FREE PROBABILITY AND RANDOM MATRICES

Lecture 6: Applications to Free Group Factors, October 18, 2007

Let G be a countable discrete group, $\ell^2(G)$ the Hilbert space where elements of G, denoted ξ_g, form an orthonormal basis, and $\lambda : G \to B(\ell^2(G))$ is the left regular representation: $\lambda_g(\xi_h) = \xi_{gh}$. $\mathcal{L}(G)$ is the closure in the weak operator topology of $\{\sum_{i=1}^n \alpha_i \lambda_{g_i}\}$; $\mathcal{L}(G)$ is the group von Neumann algebra of G. For $x \in \mathcal{L}(G)$, $x \mapsto \langle x\xi_e, \xi_e\rangle$ is a faithful normal trace on $\mathcal{L}(G)$; it gives the same state considered in Lecture 2, namely

$$\langle \lambda_g \xi_e, \xi_e\rangle = \langle \xi_g, \xi_e\rangle = \begin{cases} 1 & g = e \\ 0 & g \neq e \end{cases}$$

Thus $\mathcal{L}(G)$ is always a finite von Neumann algebra. If G is infinite then $\mathcal{L}(G)$ is a von Neumann algebra of type II_1. If every non-trivial conjugacy class $\{ghg^{-1} \mid g \in G\}$ $(h \neq e)$ is infinite (i.e. G is an ICC group) then $\mathcal{L}(G) \cap \mathcal{L}(G)' = \mathbb{C}1$ and $\mathcal{L}(G)$ is a II_1 factor.

Exercise. Show that $\mathbb{F}_n$ is an ICC group.

Let M be any II_1 factor with faithful normal trace τ and e a projection in M . Let $eMe = \{exe \mid x \in M\}$; eMe is called the *compression* of M by e. It is an elementary fact in von Neumann algebra theory that the isomorphism class of eMe depends only on $t = \tau(e)$ and we denote this isomorphism class by M_t. A deeper fact of Murray and von Neumann is that $(M_s)_t = M_{st}$. We can define M_t for all $t > 0$ as follows. For a positive integer n let $M_n = M \otimes M_n(\mathbb{C})$ and for any t, let $M_t = e(M_n)e$ for any projection e in M_n with trace t. Murray and von Neumann then defined the *fundamental group* of M, $\mathcal{G}(M)$, to be $\{t \in \mathbb{R}^+ \mid M \simeq M_t\}$ and showed that it is a multiplicative subgroup of $\mathbb{R}^+$. It is a theorem that when G is an amenable ICC group we have $\mathcal{G}(\mathcal{L}(G)) = \mathbb{R}^+$.

If $G = \mathbb{F}_\infty$ then Radulescu showed that $\mathcal{G}(\mathcal{L}(G)) = \mathbb{R}^+$. For finite n, $\mathcal{G}(\mathcal{L}(\mathbb{F}_n))$ is unknown but it is known to be either $\mathbb{R}^+$ or $\{1\}$. In 1990 D. Voiculescu showed that

$$\mathcal{L}(\mathbb{F}_n)_{1/k} \simeq \mathcal{L}(\mathbb{F}_m) \text{ where } \frac{m-1}{n-1} = k^2$$

or equivalently

$$\mathcal{L}(\mathbb{F}_n) \simeq M_k(\mathbb{C}) \otimes \mathcal{L}(\mathbb{F}_m) \text{ where } \frac{m-1}{n-1} = k^2$$

So if we embed $\mathcal{L}(\mathbb{F}_m)$ into $M_k(\mathbb{C}) \otimes \mathcal{L}(\mathbb{F}_m) \simeq \mathcal{L}(\mathbb{F}_n)$ as $x \mapsto 1 \otimes x$ then $\mathcal{L}(\mathbb{F}_m)$ is a subfactor of $\mathcal{L}(\mathbb{F}_n)$ of Jones index[1] k^2. Thus

$$\frac{m-1}{n-1} = [\mathcal{L}(\mathbb{F}_n); \mathcal{L}(\mathbb{F}_m)]$$

Now the similarity to Schreier's index formula is apparent. Indeed, suppose G is a free group of rank n and H is a subgroup of G of finite index. Then H is a free group of rank m and

$$\frac{m-1}{n-1} = [G; H]$$

In order to prove that a II_1 factor M is isomorphic to $\mathcal{L}(\mathbb{F}_n)$ we must show that we can find n Haar unitaries $u_1, \ldots, u_n$ in M which are free with respect to the trace and generate M. To do this however, it suffices to find elements $x_1, \ldots, x_m$ in M which generate M, are free with respect to the trace, and such that for each i there is a Haar unitary u_i such that $\overline{\mathrm{alg}\{1, x_i\}} = \overline{\mathrm{alg}\{u_i, u_i^*\}}$; for then the u_i's will be free Haar unitaries generating M. If x is a self-adjoint element and the spectral measure of x is diffuse, i.e. has no atoms, then $\overline{\mathrm{alg}\{1, x\}} \simeq \mathcal{L}^\infty([0,1], m)$ where m is Lebesgue measure and, moreover, $u(t) = \exp(2\pi i t)$ is a Haar unitary that generates $\mathcal{L}^\infty([0,1], m)$. Thus we have the following theorem.

Theorem. *Let M be a II_1 factor with $x_1, \ldots, x_n$ free and generating M, such that the spectral measure of each x_i is diffuse, then $M \simeq \mathcal{L}(\mathbb{F}_n)$.*

Example. Let $s \in M$ be a semi-circular operator. The spectral measure of s is $\sqrt{4-t^2}/(2\pi)\,dt$ i.e. $\tau(f(s)) = \int_{-2}^{2} f(t)\sqrt{4-t^2}/(2\pi)\,dt$. If $f(t) = 2(t\sqrt{4-t^2} + \sin^{-1}(t))$ and $u = \exp(if(s))$, then u is a Haar unitary i.e. $\int_{-2}^{2} e^{ikf(t)}\sqrt{4-t^2}/(2\pi)\,dt = \delta_{0,k}$ which generates the same von Neumann subalgebra as s.

Rather than proving Voiculescu's theorem in full generality we shall first prove a special case which illustrates the main ideas of the proof, and then sketch the general case.

Theorem. $\mathcal{L}(\mathbb{F}_3)_{1/2} \simeq \mathcal{L}(\mathbb{F}_9)$

[1]§2.3, V. F. R. Jones, Index for Subfactors, *Invent. Math.* **72** (1983), 1–25.

We must find in $\mathcal{L}(\mathbb{F}_3)_{1/2}$ nine free elements with diffuse spectral measure which generate $\mathcal{L}(\mathbb{F}_3)_{1/2}$.

To prove this theorem we will find a von Neumann algebra M with faithful normal state ϕ and $x_1, x_2, x_3 \in M$ such that

- the spectral measure of each x_i is diffuse and
- $\{x_1, x_2, x_3\}$ are free.

Let N be the von Neumann subalgebra of M generated by x_1, x_2 and x_3. Then $N \simeq \mathcal{L}(\mathbb{F}_3)$. We will then show that there is a projection p in N such that

- $\phi(p) = 1/2$
- there are 9 free and diffuse elements in pNp which generate pNp.

Thus $\mathcal{L}(\mathbb{F}_3)_{1/2} \simeq pNp \simeq \mathcal{L}(\mathbb{F}_9)$.

Circular Operators and Complex Gaussian Random Matrices. To construct the elements x_1, x_2, x_3 as required above we need to make a digression into circular operators. Let X be an $2N \times 2N$ GUE random matrix. Let $P = \begin{pmatrix} I_n & 0_n \\ 0_n & 0_n \end{pmatrix}$ and $G = \sqrt{2}\, PX(1-P)$. Then G is a $N \times N$ matrix with independent identically distributed entries which are centred complex Gaussian random variables with complex variance $1/N$, such a matrix we call a *complex Gaussian random matrix*. We can determine the limiting $*$-moments of G as follows.

Exercise. Write $Y_1 = (G + G^*)/\sqrt{2}$ and $Y_2 = -i(G - G^*)/\sqrt{2}$ then $G = (Y_1 + iY_2)/\sqrt{2}$ and Y_1 and Y_2 are independent $N \times N$ GUE random matrices. Therefore be the asymptotic freeness theorem of Lecture 1, Y_1 and Y_2 converge as $N \to \infty$ to $\{s_1, s_2\}$, a free and semi-circular family.

Definition. Let s_1 and s_2 be free and semi-circular; $c = (s_1 + is_2)/\sqrt{2}$ is a *circular operator*, (also called *Voiculescu's circular operator*).

Since s_1 and s_2 are free we can easily calculate the free cumulants of c. If $\varepsilon = \pm 1$ let us adopt the following notation $x^{(-1)} = x^*$, and $x^{(1)} = x$. Recall that for a semi-circular operator s

$$\kappa_n(s, s, \ldots, s) = \begin{cases} 1 & n = 2 \\ 0 & n \neq 2 \end{cases}$$

Thus

$$\begin{aligned}
&\kappa_n(c^{(\varepsilon_1)}, c^{(\varepsilon_2)}, \dots, c^{(\varepsilon_n)})\\
&\quad = 2^{-n/2}\kappa_n(s_1 + \varepsilon_1 i s_2, \dots, s_1 + i\varepsilon_n s_2)\\
&\quad = 2^{-n/2}(\kappa_n(s_1, \dots, s_1) + i^n \varepsilon_1 \cdots \varepsilon_n \kappa_n(s_2, \dots, s_2))
\end{aligned}$$

since all mixed cumulants are 0. Thus $\kappa_n(c^{(\varepsilon_1)}, \dots, c^{(\varepsilon_n)}) = 0$ for $n \neq 2$, and

$$\begin{aligned}
\kappa_2(c^{(\varepsilon_1)}, c^{(\varepsilon_2)}) &= 2^{-1}(\kappa_2(s_1, s_1) - \varepsilon_1\varepsilon_2\kappa_2(s_2, s_2))\\
&= \frac{1 - \varepsilon_1\varepsilon_2}{2} = \begin{cases} 1 & \varepsilon_1 \neq \varepsilon_2 \\ 0 & \varepsilon_1 = \varepsilon_2 \end{cases}
\end{aligned}$$

Hence $\kappa_2(c, c^*) = \kappa_2(c^*, c) = 1$, $\kappa_2(c, c) = \kappa_2(c^*, c^*) = 0$ and all other $*$-cumulants are 0. Also

$$\begin{aligned}
\tau((c^*c)^n) &= \tau(c^*cc^*c \cdots c^*c) = \sum_{\pi \in NC(2n)} \kappa_\pi(c^*, c, c^*, c, \dots, c^*, c)\\
&= \sum_{\pi \in NC_2(2n)} \kappa_\pi(c^*, c, c^*, c, \dots, c^*, c) = |NC_2(2n)| = \tau(s^{2n})
\end{aligned}$$

Thus by the Stone-Weierstrass theorem $|c| = \sqrt{c^*c}$ and $|s| = \sqrt{s^2}$ have the same distribution. The operator $|c| = |s|$ is called a *quarter-circular* operator and $\tau(|c|^k) = \int_0^2 t^k \sqrt{4 - t^2}/\pi \, dt$. An additional result which we will need is Voiculescu's theorem on the polar decomposition of a circular operator.

Definition. Let $x_1, \dots, x_n \in (\mathcal{A}, \phi)$ a $*$-probability space, and let $\mathcal{A}_i = \mathrm{alg}(1, x_i, x_i^*)$. If the algebras $\mathcal{A}_1, \dots, \mathcal{A}_n$ are free we say the elements $x_1, \dots, x_n$ are $*$-free.

Theorem (Voiculescu 1990)**.** *Let $c \in (M, \tau)$ be a circular operator and $c = u\,|c|$ be its polar decomposition in M. Then*

i) u and $|c|$ are $*$-free
ii) u is a Haar unitary
iii) $|c|$ is a quarter circular operator

Proof. The proof of (i) and (ii) can either be done using random matrix methods (as was done by Voiculescu) or by showing that if u is a Haar unitary and q is a quarter-circular operator such that u and q are $*$-free then uq has the same $*$-moments as a circular operator. This is achieved by using the formula for cumulants of products[2]. □

[2]Cor. 15.14, A. Nica and R. Speicher, *Lectures on the Combinatorics of Free Probability*, Cambridge U. Press, 2006

Theorem. *Let (A, ϕ) be a unital algebra with a state ϕ. Suppose $s_1, s_2, c \in A$ are $*$-free and s_1 and s_2 semi-circular and c circular. Then $x = \frac{1}{\sqrt{2}}\begin{pmatrix} s_1 & c \\ c^* & s_2 \end{pmatrix} \in (M_2(A), \phi_2)$ is semi-circular.*

Proof. Let $\mathbb{C}\langle x_{11}, x_{12}, x_{21}, x_{22}\rangle$ be the polynomials in the non-commuting variables $x_{11}, x_{12}, x_{21}, x_{22}$. Let

$$p_k(x_{11}, x_{12}, x_{21}, x_{22}) = \frac{1}{2}\mathrm{Tr}\Big(\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}^k \Big)$$

Now let $\mathcal{A}_N = M_N(\mathcal{L}(\Omega))$ be the $N \times N$ matrices with entries in $\mathcal{L}(\Omega) = \cap_{p\geq 1} L^p(\Omega)$. On $\mathcal{A}_N$ we have the state $\phi_N(x) = \mathrm{E}(N^{-1}\mathrm{Tr}(X))$. Now suppose in $\mathcal{A}_N$ we have S_1, S_2, and C with S_1 and S_2 GUE random matrices and C a complex Gaussian random matrix with the entries of S_1, S_2, C independent. Then we know that there is a $*$-algebra $\mathcal{A}$ with state ϕ and $s_1, s_2, c \in \mathcal{A}$, $*$-free with s_1 and s_2 semi-circular and c circular such that for every polynomial in non-commuting variables $p(x, y, z, w)$ we have $\phi_N(p(S_1, S_2, C, C^*)) \to \phi(p(s_1, s_2, c, c^*))$.

Now let $X = \frac{1}{\sqrt{2}}\begin{pmatrix} S_1 & C \\ C^* & S_2 \end{pmatrix}$. Then X is in $\mathcal{A}_{2N}$, and

$$\begin{aligned} \phi_{2N}(X^k) &= \phi_N(p_k(S_1, S_2, C, C^*)) \to \phi(p_k(s_1, s_2, c, c^*)) \\ &= \phi(\frac{1}{2}\mathrm{Tr}(x^k)) = \mathrm{tr} \otimes \phi(x^k) \end{aligned}$$

On the other hand X is a $2N \times 2N$ GUE random matrix; so $\phi_{2N}(X^k)$ converges to the k^{th} moment of a semi-circular operator. Hence x in $M_2(\mathcal{A})$ is semi-circular. □

Lemma. Let $\mathcal{A}$ be a unital $*$-algebra and ϕ a state on $\mathcal{A}$. Suppose $s_1, s_2, s_3, s_4, c_1, c_2, u \in \mathcal{A}$ are $*$-free with s_1, s_2, s_3, and s_4 semi-circular, c_1 and c_2 circular and u a Haar unitary. Let

$$x_1 = \begin{pmatrix} s_1 & c_1 \\ c_1^* & s_2 \end{pmatrix}, x_2 = \begin{pmatrix} s_3 & c_2 \\ c_2^* & s_4 \end{pmatrix}, x_3 = \begin{pmatrix} u & 0 \\ 0 & 2u \end{pmatrix}$$

Then x_1, x_2, x_3 are $*$-free in $M_2(\mathcal{A})$ with state $\mathrm{tr} \otimes \phi$.

Proof. We model x_1 by X_1, x_2 by X_2 and x_3 by X_3 where

$$X_1 = \begin{pmatrix} S_1 & C_1 \\ C_1^* & S_2 \end{pmatrix}, X_2 = \begin{pmatrix} S_3 & C_2 \\ C_2^* & S_3 \end{pmatrix}, X_3 = \begin{pmatrix} U & 0 \\ 0 & 2U \end{pmatrix}$$

and S_1, S_2, S_3, S_4 are $N \times N$ GUE random matrices, C_1 C_2 are $N \times N$ complex Gaussian random matrices and U is a diagonal constant unitary matrix, chosen so that the entries of X_1 are independent from those of X_2 and that the diagonal entries of U converge in distribution to the uniform distribution on the unit circle. Then X_1, X_2, X_3 are asymptotically free by Lecture 5. Thus x_1, x_2, and x_3 are free because they have the same distribution as the limiting distribution of X_1, X_2, and X_3. □

Proof of main theorem. We have shown the existence of four semi-circular operators s_1 s_2, s_3, s_4, two circular operators c_1, c_2, and a Haar unitary u in a von Neumann algebra M with trace τ such that

- s_1, s_2, s_3, s_4, c_1, c_2, u are $*$-free, and
- $x_1 = \begin{pmatrix} s_1 & c_1 \\ c_1^* & s_2 \end{pmatrix}$, $x_2 = \begin{pmatrix} s_3 & c_2 \\ c_2^* & s_4 \end{pmatrix}$, $x_3 = \begin{pmatrix} u & 0 \\ 0 & 2u \end{pmatrix}$ are $*$-free in $(M_2(M), \mathrm{tr} \otimes \tau)$
- x_1 and x_2 are semi-circular and x_3 has diffuse spectral measure.

Let $N = W^*(x_1, x_2, x_3) \subseteq M_2(M)$. Then $N \simeq \mathcal{L}(\mathbb{F}_3)$ because x_1, x_2, and x_3 are free and diffuse. Also x_3 has a spectral projection

$$p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in N$$

Let c_1 be such that $\begin{pmatrix} 0 & c_1 \\ 0 & 0 \end{pmatrix} = px_1(1-p)$. The polar decomposition of

$$\begin{pmatrix} 0 & c_1 \\ 0 & 0 \end{pmatrix} \quad \text{is} \quad \begin{pmatrix} 0 & v_1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & |c_1| \end{pmatrix}$$

where $v_1|c_1|$ is the polar decomposition of c_1 in M. Let

$$v = \begin{pmatrix} 0 & v_1 \\ 0 & 0 \end{pmatrix} \text{ then } v^*v = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } vv^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = p$$

Claim: $\cup_{i=1}^{3}\{px_ip, px_iv^*, vx_ip, vx_iv^*\}$ generate pNp.

Consider for example $px_{i_1}x_{i_2}x_{i_3}p \in pNp$. We have

$$\begin{aligned} px_{i_1}x_{i_2}x_{i_3}p &= px_{i_1}(p \cdot p + v^*v)x_{i_2}(p \cdot p + v^*v) \cdot x_{i_3}p \\ &= px_{i_1}p \cdot px_{i_2}(p \cdot p + v^*v)x_{i_3}p \\ &\quad + px_{i_1}v^* \cdot vx_{i_2}(p \cdot p + v^*v)x_{i_3}p \\ &= px_{i_1}p \cdot (px_{i_2}p \cdot px_{i_3}p + px_{i_2}v^* \cdot vx_{i_3}p) \\ &\quad + px_{i_1}v^*(vx_{i_2}p \cdot px_{i_3}p + vx_{i_2}v^* \cdot vx_{i_3}p) \end{aligned}$$

is in the subalgebra generated by $\cup_{i=1}^{3}\{px_ip, px_iv^*, vx_ip, vx_iv^*\}$.

In general we write $px_{i_1}\cdots x_{i_n}p$ as $px_{i_1}1x_{i_2}1\cdots 1x_{i_n}p$ and replace each 1 by $p \cdot p + v^*v$.

Since $v \in N$, $N = W^*(x_1, x_2, x_3)$ is generated by

$$\begin{pmatrix} s_1 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & s_2 \end{pmatrix} \quad \begin{pmatrix} 0 & v_1 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & |c_1| \end{pmatrix} \quad \begin{pmatrix} s_3 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & s_4 \end{pmatrix} \quad \begin{pmatrix} 0 & v_2 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & |c_2| \end{pmatrix} \quad \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & u \end{pmatrix}$$

Thus pNp is generated by s_1, s_2, u, $v_1 s_2 v_1^*$, $v_1 s_4 v_1^*$, $v_1 |c_1| v_1^*$, $v_1 |c_2| v_1^*$, $v_1 u v_1^*$, and $v_2 v_1^*$. To check that this set is free we recall a few elementary facts about freeness

Exercise.

i) if $\mathcal{A}_1$ and $\mathcal{A}_2$ free subalgebras of $\mathcal{A}$, $\mathcal{A}_{11}$ and $\mathcal{A}_{12}$ are free subalgebras of $\mathcal{A}_1$, and $\mathcal{A}_{21}$ and $\mathcal{A}_{22}$ free subalgebras of $\mathcal{A}_2$; then $\mathcal{A}_{11}, \mathcal{A}_{12}, \mathcal{A}_{21}, \mathcal{A}_{22}$ are free;

ii) if u is a Haar unitary free from $\mathcal{A}$, then $\mathcal{A}$ is free from $u\mathcal{A}u^*$;

iii) if v_1 and v_2 are Haar unitaries and v_2 is free from $\{v_1\} \cup \mathcal{A}$ then $v_2 v_1^*$ is free from $v_1 \mathcal{A} v_1^*$.

By construction

$$s_1, s_2, s_3, s_4, |c_1|, |c_2|, v_1, v_2, u$$

are free. Thus in particular

$$s_3, s_4, |c_1|, |c_2|, v_2, u$$

are free. Hence by (ii)

$$v_1 s_2 v_1^*, v_1 s_4 v_1^*, v_1 |c_1| v_1^*, v_1 |c_1| v_1^*, v_1 u v_1^*$$

are free and, in addition, free from

$$u, s_1, s_3, v_2$$

Thus

$$u, s_1, s_3, v_1 s_2 v_1^*, v_1 s_4 v_1^*, v_1 |c_1| v_1^*, v_1 |c_2| v_1^*, v_1 u v_1^*, v_2$$

are free. Let $\mathcal{A} = \text{alg}(s_1, s_2, s_3, s_4, |c_1|, |c_2|, u)$. We have that v_2 is free from $\{v_1\} \cup \mathcal{A}$, so by (iii), $v_2 v_1^*$ is free from $v_1 \mathcal{A} v_1^*$. Thus $v_2 v_1^*$ is free from

$$v_1 s_2 v_1^*, v_1 s_4 v_1^*, v_1 |c_1| v_1^*, v_1 |c_1| v_1^*, v_1 u v_1^*$$

and it was already free from s_1, s_3 and u. Thus by (i)

$$s_1, s_3, v_1 s_2 v_1^*, v_1 s_4 v_1^*, v_1 |c_1| v_1^*, v_1 |c_2| v_1^*, u, v_1 u v_1^*, v_2 v_1^*$$

are free. Since they are diffuse and generate pNp, we have that $pNp \simeq \mathcal{L}(\mathbb{F}_9)$. Hence $\mathcal{L}(\mathbb{F}_3)_{1/2} \simeq \mathcal{L}(\mathbb{F}_9)$. □

The general case. We write $\mathcal{L}(\mathbb{F}_n) = W^*(x_1, \ldots, x_n)$ where for $1 \leq i \leq n-1$ each x_i is a semi-circular of the form

$$x_i = \frac{1}{\sqrt{k}} \begin{pmatrix} s_1^{(i)} & c_{12}^{(i)} & \cdots & c_{1k}^{(i)} \\ c_{12}^{(i)*} & \ddots & & \vdots \\ \vdots & & \ddots & c_{k-1,k}^{(i)} \\ c_{1k}^{(i)*} & \cdots & \cdots & s_k^{(i)} \end{pmatrix} \text{ and } x_n = \begin{pmatrix} u & & & \\ & 2u & & \\ & & \ddots & \\ & & & ku \end{pmatrix}$$

with $s_j^{(i)}$ semi-circular, $c_i^{(i)}$ circular, and u a Haar unitary, so that $\{s_j^{(i)}\}_{i,j} \cup \{c_j^{(i)}\}_{i,j} \cup \{u\}$ are $*$-free.

So we have $(n-1)k$ semi-circular operators, $(n-1)\binom{k}{2}$ circular operators and one Haar unitary. Each circular operator produces two free elements so we have in total

$$(n-1)k + 2(n-1)\binom{k}{2} + 1 = (n-1)k^2 + 1$$

free and diffuse generators. Thus $\mathcal{L}(\mathbb{F}_n)_{1/k} \simeq \mathcal{L}(\mathbb{F}_m)$ where $\frac{m-1}{n-1} = k^2$.