## Free Probability and Random Matrices

## Exercise 1, Due September 27

**1)** Let $\nu$ be a probability measure on $\mathbb{R}$. show that if $\nu$ has a moment of order $k$ then $\nu$ has moments of order $m$ for $m < k$.

**2)** Suppose that the probability measure $\nu$ has a fourth moment, then its characteristic function $\phi(t) = 1 + \alpha_1 \frac{(it)}{1!} + \alpha_2 \frac{(it)^2}{2!} + \alpha_3 \frac{(it)^3}{3!} + \alpha_4 \frac{(it)^4}{4!} + o(t)$, where $\alpha_1$, $\alpha_2$, $\alpha_3$, and $\alpha_4$ are the first four moments. Let $\log(\phi(t)) = k_1(it) + k_2 \frac{(it)^2}{2!} + k_3 \frac{(it)^3}{3!} + k_4 \frac{(it)^4}{4!} + o(t)$. Using the Taylor series for $\log(1+x)$, find a formula for $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ in terms of $\{k_1, k_2, k_3, k_4\}$.

**3)** Let $\vec{X} = (X_1, X_2, \ldots, X_n)$ be a Gaussian random vector with density $\exp(-\frac{1}{2}\langle B\vec{t}, \vec{t}\rangle) \det(B)^{1/2}(2\pi)^{-n/2}$. Let $C = B^{-1}$ normalized so that $\mathrm{E}(|f_{ij}|^2) = 1$.

*i)* Show that $B$ is diagonal if and only if $\{X_1, \ldots X_n\}$ is independent.

*ii)* by first diagonalizing $B$ show that $c_{ij} = \mathrm{E}((X_i - \mathrm{E}(X_i))(X_j - \mathrm{E}(X_j)))$.

**4)** Let $Z = \frac{1}{\sqrt{2}}(X + iY)$ with $X$ and $Y$ independent real Gaussian random variables with mean 0 and variance 1.

*i)* By making the substitution $\vec{t} = O\vec{s}$ where $O = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, show that for $m \neq n$, $\displaystyle\int_{\mathbb{R}^2} (t_1 + it_2)^m (t_1 - it_2)^n e^{-(t_1^2 + t_2^2)}\, dt_1 dt_2 = 0$.

*ii)* Show that $\mathrm{E}(Z^m \bar{Z}^n) = 0$ for $m \neq n$.

*iii)* By switching to polar coordinates, show that $\mathrm{E}(|Z|^n) = n!$.

**5)** Let $X$ be an $N \times N$ GUE random matrix, with entries $f_{i,j} = x_{i,j} + \sqrt{-1}y_{i,j}$.

*i)* Consider the random $N^2$-vector:

$(x_{11}, x_{22}, \ldots, x_{NN}, x_{1,2}, \ldots, x_{1,N}, \ldots, x_{N-1,N}, y_{12}, \ldots, y_{1N}, \ldots, y_{N-1,N})$

Show that the density of the vector is $c\exp(-\frac{1}{2}\mathrm{Tr}(X^2))\, dX$, where $dX = \displaystyle\prod_{i=1}^{N} dx_{ii} \prod_{i<j} dx_{i,j} dy_{i,j}$ and $c$ is some constant.

*ii)* Evaluate the constant $c$.