The Fields Institute
Free Probability and Random Matrices

Problem Set 4, Due November 22, 2007

Let s_1 and s_2 be free and semi-circular and let $c = (s_1 + is_2)/\sqrt{2}$. We call c a *circular operator*. Recall that $\varphi(cc^*) = 1$ and all free cumulants of c and c^* are 0 with the exception of $\kappa_2(c^*, c) = \kappa_2(c, c^*) = 1$.

1) Show that the norm of c is 2.

In the next few questions we will calculate the spectral radius of c. By the formula for free cumulants with products as arguments one can show that for all positive integers m and n

$$\kappa_m(c^n c^{*n}, \ldots, c^n c^{*n}) = \varphi((c^{n-1} c^{*(n-1)})^m) \qquad (*)$$

2) Verify equation $(*)$ directly for $n = 1$ and $n = 2$.

3) Denote by

$$M_n(z) := \sum_{m=0}^{\infty} \varphi((c^n c^{*n})^m) z^n$$

the moment generating series of $c^n c^{*n}$. Using equation $(*)$ (for general m) and the moment-cumulant formulas, show that M_n satisfies the equation

$$M_n(z) = 1 + zM_n(z)^{n+1}.$$

4) Show that the solution of the functional equation for M_n in question 3 is given by

$$M_n(z) = \sum_{m=0}^{\infty} C_n^{(m)} z^n,$$

where $C_n^{(m)}$ are the Fuss-Catalan numbers

$$C_n^{(m)} = \frac{1}{nm+1} \binom{m(n+1)}{m}.$$

5) Calculate the norm of c^n via

$$\|c^n\| = \lim_{m \to \infty} \sqrt[2m]{\varphi((c^n c^{*n})^m)}.$$

6) Calculate the spectral radius $r(c)$ of c by

$$r(c) = \lim_{n \to \infty} \sqrt[n]{\|c^n\|}$$