FREE PROBABILITY AND RANDOM MATRICES

Lectures given at the Fields Institute

Lecture 1: September 13, 2007

Moments and Cumulants of Random Variables

Let ν be a probability measure on $\mathbb{R}$. If $\int_{\mathbb{R}} |t|^k \, d\nu(t) < \infty$ we say that ν has a moment of order k, the k^{th} moment is denoted $\alpha_k = \int_{\mathbb{R}} t^k d\nu(t)$.

Exercise 1. If ν has a moment of order k then ν has all moments of order m for $m < k$.

The integral $\phi(t) = \int e^{ist} \, d\nu(t)$ is always convergent and is called the characteristic function of ν. It is always uniformly continuous on $\mathbb{R}$ and $\phi(0) = 1$, so for $|t|$ small enough $\phi(t) \notin (-\infty, 0]$ and we can define the continuous function $\log(\phi(t))$. If ν has a moment of order k then ϕ has a derivative of order k, and conversely. Moreover $\alpha_k = i^{-k}\phi^{(k)}(0)$, so at the level of formal power series $\phi(t) = \sum_{k\geq 0} \alpha_k \frac{(it)^k}{k!}$. Thus if ν has a moment of order k we can write $\log(\phi(t)) = \sum_{j=0}^{k} k_j \frac{(it)^j}{j!} + o(t^k)$ with

$$k_j = i^{-j} \, \frac{d^j}{dt^j} \log(\phi(t)) \Big|_{t=0}$$

The numbers (k_j) are the *cumulants* of ν. To distinguish them from the free cumulants which will be defined below, we will call (k_j) the classical cumulants of ν. The moments $(\alpha_j)_j$ of ν and the cumulants (k_j) of ν each determine the other through the *moment-cumulant formulas*:

$$k_n = \sum_{\substack{1\cdot r_1+\cdots+n\cdot r_n=n \\ r_1,\ldots,r_n\geq 0 \\ r=r_1+\cdots+r_n}} (-1)^{r-1}(r-1)! \frac{n!}{(1!)^{r_1}\cdots(n!)^{r_n} r_1!\cdots r_n!} \alpha_1^{r_1}\cdots\alpha_n^{r_n}$$

$$\alpha_n = \sum_{\substack{1\cdot r_1+\cdots+nr_n=n \\ r_1,\ldots,r_n\geq 0}} \frac{n!}{(1!)^{r_1}\cdots(n!)^{r_n} r_1!\cdots r_n!} k_1^{r_1}\cdots k_n^{r_n}$$

We shall see below how to use partitions to simplify these equations.

A very important random variable is the Gaussian or normal random variable. It has the distribution $P(t_1 \le X \le t_2) = \displaystyle\int_{t_1}^{t_2} \frac{e^{-(t-a)^2/(2\sigma^2)}}{\sqrt{2\pi\sigma^2}}\,dt$ where a is the mean and σ^2 is the variance. The characteristic function of a Gaussian random variable is $\phi(t) = \exp(iat - \frac{\sigma^2 t^2}{2})$. Thus $\log\phi(t) = a\frac{(it)^1}{1!} + \sigma^2\frac{(it)^2}{2!}$. Hence for a Gaussian random variable all cumulants beyond the second are 0.

Exercise 2. Suppose ν has a fourth moment and we write $\phi(t) = 1 + \alpha_1\frac{(it)}{1!} + \alpha_2\frac{(it)^2}{2!} + \alpha_3\frac{(it)^3}{3!} + \alpha_4\frac{(it)^4}{4!} + o(t^4)$ where α_1, α_2, α_3, and α_4 are the first four moments of ν. Let $\log(\phi(t)) = k_1(it) + k_2\frac{(it)^2}{2!} + k_3\frac{(it)^3}{3!} + k_4\frac{(it)^4}{4!} + o(t^4)$. Using the Taylor series for $\log(1+x)$ find a formula for $\alpha_1, \alpha_2, \alpha_3$, and α_4 in terms of k_1, k_2, k_3, and k_4.

Moments of a Gaussian Random Variable. Let X be a Gaussian random variable with mean 0 and variance 1. Then

$$P(t_1 \le X \le t_2) = \int_{t_1}^{t_2} e^{-t^2/2}\frac{dt}{\sqrt{2\pi}}.$$

Let us find the moments of X. $\alpha_1 = 0$, $\alpha_2 = 1$, and by integration by parts $\alpha_k = \mathrm{E}(X^k) = \int_{\mathbb{R}} t^k e^{-t^2/2}\frac{dt}{\sqrt{2\pi}} = (k-1)\alpha_{k-2}$ for $k \ge 3$. Thus $\alpha_{2k} = (2k-1)(2k-3)\cdots 5\cdot 3\cdot 3\cdot 1 = (2k-1)!!$ and $\alpha_{2k-1} = 0$ for all k.

Let us find a combinatorial interpretation of these numbers. For a positive integer n let $[n] = \{1, 2, 3, \ldots, n\}$, and $\mathcal{P}(n)$ denote all partitions of the set $[n]$ i.e. $\pi = \{V_1, \ldots, V_k\} \in \mathcal{P}(n)$ means $V_1, \ldots, V_k \subseteq [n]$, $V_1 \cup \cdots \cup V_k = [n]$, and $V_i \cap V_j = \emptyset$ for $i \ne j$; $V_1, \ldots, V_k$ are called the blocks of π. We let $\#(\pi)$ denote the number of blocks of π and $\#(V_i)$ the number of elements in the block V_i. A partition is a pairing if each block has size 2. The pairings of $[n]$ will be denoted $\mathcal{P}_2(n)$.

Let us count the number of pairings of $[n]$. 1 must be paired with something and there are $n-1$ ways of choosing it. Thus $\#(\mathcal{P}_2(n)) = (n-1)\#(\mathcal{P}_2(n-2)) = (n-1)!!$. So $\mathrm{E}(X^{2n}) = \#(\mathcal{P}_2(2n))$, but the analogy runs deeper and is known as Wick's formula.

Gaussian vectors. Let $\vec{X} : \Omega \to \mathbb{R}^n$, $\vec{X} = (X_1, \ldots, X_n)$ be a random vector. We say that $\vec{X}$ is Gaussian if there is a positive definite $n \times n$ real matrix B such that

$$\mathrm{E}(X_{i_1}\cdots X_{i_k}) = \int_{\mathbb{R}^n} t_{i_1}\cdots t_{i_k}\frac{\exp\frac{-1}{2}\langle B\vec{t}, \vec{t}\rangle d\vec{t}}{(2\pi)^{n/2}\det(B)^{-1/2}}$$

where $\langle\cdot,\cdot\rangle$ denotes the standard inner product on $\mathbb{R}^n$. Let $C = (c_{ij})$ be the covariance matrix, that is $c_{ij} = \mathrm{E}\big([X_i - \mathrm{E}(X_i)]\,[X_j - \mathrm{E}(X_j)]\big)$. In fact $C = B^{-1}$ and if $X_i, \dots, X_n$ are independent then B is a diagonal matrix, see Exercise 3. If $Y_1, \dots, Y_k$ are independent Gaussian random variables and $\vec{X} = A\vec{Y}$, then $\vec{X}$ is a Gaussian random vector and every Gaussian random vector is obtained in this way. If $\vec{X} = (X_1, \dots, X_n)$ is a complex random vector we say that $\vec{X}$ is a complex Gaussian random vector if $(\mathrm{Re}(X_1), \mathrm{Im}(X_1), \dots, \mathrm{Re}(X_n), \mathrm{Im}(X_n))$ is a real Gaussian random vector.

Exercise 3. Let $\vec{X} = (X_1, \dots, X_n)$ be a Gaussian random vector with density $\dfrac{\exp(-\frac{1}{2}\langle B\vec{t}, \vec{t}\rangle)}{(2\pi)^{n/2}\det(B)^{-1/2}}$. Let $C = B^{-1}$.

i) Show that B is diagonal if and only if $\{X_1, \dots, X_n\}$ are independent [1]

ii) By first diagonalizing B show that $c_{ij} = \mathrm{E}((X_i - \mathrm{E}(X_i)) \cdot (X_j - \mathrm{E}(X_j)))$.

The Moments of a Complex Gaussian Random Variable. Suppose X and Y are independent real Gaussian random variables with mean 0 and variance 1. Then $Z = (X + iY)/\sqrt{2}$ is a complex Gaussian random variable with mean 0 and variance $\mathrm{E}(Z\overline{Z}) = \frac{1}{2}\mathrm{E}(X^2 + Y^2) = 1$ moreover

$$\mathrm{E}(Z^m\overline{Z}^n) = \begin{cases} 0 & m \neq n \\ m! & m = n \end{cases}$$

Exercise 4. Let $Z = (X + iY)/\sqrt{2}$ be a complex Gaussian random variable with mean 0 and variance 1.

i) By making the substitution $\vec{t} = O\vec{s}$, show that for $m \neq n$ $\int_{\mathbb{R}^2}(t_1 + it_2)^m(t_1 - it_2)^n e^{-(t_1^2+t_2^2)}dt_1dt_2 = 0$, where

$$O = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

ii) Show that $\mathrm{E}(Z^m\overline{Z}^n) = 0$ for $m \neq n$.

iii) By switching to polar coordinates show that $\mathrm{E}(|Z|^{2n}) = n!$.

Wick's Formula[2]**.** Let $(X_1, \dots X_n)$ be a real Gaussian random vector and $i_1, \dots i_k \in [n]$. Gian Carlo Wick found in 1950 a simple expression for $\mathrm{E}(X_{i_1} \cdots X_{i_k})$. If k is even and $\pi \in \mathcal{P}_2(k)$ let $\mathrm{E}_\pi(X_1, \dots, X_k) = \prod_{(r,s)\in\pi} \mathrm{E}(X_rX_s)$. For example if $\pi = \{(1,3)(2,6)(4,5)\}$ then $\mathrm{E}_\pi(X_1, X_2, X_3, X_4, X_5, X_6) = \mathrm{E}(X_1X_3)\mathrm{E}(X_2X_6)\mathrm{E}(X_4X_5)$. E_π is a k-linear functional.

[1]i.e. if $i_1, \dots, i_k$ are distinct and $n_1, \dots, n_k$ are positive integers then $\mathrm{E}(X_{i_1}^{n_1} \cdots X_{i_k}^{n_k}) = \mathrm{E}(X_{i_1}^{n_1}) \cdots \mathrm{E}(X_{i_k}^{n_k})$

[2]Gian Carlo Wick, The Evaluation of the Collision Matrix (1950)

Theorem 1. *Let $(X_1, \ldots, X_n)$ be a real Gaussian random vector and $i_1, \ldots, i_k \in [n]$. Then* $\mathrm{E}(X_{i_1} \cdots X_{i_k}) = \sum_{\pi \in \mathcal{P}_2(k)} \mathrm{E}_\pi(X_{i_1}, \ldots, X_{i_k})$.

Proof. Suppose that the covariance matrix C of $(X_1, \ldots, X_n)$ is diagonal, i.e. the X_i's are independent. Consider $(i_1, \ldots i_k)$ as a function $[k] \to [n]$. Let $\{a_1, \ldots, a_r\}$ be the range of i and $A_j = i^{-1}(a_j)$. Then $\{A_1, \ldots, A_r\}$ is a partition of $[k]$ which we denote $\ker(i)$. Then $\mathrm{E}(X_{i_1} \cdots X_{i_k}) = \prod_{t=1}^r \mathrm{E}(X_{a_t}^{\#(A_t)})$. Let us recall that if X is a real random variable of mean 0 and variance c then for k even $\mathrm{E}(X^k) = c^{k/2} \times \#(\mathcal{P}_2(k)) = \sum_{\pi \in \mathcal{P}_2(k)} \mathrm{E}_\pi(X, \ldots, X_k)$ and for k odd $\mathrm{E}(X^k) = 0$. Thus we can write the product $\prod_t \mathrm{E}(X_{a_t}^{\#(A_t)})$ as a sum $\sum_{\pi \in \mathcal{P}_2(k)} \mathrm{E}_\pi(X_{i_1}, \ldots, X_{i_k})$ where the sum runs over all π's which only connect elements in the same block of $\ker(i)$. Since $\mathrm{E}(X_{i_r} X_{i_s}) = 0$ for $i_r \neq i_s$ we can relax the condition that π only connect elements in the same block of $\ker(i)$. Hence

$$\mathrm{E}(X_{i_1} \cdots X_{i_k}) = \sum_{\pi \in \mathcal{P}_2(k)} \mathrm{E}_\pi(X_{i_1}, \ldots, X_{i_k}) \tag{1}$$

Finally let us suppose that C is arbitrary. Let the density of $(X_1, \ldots, X_n)$ be $\frac{\exp(-\frac{1}{2}\langle B\vec{t}, \vec{t}\rangle)}{(2\pi)^{n/2}\det(B)^{-1/2}}$ and choose an orthogonal matrix O such that $D = O^{-1}BO$ is diagonal. Let $\begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix} = O^{-1} \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}$. Then $(Y_1, \ldots, Y_n)$ is a real Gaussian random vector with the diagonal covariance matrix D. Then

$$\begin{aligned} \mathrm{E}(X_{i_1} \cdots X_{i_k}) &= \sum_{j_1, \ldots, j_k = 1}^n o_{i_1 j_1} o_{i_2 j_2} \cdots o_{i_k j_k} \mathrm{E}(Y_{j_1} \cdots Y_{j_k}) \\ &= \sum_{j_1, \ldots, j_k = 1}^n o_{i_1 j_1} \cdots o_{i_k j_k} \sum_{\pi \in \mathcal{P}_2(k)} \mathrm{E}_\pi(Y_{i_1}, \ldots, Y_{i_k}) \\ &= \sum_{\pi \in \mathcal{P}_2(k)} \mathrm{E}_\pi(X_{i_1}, \ldots, X_{i_k}) \end{aligned}$$

□

Since both sides of equation (1) are k-linear we can extend by linearity to the complex case.

Corollary 2. *Suppose* $(X_1, \dots, X_n)$ *is a complex Gaussian random vector then*

$$\mathrm{E}(X_{i_1} \cdots X_{i_k}) = \sum_{\pi \in \mathcal{P}_2(k)} \mathrm{E}_\pi(X_{i_1}, \dots, X_{i_k})$$

Gaussian Random Matrices. Let X be an $N \times N$ matrix with entries f_{ij} where $f_{ij} = x_{ij} + \sqrt{-1}\, y_{ij}$ is a complex Gaussian random variable such that

i) $\{x_{ij}\}_{i \geq j} \cup \{y_{ij}\}_{i > j}$ is independent

ii) $\mathrm{E}(f_{ij}) = 0$, $\mathrm{E}(|f_{ij}|^2) = \frac{1}{N}$

Then X is a self-adjoint Gaussian random matrix. Such a random matrix is often called a GUE random matrix (GUE = Gaussian unitary ensemble).

Exercise 5. Let X be an $N \times N$ GUE random matrix, with entries $f_{ij} = x_{ij} = \sqrt{-1}\, y_{ij}$ normalized so that $\mathrm{E}(|f_{ij}|^2) = 1$.

i) Consider the random N^2-vector $(x_{11}, \dots, x_{NN}, x_{12}, \dots, x_{1N}, \dots, x_{N-1,N}, y_{12}, \dots, y_{N-1,N})$. Show that the density of this vector is $ce^{-\frac{1}{2}\mathrm{Tr}(X^2)}dX$ where $dX = \prod_{i=1}^N dx_{ii} \prod_{i<j} dx_{ij} dy_{ij}$, for some constant c.

ii) Evaluate the constant c.

A genus expansion for the GUE. Let us calculate $\mathrm{E}(\mathrm{Tr}(X^k))$, for X a $N \times N$ GUE random matrix. We first suppose for convenience that the entries of X have been normalized so that $\mathrm{E}(|f_{ij}|^2) = 1$. Now

$$\mathrm{E}(\mathrm{Tr}(X^k)) = \sum_{i_1, \dots, i_k = 1}^{N} \mathrm{E}(f_{i_1 i_2} f_{i_2 i_3} \cdots f_{i_k i_1}).$$

By Wick's formula $\mathrm{E}(f_{i_1 i_2} f_{i_2 i_3} \cdots f_{i_k i_1}) = 0$ whenever k is odd. So let us evaluate

$$\mathrm{E}(f_{i_1 i_2} f_{i_2 i_3} \cdots f_{i_{2k} i_1}) = \sum_{\pi \in \mathcal{P}_2(2k)} \mathrm{E}_\pi(f_{i_1 i_2}, f_{i_2 i_3}, \cdots, f_{i_{2k} i_1})$$

Now $\mathrm{E}(f_{i_r i_{r+1}} f_{i_s i_{s+1}})$ will be 0 unless $i_r = i_{s+1}$ and $i_s = i_{r+1}$ (using the convention that $i_{2k+1} = i_1$). If $i_r = i_{s+1}$ and $i_s = i_{r+1}$ then $\mathrm{E}(f_{i_r i_{r+1}} f_{i_s i_{s+1}}) = \mathrm{E}(\left|f_{i_r i_{r+1}}\right|^2) = 1$. Thus given $(i_1, \dots, i_{2k})$, $\mathrm{E}(f_{i_1 i_2} f_{i_2 i_3} \cdots f_{i_{2k} i_1})$ will be the number of pairings π of $[2k]$ such that for each pair (r, s) of π, $i_r = i_{s+1}$ and $i_s = i_{r+1}$.

In order to easily count these we introduce the following notation. We regard the $2k$-tuple $(i_1, \dots, i_{2k})$ as a function $i : [2k] \to [N]$. A pairing $\pi = \{(r_1, s_1)(r_2, s_2), \dots, (r_k, s_k)\}$ of $[2k]$ will be regarded as a permutation of $[2k]$ by letting (r_i, s_i) be the transposition that switches

r_i with s_i and $\pi = (r_1, s_1) \cdots (r_k, s_k)$ as the product of these transpositions. We also let γ_{2k} be the permutation of $[2k]$ which has the one cycle $(1, 2, 3, \ldots, 2k)$. With this notation our condition on the pairings has a simple expression. Let π be a pairing of $[2k]$ and (r, s) be a pair of π. The condition $i_r = i_{s+1}$ can be written as $i(r) = i(\gamma_{2k}(\pi(r)))$ since $\pi(r) = s$ and $\gamma_{2k}(\pi(r)) = s + 1$. Thus $\mathrm{E}_\pi(f_{i_1 i_2}, f_{i_2 i_3}, \ldots, f_{i_{2k} i_1})$ will be 1 if i is constant on the orbits of $\gamma_{2k}\pi$ and 0 otherwise. Thus

$$\begin{aligned}
\mathrm{E}(\mathrm{Tr}(X^{2k})) &= \sum_{i_1,\ldots,i_{2k}=1}^{N} \mathrm{n^o} \left\{ \pi \in \mathcal{P}_2(2k) \;\middle|\; \begin{array}{l} i \text{ is constant on the} \\ \text{orbits of } \gamma_{2k}\pi \end{array} \right\} \\
&= \sum_{\pi \in \mathcal{P}_2(2k)} \mathrm{n^o} \left\{ i : [2k] \to [N] \;\middle|\; \begin{array}{l} i \text{ is constant on the} \\ \text{orbits of } \gamma_{2k}\pi \end{array} \right\} \\
&= \sum_{\pi \in \mathcal{P}_2(2k)} N^{\#(\gamma_{2k}\pi)}
\end{aligned}$$

Now let π and γ be any two permutations of $[n]$ and suppose that the group generated by π and γ acts transitively on $[n]$, then by a theorem of Alain Jacques (1968) and Robert Cori (1969) there is a positive integer g such that $\#(\pi) + \#(\pi^{-1}\gamma) + \#(\gamma) = n + 2(1 - g)$, and g is the minimal genus of a surface upon which a 'graph' of π relative to γ can be embedded.

Example 3. $\gamma = (1, 2, 3, 4, 5, 6)$, $\pi = (1, 4)(2, 5)(3, 6)$, $\#(\pi) = 3$, $\#(\gamma) = 1$, $\#(\pi^{-1}\gamma) = 2$, $\#(\pi) + \#(\pi^{-1}\gamma) + \#(\gamma) = 6$, $\therefore g = 1$
Conclusion: for a pairing π of $[2k]$

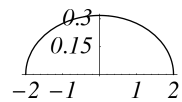
- $\#(\pi^{-1}\gamma) = k + 1 - 2g$ for some $g \geq 0$,
- $\#(\pi^{-1}\gamma) \leq k + 1$ with equality only when $g = 0$ i.e. when the graph is planar.

Now change the normalization of X so $\mathrm{E}(|f_{ij}|^2) = \frac{1}{N}$, and let $\mathrm{tr} = \frac{1}{N}\mathrm{Tr}$. Then

$$\begin{aligned}
\mathrm{E}(\mathrm{tr}(X^{2k})) &= N^{-(k+1)} \sum_{\pi \in \mathcal{P}_2(2k)} N^{\#(\gamma_{2k}\pi)} \\
&= \sum_{\pi \in \mathcal{P}_2(2k)} N^{-2g_\pi} \qquad (2)
\end{aligned}$$

because $\#(\pi^{-1}\gamma) = \#(\gamma\pi^{-1})$ in general and if π is a pairing $\pi = \pi^{-1}$. Thus $c_k = \lim_{N\to\infty} \mathrm{E}(\mathrm{tr}(X^{2k}))$ is the number of non-crossing pairings of $[2k]$, i.e. the cardinality of $|NC_2(2k)|$, which is the k-th Catalan number $\frac{1}{k+1}\binom{2k}{k}$. c_k is also the $2k^{th}$ moment of the semi-circle law:

$$c_k = \frac{1}{2\pi}\int_{-2}^{2} t^{2k}\sqrt{4-t^2}dt$$



This is Wigner's famous semi-circle law, which says that the spectral measures of $\{X_N\}_N$, relative to the state $\mathrm{E}(\mathrm{tr}(\cdot))$, converge to $\frac{\sqrt{4-t^2}}{2\pi}dt$ i.e. the expected proportion of eigenvalues of X between a and b is asymptotically $\int_a^b \frac{\sqrt{4-t^2}}{2\pi}dt$. If we regard $X : \Omega \to M_N(\mathbb{C})$ we can say something stronger. On the complement of a set of probability 0, the same result holds for $X(\omega)$.

Asymptotic Freeness of Independent GUE's. Suppose that for each N we have $X_1, \ldots, X_s$ are independent $N \times N$ GUE's. For notational simplicity we suppress the dependence on N. Suppose $m_1, \ldots,$ m_r are positive integers and $i_1, i_2, \ldots, i_r \in [s]$ such that $i_1 \neq i_2, i_2 \neq i_3, \ldots, i_{r-1} \neq i_r$. Consider the random matrix (which depends on N)

$$Y := (X_{i_1}^{m_1} - c_{m_1}I)(X_{i_2}^{m_2} - c_{m_2}I)\cdots(X_{i_r}^{m_r} - c_{m_r}I)$$

Each factor is centred asymptotically and adjacent factors involve independent matrices. We shall show that $\mathrm{E}(\mathrm{tr}(Y)) \to 0$ and we shall call this property asymptotic freeness. This will then motivate Voiculecu's definition of freeness.

First let us recall the principle of inclusion-exclusion (Da Silva, 1853). Let S be a set and $E_1, \ldots, E_r \subseteq S$. Then

$$|S \setminus (E_1 \cup \cdots \cup E_r)| = |S| - \sum_{i+1}^{r} |E_i| + \sum_{i_1 \neq i_2} |E_{i_1} \cap E_{i_2}| + \cdots$$

$$+ (-1)^k \sum_{\substack{i_1,\ldots,i_k \\ \text{distinct}}} |E_{i_1} \cap \cdots \cap E_{i_k}| + \cdots + (-1)^r |E_1 \cap \cdots \cap E_r| \tag{3}$$

for example, $|S \setminus (E_1 \cup E_2)| = |S| - (|E_1| + |E_2|) + |E_1 \cap E_2|$.

We can rewrite the right hand side of (3) as

$$\begin{aligned}|S \setminus (E_1 \cup \cdots \cup E_r)| &= \sum_{\substack{M \subseteq [r] \\ M=\{i_1,\ldots,i_m\}}} (-1)^m |E_{i_1} \cap \cdots \cap E_{i_m}| \\ &= \sum_{M \subseteq [r]} (-1)^{|M|} \left| \bigcap_{i \in M} E_i \right|\end{aligned}$$

provided we make the convention that $\bigcap_{i \in \emptyset} E_i = S$ and $(-1)^{|\emptyset|} = 1$.

Notation 4. Let $i_1, \ldots, i_m \in [s]$. We regard these labels as the colours of the matrices $X_{i_1}, X_{i_2}, \ldots, X_{i_m}$. Given a pairing $\pi \in \mathcal{P}_2(m)$, we say that π respects the colours $\vec{i} := (i_1, \ldots, i_m)$, or to be brief: π respects i, if $i_r = i_s$ whenever (r, s) is a pair of π. Thus π respects i if and only if π only connects matrices of the same colour.

Lemma 5. *Suppose $i_1, \ldots, i_m \subseteq [s]$ are positive integers. Then*

$$\mathrm{E}(\mathrm{tr}(X_{i_1} \cdots X_{i_m})) = \left|\left\{\pi \in NC_2(m) \mid \pi \textit{ respects } \vec{i}\right\}\right| + O(N^{-2})$$

Proof.

$$\begin{aligned}
\mathrm{E}(\mathrm{tr}(X_{i_1} \cdots X_{i_m})) &= \sum_{j_1,\ldots,j_m} \mathrm{E}(f^{(i_1)}_{j_1 j_2} \cdots f^{(i_m)}_{j_m, j_1}) \\
&= \sum_{j_1,\ldots,j_m} \sum_{\pi \in \mathcal{P}_2(m)} \mathrm{E}_\pi(f^{(i_1)}_{j_1,j_2}, \ldots, f^{(i_k)}_{j_m,j_1}) \\
&= \sum_{\substack{\pi \in \mathcal{P}_2(m) \\ \pi \text{ respects } i}} \sum_{j_1,\ldots,j_m} \mathrm{E}_\pi(f^{(i_1)}_{j_1,j_2}, \ldots, f^{(i_k)}_{j_m,j_1}) \\
&\overset{\text{by (2)}}{=} \sum_{\substack{\pi \in \mathcal{P}_2(m) \\ \pi \text{ respects } i}} N^{-2g_\pi} \\
&= |\{\pi \in NC_2(m) \mid \pi \text{ respects } i\}| + O(N^{-2})
\end{aligned}$$

□

Theorem 6. *If $i_1 \neq i_2, i_2 \neq i_3, \ldots, i_{r-1} \neq i_r$ then $\lim_N \mathrm{E}(\mathrm{tr}(Y)) = 0$.*

Proof. Let let $I_1 = \{1, \ldots, m_1\}$, $I_2 = \{m_1 + 1, \ldots, m_1 + m_2\}$, $\ldots$, $I_r = \{m_1 + \cdots + m_{r-1} + 1, \ldots, m_1 + \cdots + m_r\}$.

$$\begin{aligned}
&\mathrm{E}(\mathrm{tr}((X^{m_1}_{i_1} - c_{m_1} I) \cdots (X^{m_r}_{i_r} - c_{m_r} I))) \\
&= \sum_{M \subseteq [r]} (-1)^{|M|} \left[\prod_{i \in M} c_{m_i}\right] \mathrm{E}\Big(\mathrm{tr}\Big(\prod_{j \notin M} X^{m_j}_{i_j}\Big)\Big) \\
&= \sum_{M \subseteq [r]} (-1)^{|M|} \left[\prod_{i \in M} c_{m_i}\right] |\{\pi \in NC_2(\cup_{j \notin M} I_j) \mid \pi \text{ respects } i\}| \\
&\quad + O(N^{-2})
\end{aligned}$$

Let $S = \{\pi \in NC_2(m) \mid \pi \text{ respects } i\}$ and $E_j = \{\pi \in S \mid$ elements of I_j are only paired amongst themselves $\}$. Then

$$\Big|\bigcap_{j \in M} E_j\Big| = \left[\prod_{j \in M} c_{m_j}\right] \Big|\{\pi \in NC_2(\cup_{j \notin M} I_j) \mid \pi \text{ respects } i\}\Big|$$

Thus

$$\mathrm{E}(\mathrm{tr}((X_{i_1}^{m_1} - c_{m_1} I) \cdots (X_{i_r}^{m_r} - c_{m_r} I))) = \sum_{M \subseteq [r]} (-1)^{|M|} \Big| \bigcap_{j \in M} E_j \Big| + O(N^{-2})$$

So we must show that $\sum_{M\subseteq[r]}(-1)^{|M|}\big|\bigcap_{j\in M} E_J\big| = 0$. However by inclusion-exclusion this sum equals $|S \setminus (E_1 \cup \cdots \cup E_r)|$. Now $S \setminus (E_1 \cup \cdots \cup E_r)$ is the set of pairings of $[m]$ respecting i such that at least one element of each interval is connected to another interval. However this set is empty because elements of $|S \setminus (E_1 \cup \cdots \cup E_r)|$ must connect each interval to at least one other interval in a non-crossing way and thus form a non-crossing partition of the intervals $\{I_1, \ldots, I_r\}$ without singletons, in which no pair of adjacent intervals are in the same block, and this is impossible. □

Asymptotic Freeness. For each N let $\mathcal{A}_{N,i}$ be the polynomials in $X_{N,i}$ with complex coefficients. Let $\mathcal{A}_N$ be the algebra generated by $\mathcal{A}_{N,1}, \ldots, \mathcal{A}_{N,s}$. For $A \in \mathcal{A}_N$ let $\phi_N(A) = \mathrm{E}(\mathrm{tr}(A))$. Thus $\mathcal{A}_{N,1}, \ldots,$ $\mathcal{A}_{N,s}$ are unital subalgebras of the unital algebra $\mathcal{A}_N$ with state ϕ_N. We have just shown that if we have a positive integer r such that for each N and each $A_{N,1}, \ldots, A_{N,r} \in \mathcal{A}_N$ such that

- ∘ $\lim_N \phi_N(A_{N,i}) = 0$ for $i = 1, 2, \ldots, r$
- ∘ $A_{N,i} \in \mathcal{A}_{N,j_i}$ for $i = 1, 2, \ldots, r$
- ∘ $j_1 \neq j_2$, $j_2 \neq j_3$, $\ldots$, $j_{r-1} \neq j_r$

then $\lim_N \phi_N(A_{N,1}A_{N,2} \cdots A_{N,r}) = 0$. We thus say that the subalgebras $\mathcal{A}_{N,1}, \ldots, \mathcal{A}_{N,s}$ are *asymptotically free* because, in the limit as N tends to infinity, they satisfy the freeness property of Voiculescu.

Freeness. Let $(\mathcal{A}, \phi)$ be a unital algebra with a state. Suppose $\mathcal{A}_1, \ldots,$ $\mathcal{A}_s$ are unital subalgebras. We say that $\mathcal{A}_1 \ldots \mathcal{A}_s$ are freely independent with respect to ϕ (or just free) if whenever we have $a_1, \ldots, a_r \in \mathcal{A}$ such that

- ∘ $\phi(a_i) = 0$ for $i = 1, \ldots, r$
- ∘ $a_i \in \mathcal{A}_{j_i}$ for $i = 1, \ldots, r$
- ∘ $j_1 \neq j_2, j_2 \neq j_3, \ldots, j_{r-1} \neq j_r$

we must have $\phi(a_1 \cdots a_r) = 0$.