \in cl_H(\mathcal{C})$ if and only if $Y = st(\widetilde{X}_\alpha)$ for some $\alpha \in \widetilde{R}^m$.*

Proof. Let $Y \in \mathcal{K}(\mathbb{R}^n)$. Assume $Y \in cl_H(\mathcal{C})$. Since Y is compact, for each $k > 0 \in \mathbb{N}$ we pick finite subsets $Y_k \subseteq Y$ such that $d_H(Y, Y_k) < \frac{1}{k}$, and $Y_k \subseteq Y_{k+1}$. For each $k > 0 \in \mathbb{N}$ let $\mathcal{C}_k = \{a \in \mathbb{R}^m \colon d_H(Y_k, X_a) < \frac{2}{k}\}$.

Ecercise 8.2. Every $\mathcal{C}_k$ is definable, non-empty, and $\mathcal{C}_{k+1} \subseteq \mathcal{C}_k$.

Since $\widetilde{\mathcal{R}}$ is $\aleph_1$-saturated, there is $\alpha \in \widetilde{R}^m$ such that $\alpha \in \cap\widetilde{\mathcal{C}}_k$.

Ecercise 8.3. Show that $Y = st(\widetilde{X}_\alpha)$.

Ecercise 8.4. Let $\beta \in \widetilde{R}^m$ be such that $Z = st(\widetilde{X}_\beta)$ is compact. Show that $Z \in cl_H(\mathcal{C})$.

□

Theorem 8.3. *Let $\mathcal{R} = \langle \mathbb{R}, <, +, -, \cdot, \dots, \rangle$ be an o-minimal expansion of the field of real numbers. Let $\mathcal{C}$ be a uniformly definable family of compact subsets of $\mathbb{R}^n$. Then the family*

$$\{Y \in \mathcal{K}(\mathbb{R}^n) \colon Y \in cl_H(\mathcal{C})\}$$

is uniformly definable as well.

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