FREE PROBABILITY AND RANDOM MATRICES

EXERCISE 3, DUE NOVEMBER 8

1) Let a, b, c be free and identically distributed random variables with distribution $\frac{1}{2}(\delta_{-1} + \delta_{+1})$ (i.e., the odd moments are zero, and all even moments are equal to 1). Calculate the distribution of $a + b + c$.

2) (a) Show that the number of non-crossing pairings of $2n$ elements is, for each natural n, the same as the number of non-crossing partitions of n elements.

(b) Let s be a semicircular element of variance 1, so that its free cumulants are given by

$$\kappa_n(s, s, \ldots, s) = \delta_{n2}.$$

Show, by using part (a), that the free cumulants of s^2 are given by

$$\kappa_n(s^2, s^2, \ldots, s^2) = 1 \qquad \text{for all natural } n.$$

(c) Let s be semicircular of variance 1 as before and consider in addition a variable a which is free from s. By using part (b) and the formula

$$\varphi(a_1 b_1 a_2 b_2 \cdots a_n b_n) = \sum_{\pi \in NC(n)} \kappa_\pi[a_1, a_2, \ldots, a_n] \cdot \varphi_{K(\pi)}[b_1, b_2, \ldots, b_n]$$

for $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$ free, show that the free cumulants of sas are given by the moments of a:

$$\kappa_n(sas, sas, \ldots, sas) = \varphi(a^n) \qquad \text{for all natural } n.$$

[You can use the fact that in this setting φ is automatically a trace on the unital algebra generated by s and a.]

(3) An important distribution in statistics is the Marchenko-Pastur distribution, μ_c. It is in fact a family of distributions indexed by a real number $c > 0$. This distribution arises as the limiting distribution of a family of random matrices called Wishart matrices. Let X_N be a $M \times N$ random matrix with independent identically distributed entries which are complex Gaussian random variables with mean 0 and variance N^{-1}. Suppose that as M and N tend to infinity we have $M/N \to c$. Then

$$\lim_n \mathrm{E}\Big(N^{-1}\mathrm{Tr}((X^*X)^k)\Big) = \int_{\mathbb{R}} t^k \, d\mu_c(t)$$

(a) Calculate the first three moments of μ_c, by determining the limit of the corresponding moments of the Wishart matrices. Calculate from this the first three free cumulants of μ_c.

(b) It is true in general that all free cumulants of μ_c are equal to c. [For the case $c = 1$ this can be seen, modulo replacing non-selfadjoint X_N by selfadjoint ones, by combining Wigner's semicircle theorem with part (b) of question 2.] Given this information, show that the moment generating function $M(z) = 1 + \sum_{k=1}^{\infty} \int_{\mathbb{R}} t^k \, d\mu_c(t) z^k$ satisfies the equation

$$M(z) = 1 + czM(z) + zM(z)(M(z) - 1) \tag{1}$$

(c) By using this equation and invoking the Stieltjes inversion formula show that

$$d\mu_c(t) = \begin{cases} (1-c)\delta_0 + \frac{\sqrt{(b-t)(t-a)}}{2\pi t} \, dt & \text{if } c < 1 \text{ and} \\ \frac{\sqrt{(b-t)(t-a)}}{2\pi t} \, dt & \text{if } c \geq 1 \end{cases}$$

where δ_0 is the Dirac mass at 0, $a = (1-\sqrt{c})^2$, $b = (1+\sqrt{c})^2$, and the density $\frac{\sqrt{(b-t)(t-a)}}{2\pi t}$ is assumed to be 0 outside of the interval $[a, b]$.

[This density was found in 1967 by Marchenko and Pastur.]